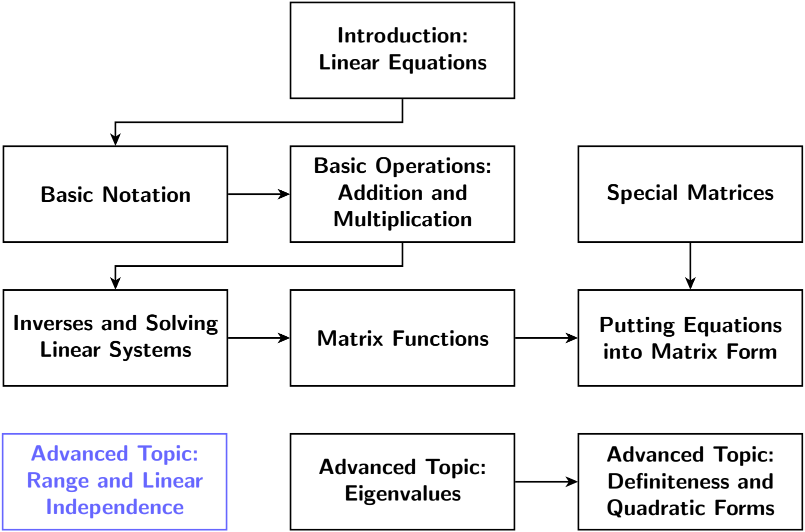
Linear Algebra Review



Range

- For $A \in \mathbb{R}^{m \times n}$, *range* of A is the set of all vectors that can be written Ax for some $x \in \mathbb{R}^n$

$$\mathcal{R}(A) \subseteq \mathbb{R}^m = \{y : y = Ax, x \in \mathbb{R}^n\}$$

- The columns of A are *linearly independent* if no column is in the range of the remaining columns

$$a_i \notin \mathcal{R}(A_{-i}), \forall i = 1, \ldots, n$$

Rank

- *Rank* of $A \in \mathbb{R}^{m \times n}$ is the number of linearly indpendent columns

- Some important properties

  - $\mathrm{rank}(A) = \mathrm{rank}(A^T)$

  - For $A \in \mathbb{R}^{n \times n}$,

$$\mathrm{rank}(A) = n \;\Leftrightarrow\; \mathcal{R}(A) = \mathbb{R}^n \;\Leftrightarrow\; A \text{ non-singular}$$

Orthogonality

- Two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* if

$$x^T y = 0$$

- They are *orthonormal* if, in addition,

$$\|x\|_2 = \|y\|_2 = 1$$

- A matrix $U \in \mathbb{R}^{n \times n}$ is orthogonal if all it's columns are orthonormal, i.e.,

$$U^T U = I = U U^T$$

- Columns of an orthogonal matrix are linearly independent

Nullspace

- for $A \in \mathbb{R}^{m \times n}$, *nullspace* of A is set of all vectors x s.t. $Ax = 0$

$$\mathcal{N}(A) \subseteq \mathbb{R}^n = \{x : Ax = 0\}$$

- $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$ are *orthogonal complements*

$$\mathcal{R}(A) \cup \mathcal{N}(A^T) = \mathbb{R}^m, \;\; \mathcal{R}(A) \cap \mathcal{N}(A^T) = \{0\}$$