A Composition Theorem for Parity Kill Number

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Joint with
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Outline

- Preliminaries
- Background
- Parity kill number
- Function composition
- Main Result
- Future Direction

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$$f(x_1,x_2,x_3,x_4) = \begin{cases} 1 & \text{if } x_1 \le x_2 \le x_3 \le x_4 \text{ or } x_1 \ge x_2 \ge x_3 \ge x_4 \\ -1 & \text{otherwise} \end{cases}$$

[Amb06]

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$$f(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4 \quad \deg[f] = 2$$

Sparsity

Definition: $\mathrm{sparsity}[\hat{f}]$

The number of non-zero coefficients in its Fourier expansion

Fourier coefficients

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$$

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$$\left(\frac{1+x_1}{2}\right)\left(\frac{1+x_2}{2}\right)\cdots\left(\frac{1+x_n}{2}\right)$$

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$$f(x) = 2\left(\frac{1+x_1}{2}\right)\left(\frac{1+x_2}{2}\right)\cdots\left(\frac{1+x_n}{2}\right) - 1$$

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- Examples:
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sparsity$[\hat{f}] = 4$

HI function

$$\mathrm{HI}(x_1, x_2, \ldots, x_6) = \begin{cases} 1 & \text{if \#1's in } x \text{ is } 1, 2, 6 \\ -1 & \text{if \#1's in } x \text{ is } 0, 4, 5 \\ ? & \text{if \#1's in } x \text{ is } 3 \end{cases}$$

[NW95]

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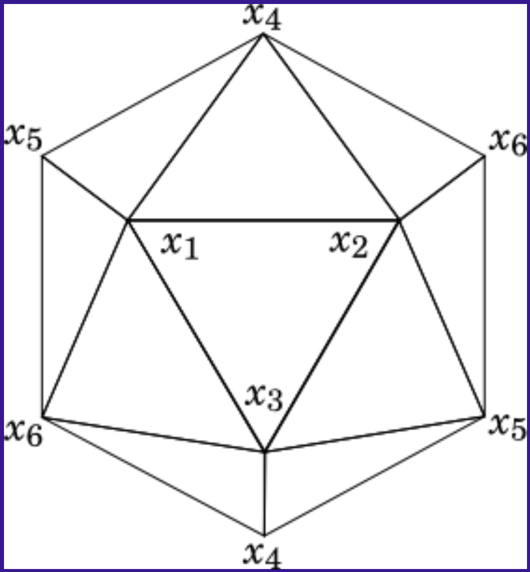
$\mathrm{HI}(x) = 1$ iff one triangle in the graph has all vertices 1



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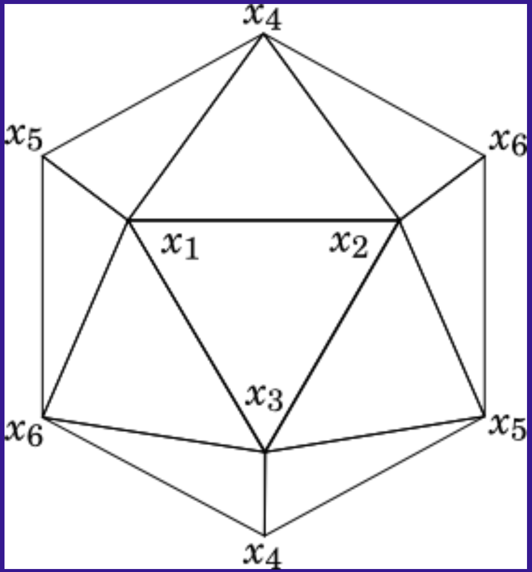

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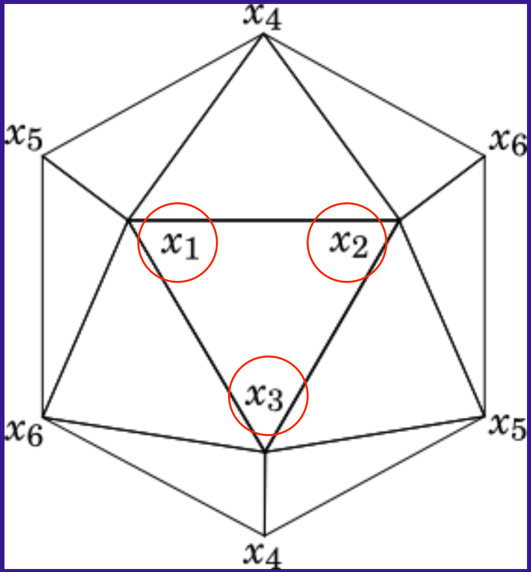

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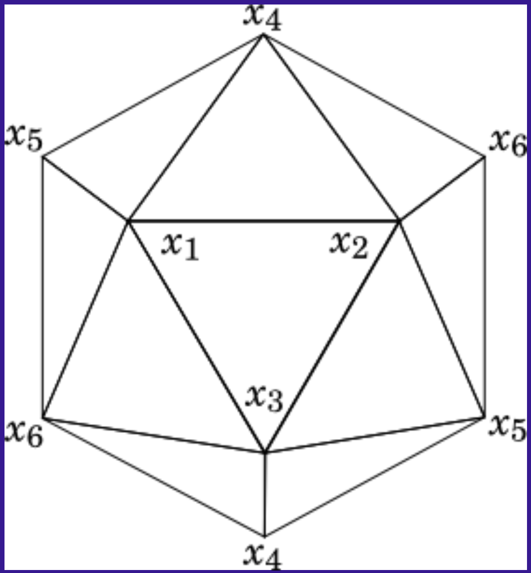

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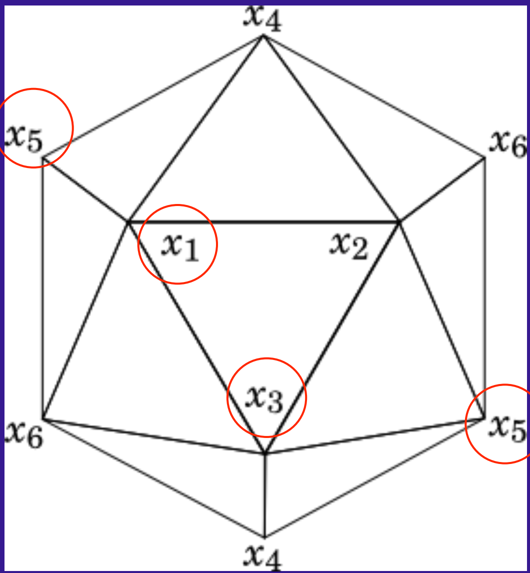

