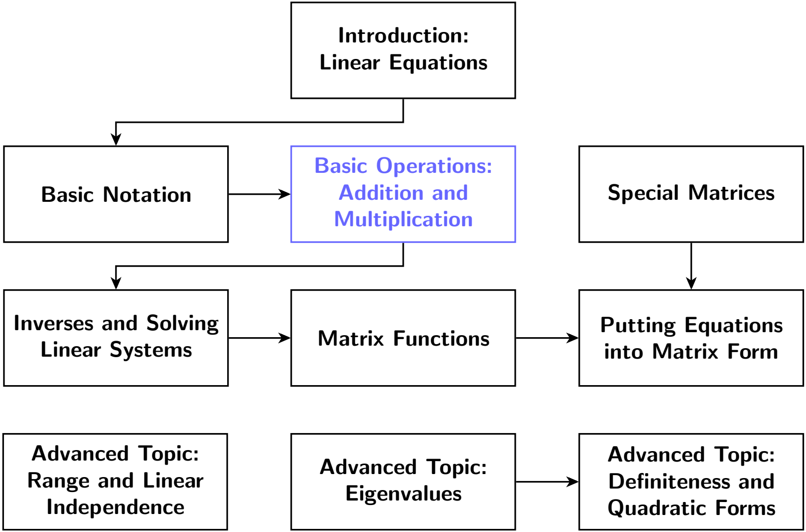
Linear Algebra Review



Matrix addition

- For two matrices *of the same size and type*, $A, B \in \mathbb{R}^{m \times n}$ addition is just sum of corresponding elements

$$A + B = C \in \mathbb{R}^{m \times n} \iff C_{ij} = A_{ij} + B_{ij}$$

- Addition is *undefined* for matrices of different sizes $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$

Matrix multiplication

- For two matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, their product is

$$AB = C \in \mathbb{R}^{m \times p} \iff C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

- Multiplication is undefined when number of columns in A doesn't equal number or rows in B (one exception: cA for $c \in \mathbb{R}$ taken to mean scaling A by c)

- Some imporant properties
  - Associative: ($A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$)

$$A(BC) = (AB)C$$

  - Distributive: ($A \in \mathbb{R}^{m \times n}$, $B, C \in \mathbb{R}^{n \times p}$)

$$A(B + C) = AB + AC$$

  - *NOT* commutative: (the dimensions might not even make sense, but this doesn't hold even when the dimensions are correct)

$$AB \neq BA$$

Vector-vector Products

- Inner product: $x, y \in \mathbb{R}^n$

$$x^T y \in \mathbb{R} = \sum_{i=1}^{n} x_i y_i$$

- Outer product: $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$xy^T \in \mathbb{R}^{n \times m} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_m \\ x_2y_1 & x_2y_2 & \cdots & x_2y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_m \end{bmatrix}$$

Matrix-vector Products

- $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n \iff Ax \in \mathbb{R}^m$

- Writing A by rows, each entry of Ax is an inner product between x and a row of A

$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}, \quad Ax \in \mathbb{R}^m = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

- Writing A by columns, Ax is a *linear combination* of the columns of A, with coefficients given by x

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}, \quad Ax \in \mathbb{R}^m = \sum_{i=1}^{n} a_i x_i$$

Matrix-matrix Products

- Write $A \in \mathbb{R}^{m \times n}$ by rows, $B \in \mathbb{R}^{n \times p}$ by columns: entries of AB are inner products of the rows of A and the columns of B

$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}, \quad B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix}$$

$$(AB)_{ij} = a_i^T b_j$$

- Write $A \in \mathbb{R}^{m \times n}$ by columns, $B \in \mathbb{R}^{n \times p}$ by rows: AB is a sum of outer products of columns of A and rows of B

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}, \quad B = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix}$$

$$AB \in \mathbb{R}^{m \times p} = \sum_{i=1}^{n} a_i b_i^T$$

- Leave $A \in \mathbb{R}^{m \times n}$ as complete matrix, write $B \in \mathbb{R}^{n \times p}$ by columns: columns of AB are matrix-vector products between A and columns of B

$$B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix}$$

$$AB \in \mathbb{R}^{m \times p} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$