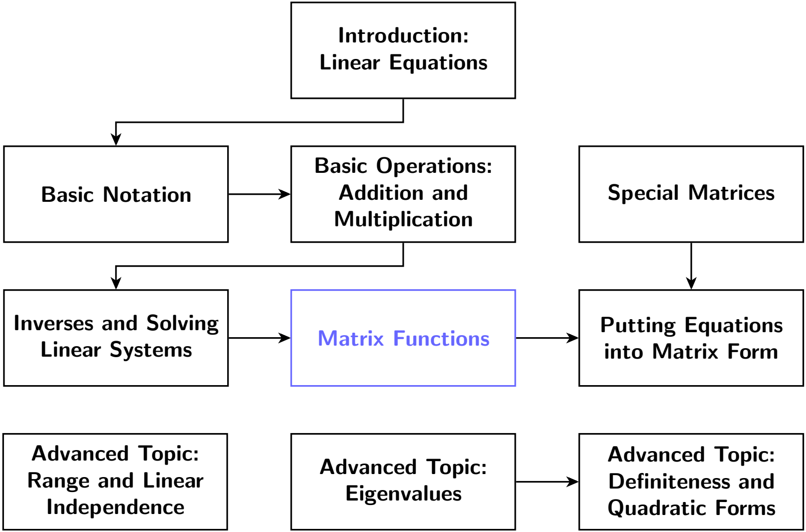
Linear Algebra Review



Notation for Functions

- $f(x) = x^2$, $f : \mathbb{R} \rightarrow \mathbb{R}$

- Function with matrix inputs/outputs

$$f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{p \times q}$$

Some Examples

- Transpose: $f(A) = A^T$

$$f : \mathbb{R}^{m \times n} \to \mathbb{R}^{n \times m}$$

- Inverse: $f(A) = A^{-1}$

$$f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$$

- Multiplication: $f(x) = Ax$ for $A \in \mathbb{R}^{m \times n}$

$$f : \mathbb{R}^{n} \to \mathbb{R}^{m}$$

The Trace

- $\mathrm{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$

$$\mathrm{tr}\, A = \sum_{i=1}^{n} A_{ii}$$

- Some properties
  - $\mathrm{tr}\, A = \mathrm{tr}\, A^T$, $A \in \mathbb{R}^{n\times n}$
  - $\mathrm{tr}(A + B) = \mathrm{tr}\, A + \mathrm{tr}\, B$, $A, B \in \mathbb{R}^{n\times n}$
  - $\mathrm{tr}\, AB = \mathrm{tr}\, BA$, $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$

Norms

- A vector norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ with

    1. $f(x) \geq 0$ and $f(x) = 0 \Leftrightarrow x = 0$

    2. $f(ax) = |a| f(x)$ for $a \in \mathbb{R}$

    3. $f(x + y) \leq f(x) + f(y)$

- ℓ_2 norm

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- ℓ_1 norm

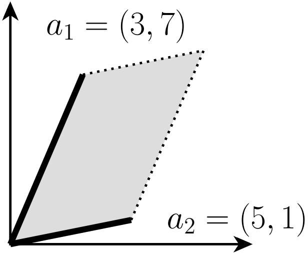
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

- ℓ_∞ norm

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

Determinant

- $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ (sometimes denoted $|\cdot|$)



$$A = \begin{bmatrix} 3 & 7 \\ 5 & 1 \end{bmatrix}$$

- $|\det A|$ is the area of the parallelogram

- Can be formally defined by three properties

  1. Determinant of identity is one: $\det I = 1$

  2. Multiplying a row by scalar $t \in \mathbb{R}$ scales determinant:

$$\det \begin{bmatrix} — & ta_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} = t \det A$$

  3. Swapping rows negates determinant:

$$\det \begin{bmatrix} — & a_2^T & — \\ — & a_1^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} = -\det A$$

- Important properties
  - $\det A = \det A^T$
  - $\det AB = \det A \det B$
  - $\det A = 0 \Leftrightarrow$ A singular (non-invertible)
  - $\det A^{-1} = 1/\det A$