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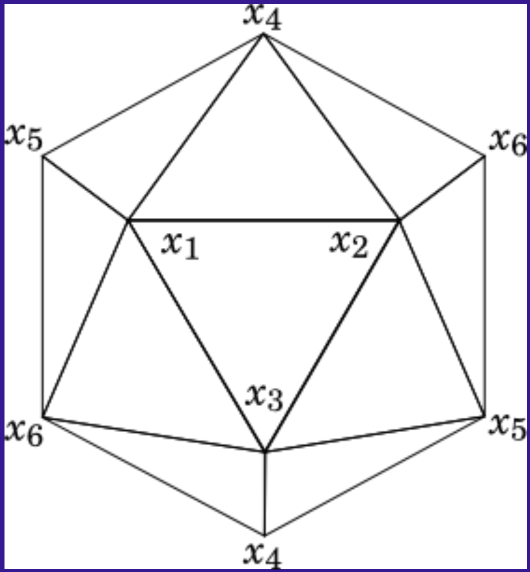

$$\mathrm{HI}(x) = \frac{1}{4}\Big(-\sum_i x_i + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 + x_1x_4x_5 + x_1x_5x_6 + x_2x_3x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_4x_6\Big)$$

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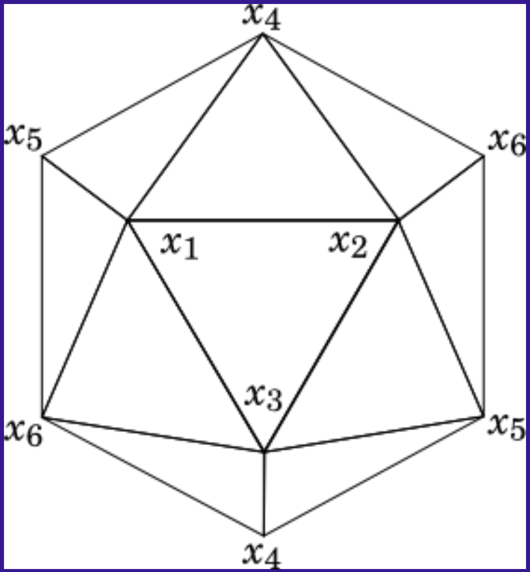

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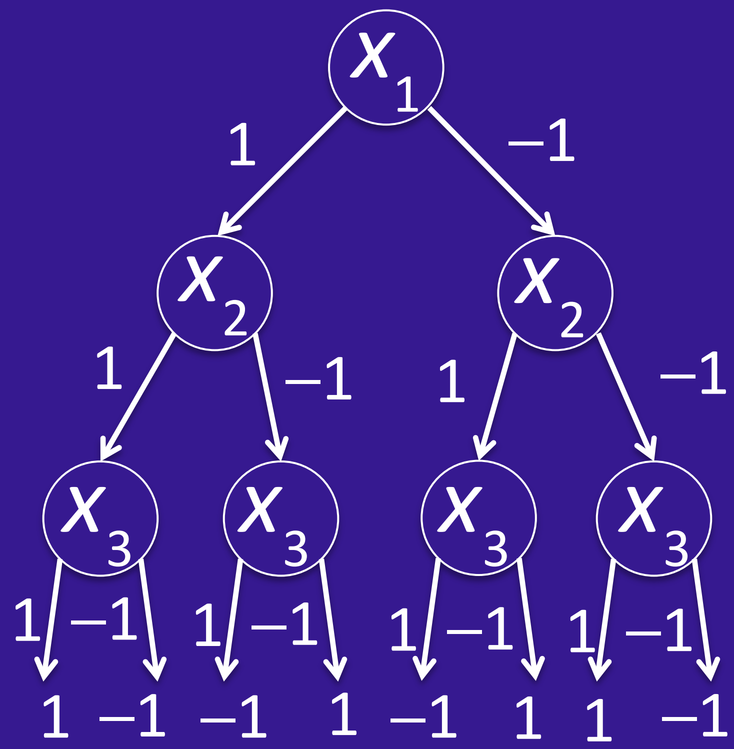
$\deg[\mathrm{HI}]=3$

$\mathrm{sparsity}[\hat{f}]=16$

Decision Tree

PARITY$_3$ $f(x) = x_1 x_2 x_3$



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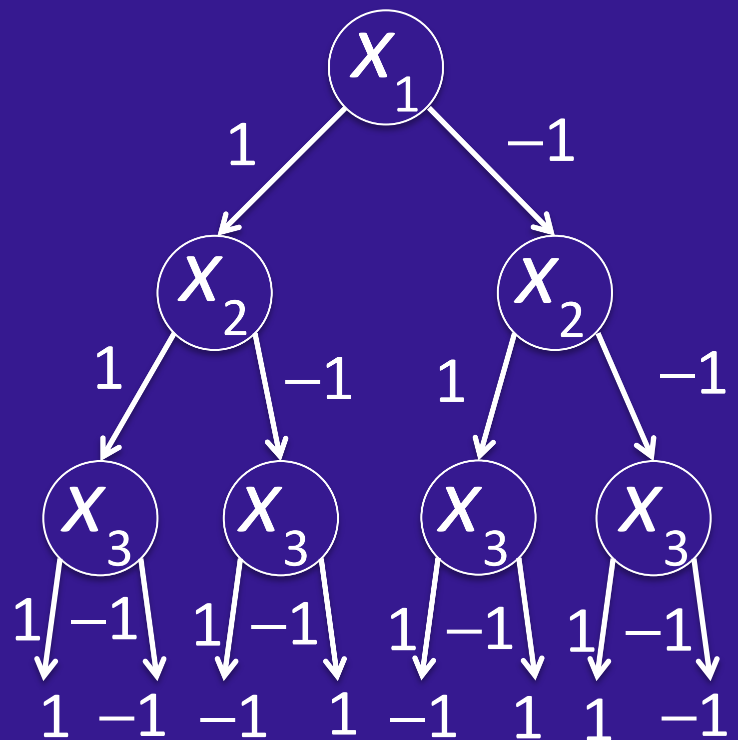

