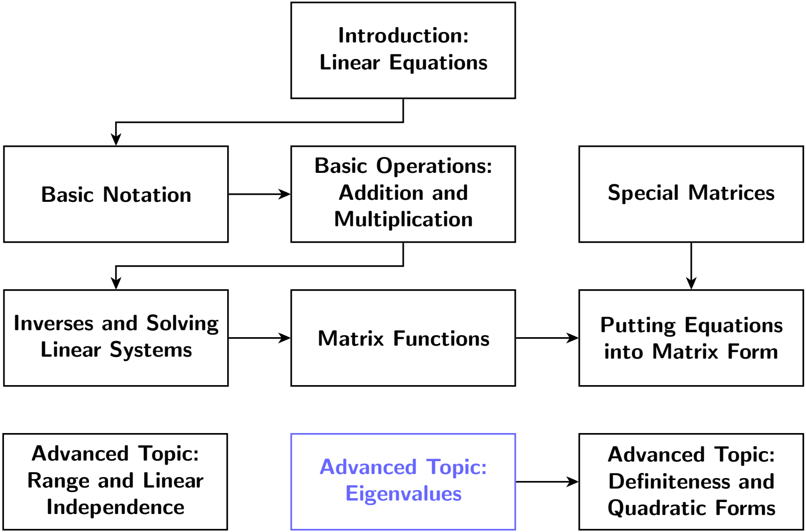
Linear Algebra Review



Eigenvalues and Eigenvectors

- For $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an *eigenvalue* and $x \in \mathbb{C}^n \neq 0$ an *eigenvector* if

$$Ax = \lambda x$$

- Satisfied if $(\lambda I - A)x = 0$, which we know exists if and only if $\det(\lambda I - A) = 0$

- $\det(\lambda I - A)$ is a polynomial (of degree n) in λ, its n roots are the n eigenvalues of A

Diagonalization

- Write equations for all n eigenvalues as

$$A \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

- Write as $AX = X\Lambda$, which implies

$$A = X\Lambda X^{-1}$$

if X is invertible (A *diagonalizable*)

- Important properties of eigenvectors/eigenvalues
  - $\mathrm{tr}\, A = \sum_{i=1}^{n} \lambda_i$
  - $\det A = \prod_{i=1}^{n} \lambda_i$
  - $\mathrm{rank}(A) =$ number of non-zero eigenvalues
  - Eigenvalues of A^{-1} are $1/\lambda_i$, $i = 1, \ldots, n$, eigenvectors are the same

- An example: Given $A \in \mathbb{R}^{n \times n}$, what can we say about A^k as $k \to \infty$?

Symmetric Matrices

- For a symmetric matrix $A \in \mathbb{R}^{n\times n})$ $(A = A^T)$, we have the following properties

    1. All eigenvalues/eigenvectors of A are real (more correctly, eigenvectors can be chosen to be real)

    2. The eigenvectors of A are orthogonal (can be chosen to be orthogonal)

- Implies that A can be diagonalized as

$$A = U\Lambda U^T$$

- Eigenvalues of symmetric matrix are real

- Eigenvectors of symmetric matrix can be chosen to be real

- Eigenvectors of symmetric matrix can be chosen to be orthogonal