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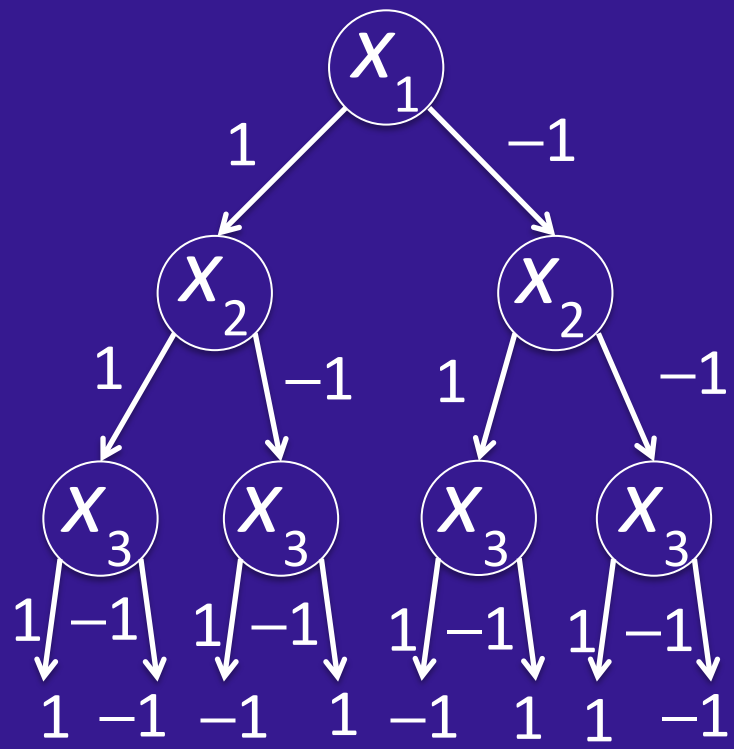

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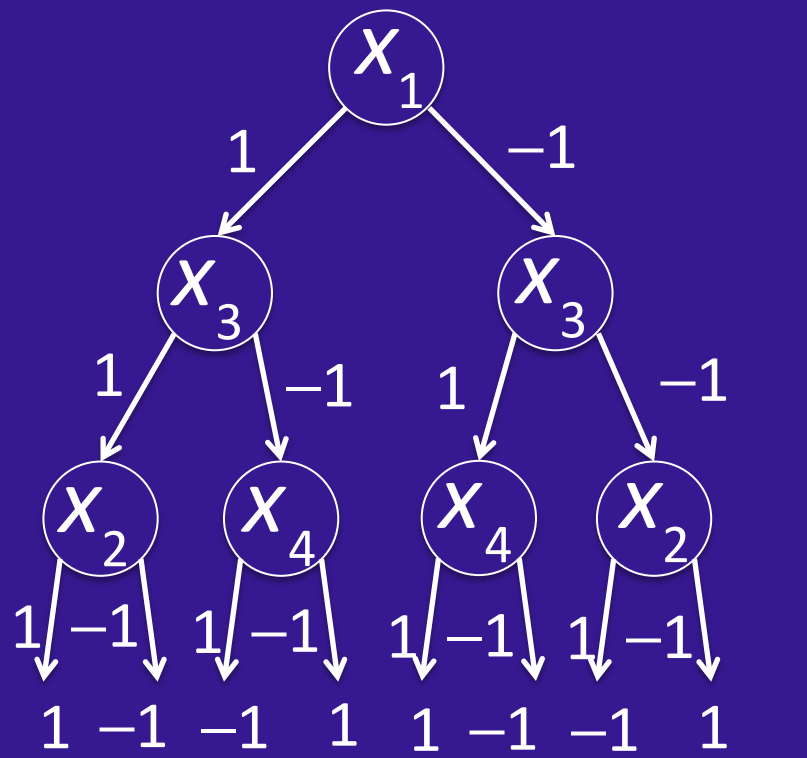
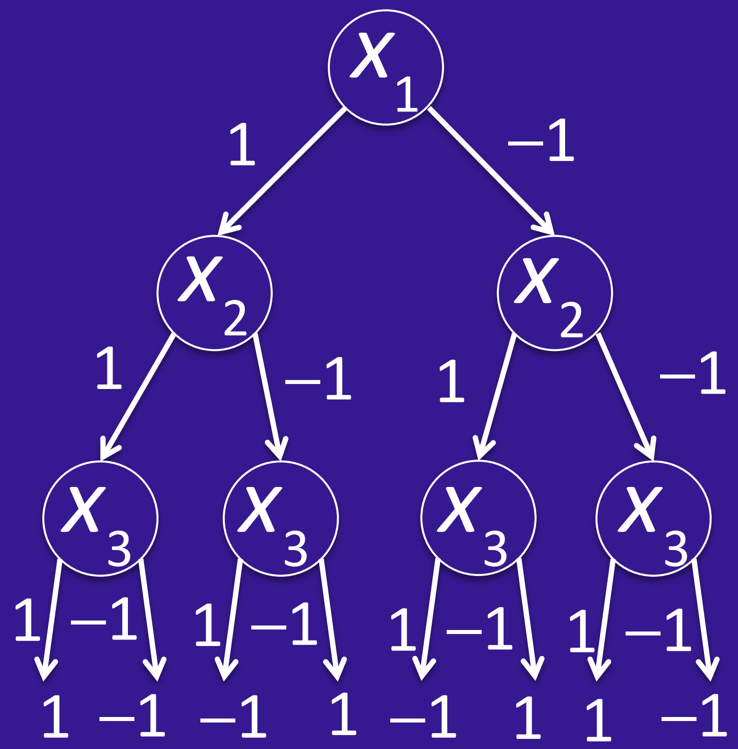


Decision Tree

x_1	x_2	x_3	x_4	**Sort**
1	?	1	?	

PARITY$_3$ $\quad f(x) = x_1 x_2 x_3$



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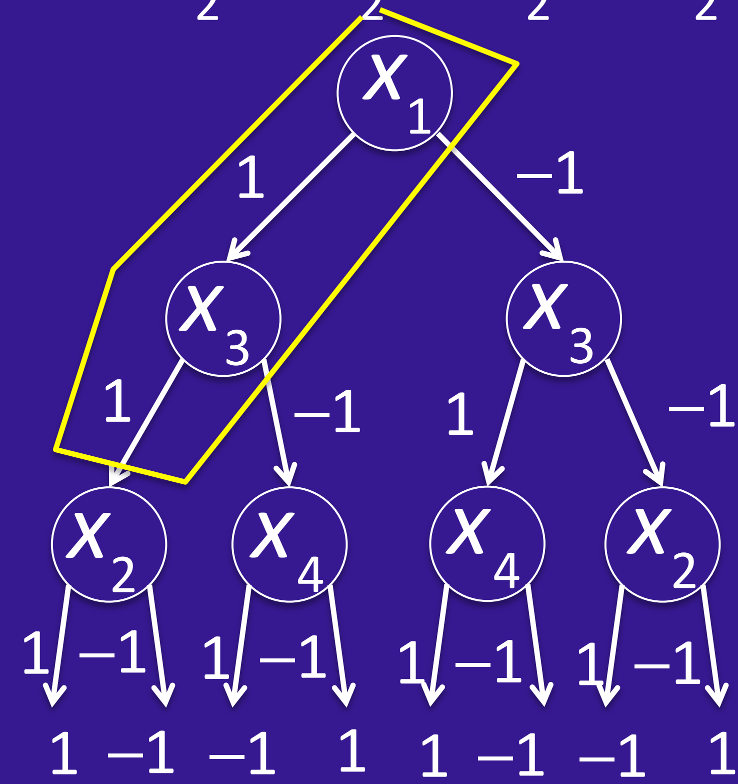

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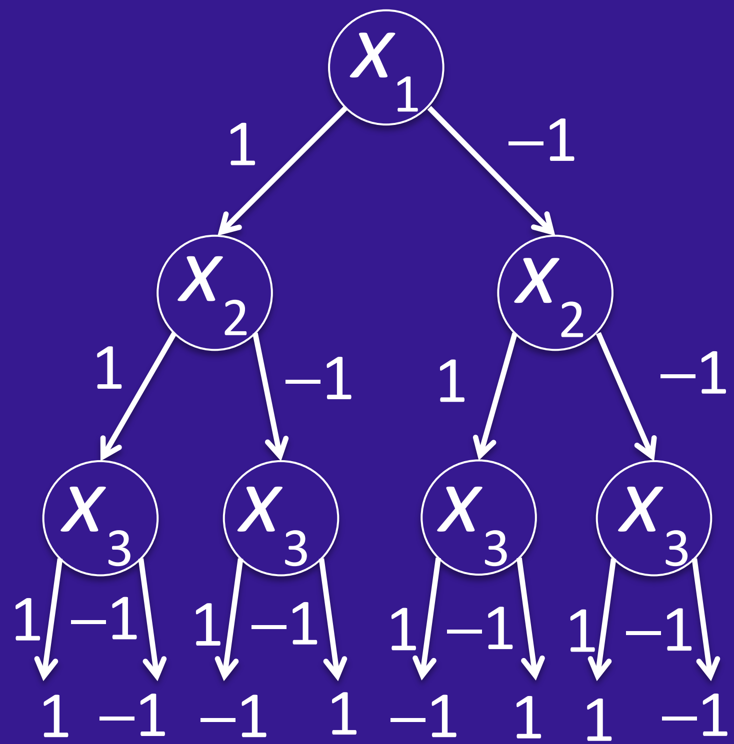

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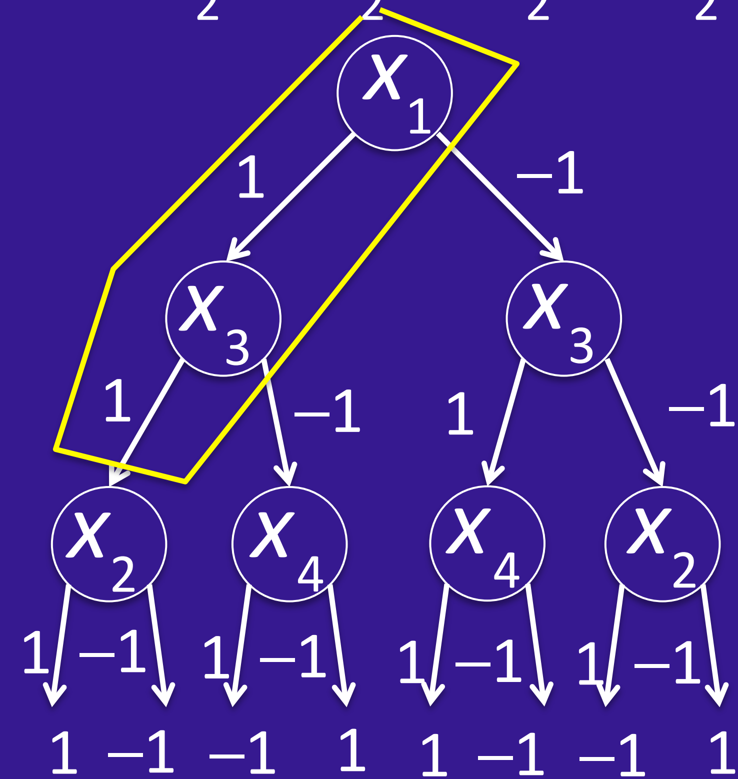

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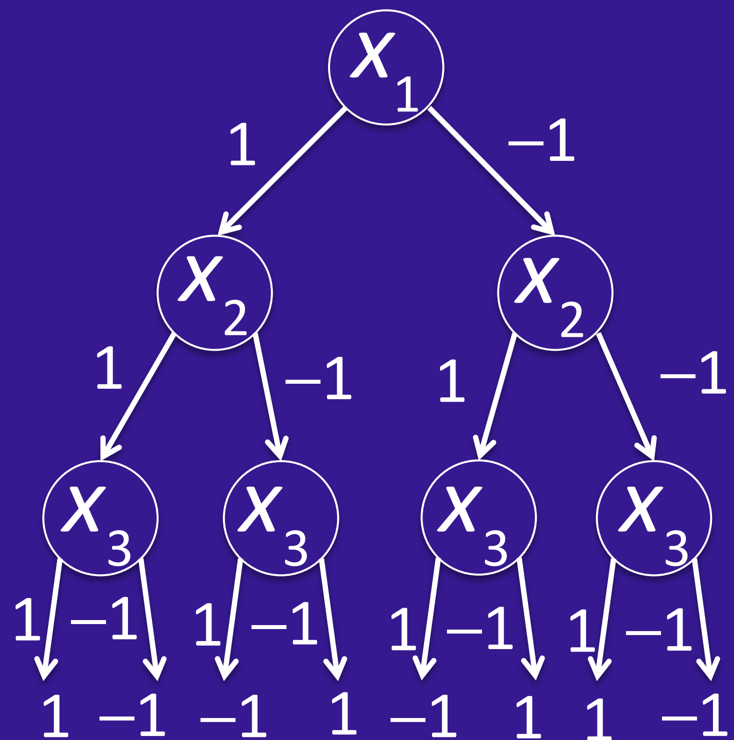

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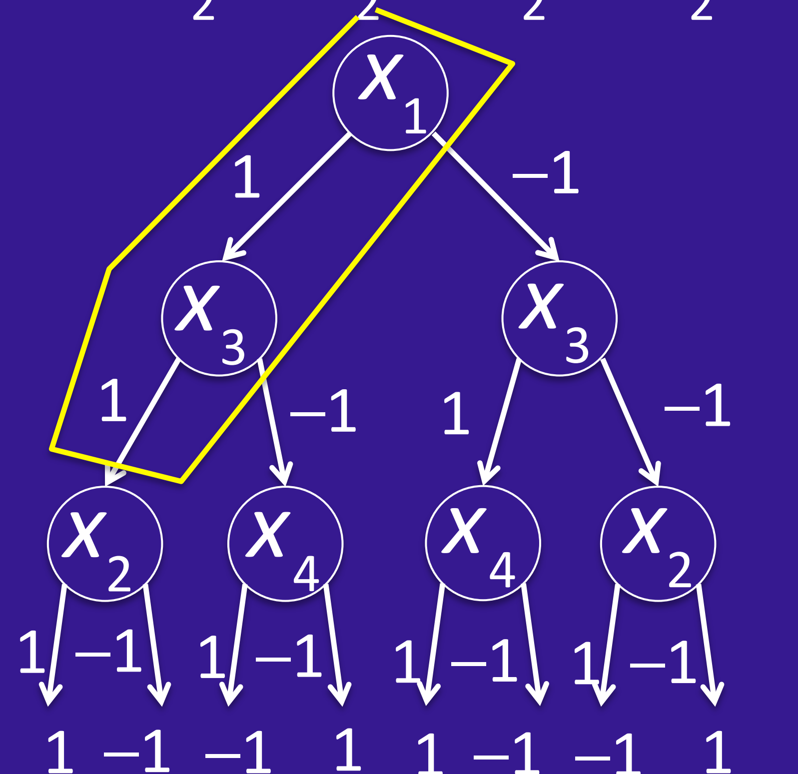

Decision Tree

$\mathrm{DT}[f]$: The depth of shortest decision tree which computes f

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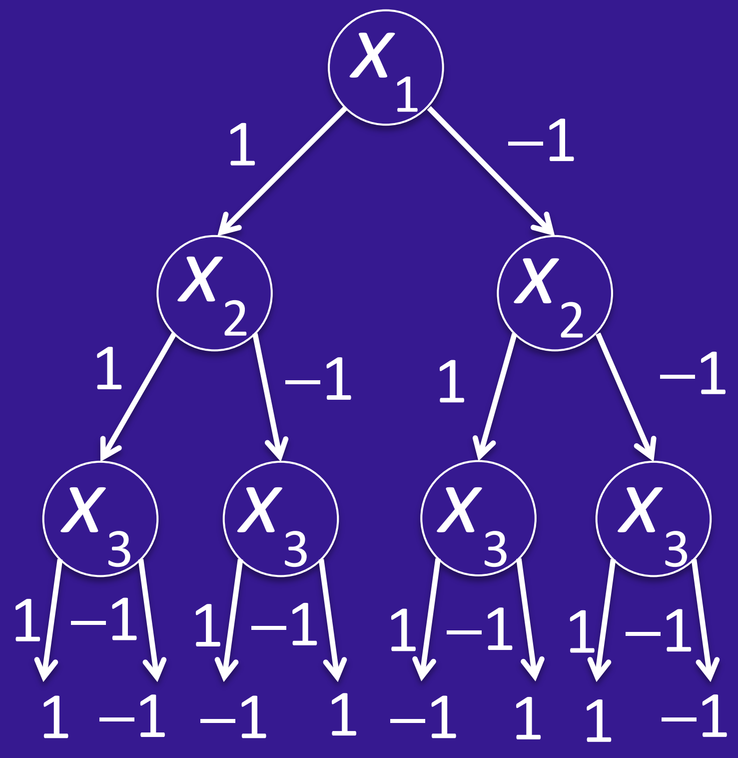

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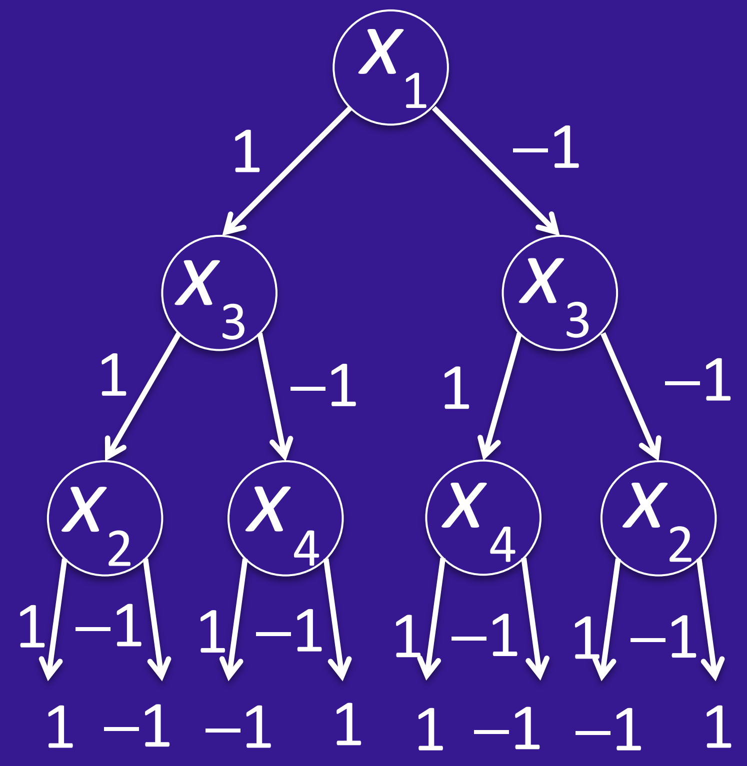

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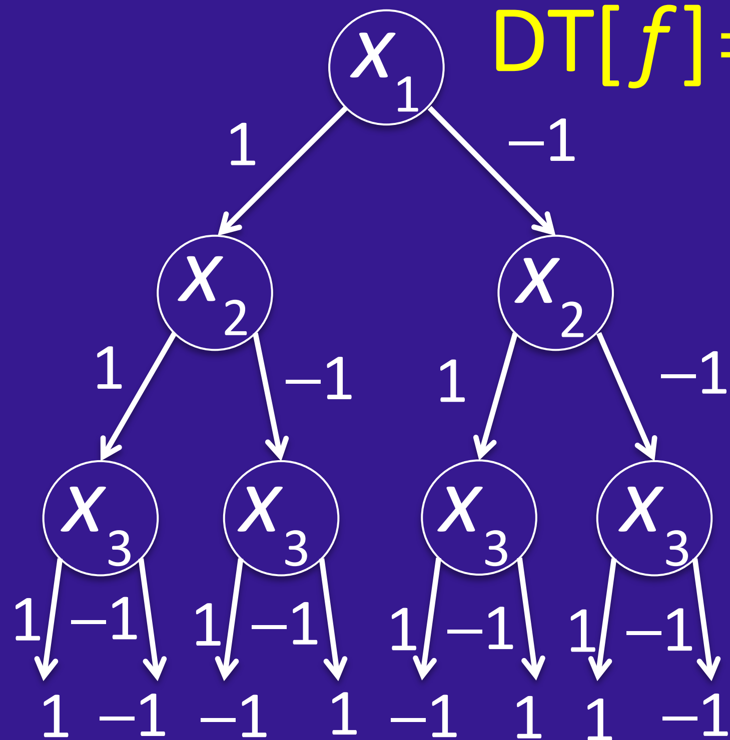
$DT[f] = 3$



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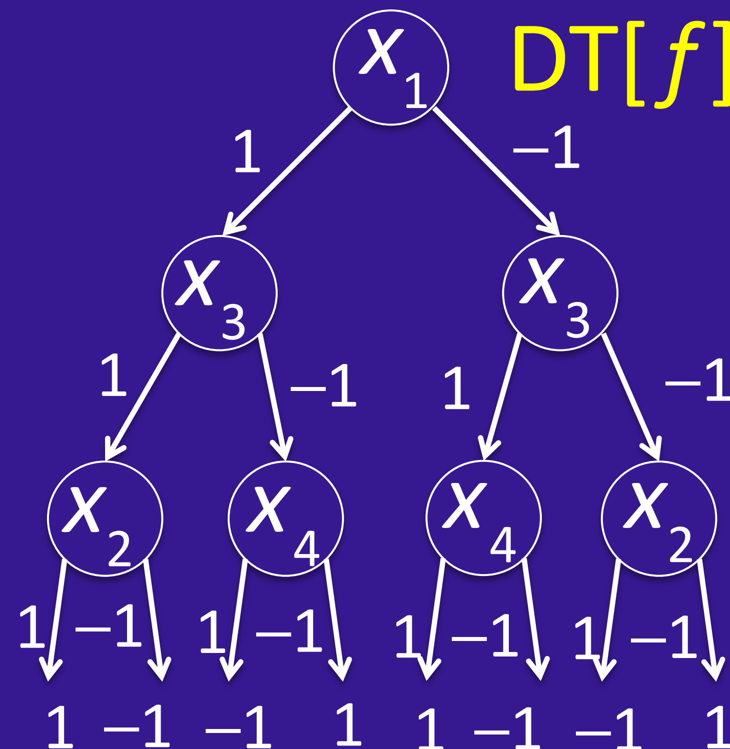

Parity Decision Tree

$\mathrm{DT}^{\oplus}[f]$: The depth of shortest parity decision tree which computes f

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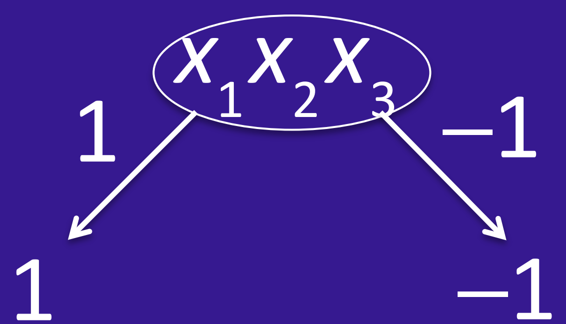
$$DT^{\oplus}[f]=1$$



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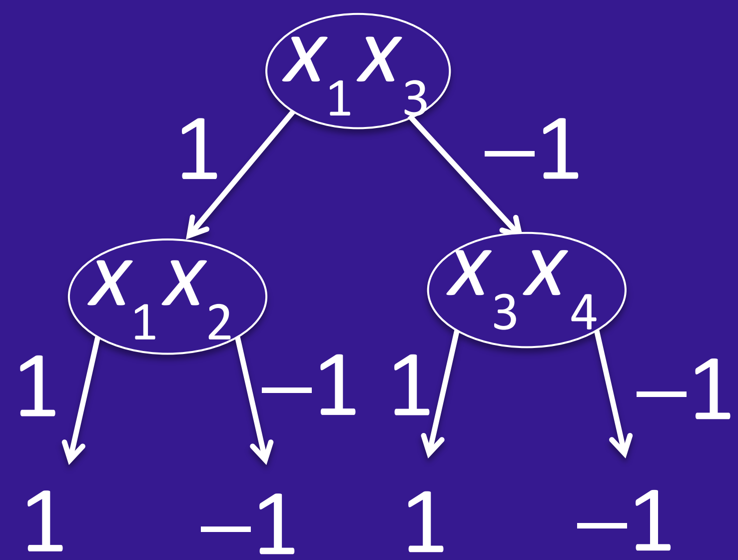
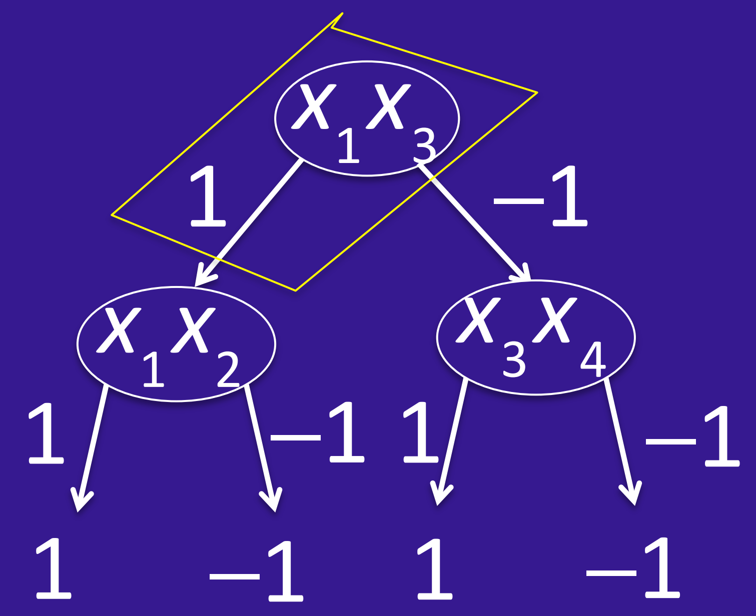


Parity Decision Tree

$\mathrm{DT}^{\oplus}[f]$: The depth of shortest parity decision tree which computes f

SORT $f(x)=\frac{1}{2}x_1x_2+\frac{1}{2}x_2x_3+\frac{1}{2}x_3x_4-\frac{1}{2}x_1x_4$

x_1	x_2	x_3	x_4	**Sort**
1	?	1	?	
-1	?	-1	?	



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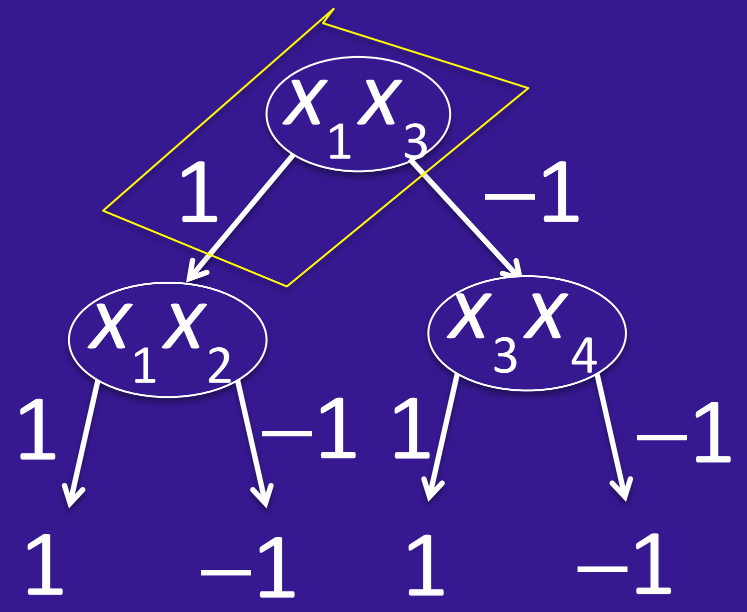
x_1	x_2	x_3	x_4	**Sort**
1	1	1	?	1
-1	-1	-1	?	1



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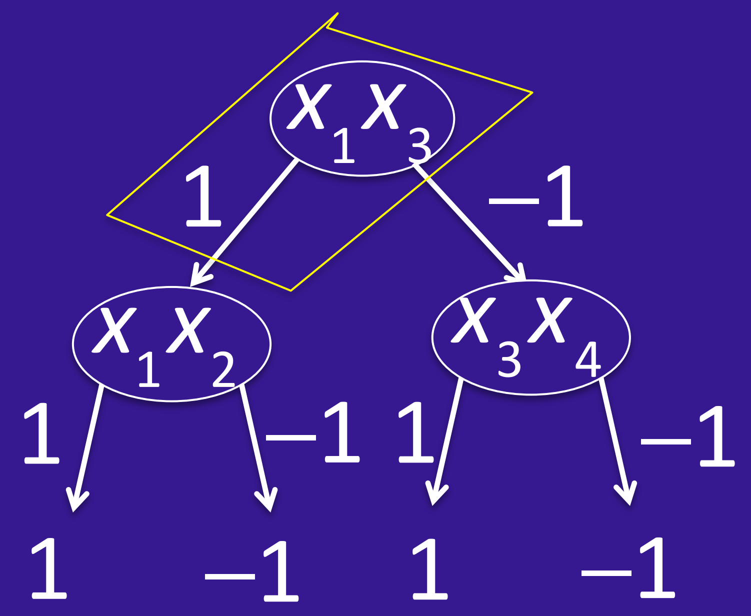
1	-1	1	?	-1
-1	1	-1	?	-1



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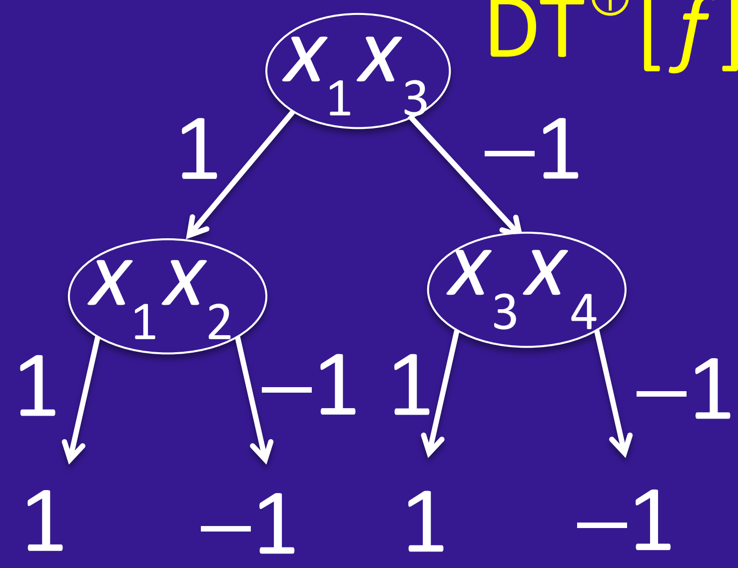
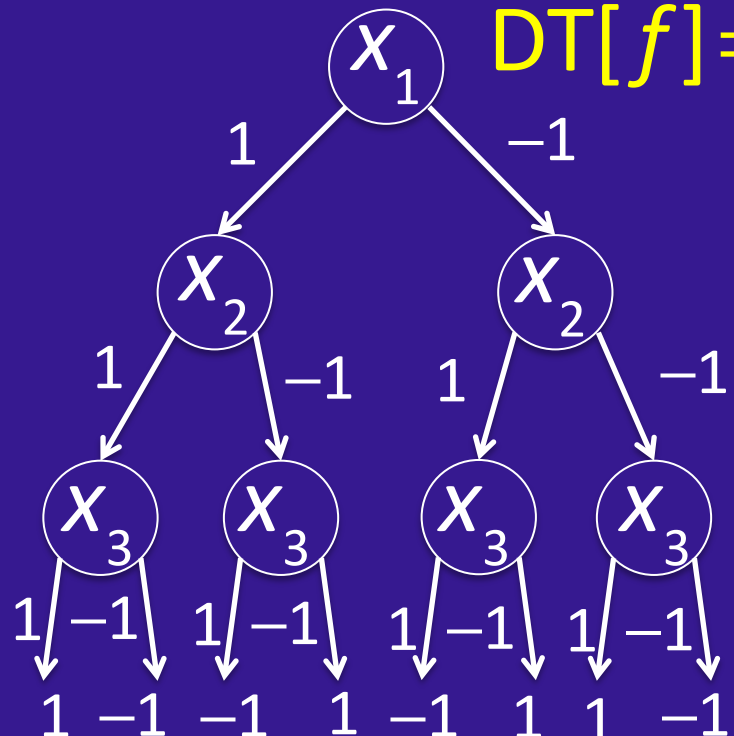
$\mathrm{DT}^{\oplus}[f] = 2$



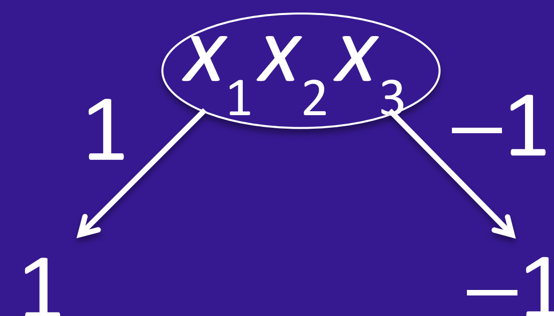
DT vs Parity DT

PARITY_3 $f(x) = x_1 x_2 x_3$ $\text{sparsity}[\hat{f}] = 1$

$\text{DT}[f] = 3$



$\text{DT}^{\oplus}[f] = 1$



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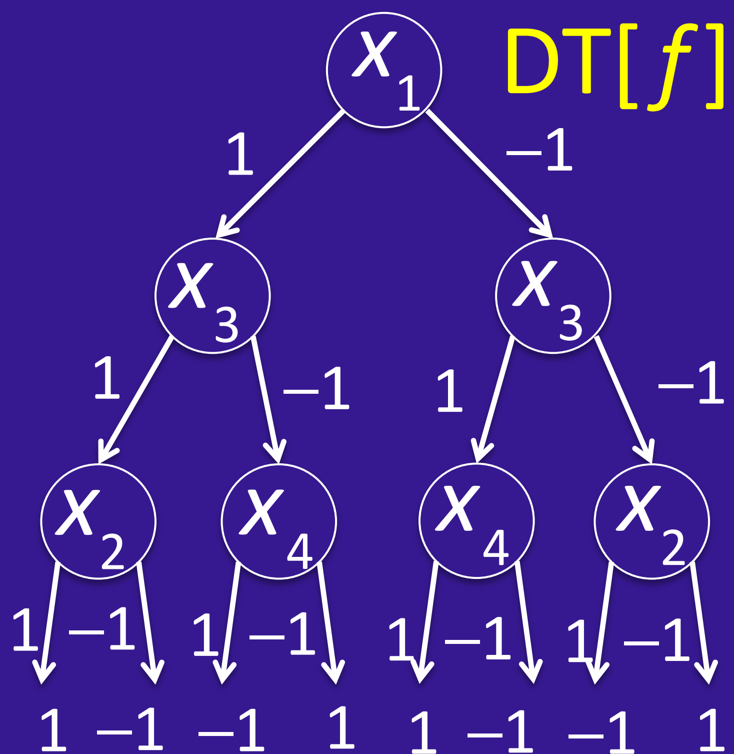
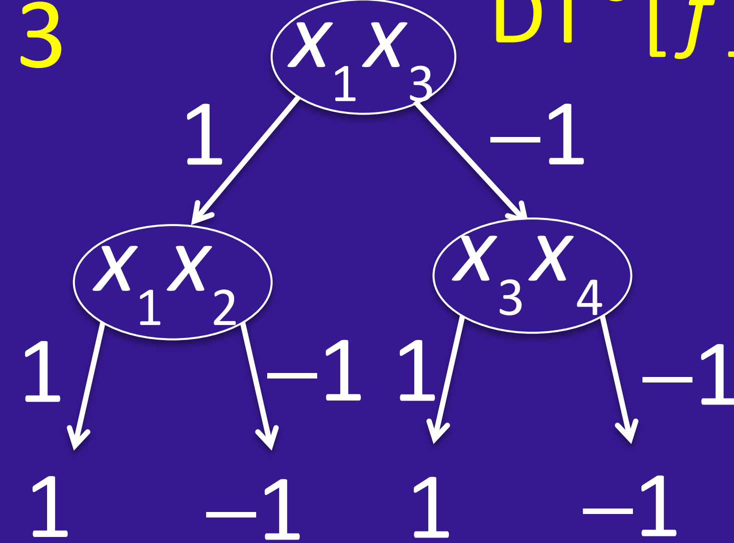
$\mathrm{sparsity}[\hat{f}]=4$

$\mathrm{DT}[f]=3$



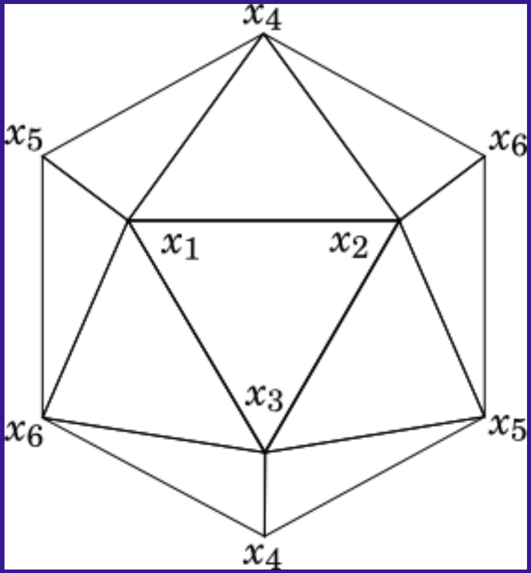
$\mathrm{DT}^{\oplus}[f]=2$



DT vs Parity DT

$$\mathrm{HI}(x_1, x_2, \ldots, x_6) = \begin{cases} 1 & \text{if \#1's in } x \text{ is } 1, 2, 6 \\ -1 & \text{if \#1's in } x \text{ is } 0, 4, 5 \\ ? & \text{if \#1's in } x \text{ is } 3 \end{cases}$$

$\mathrm{HI}(x) = 1$ iff one triangle in the graph has all vertices 1



$\mathrm{DT}[\mathrm{HI}] = 6$

DT vs Parity DT

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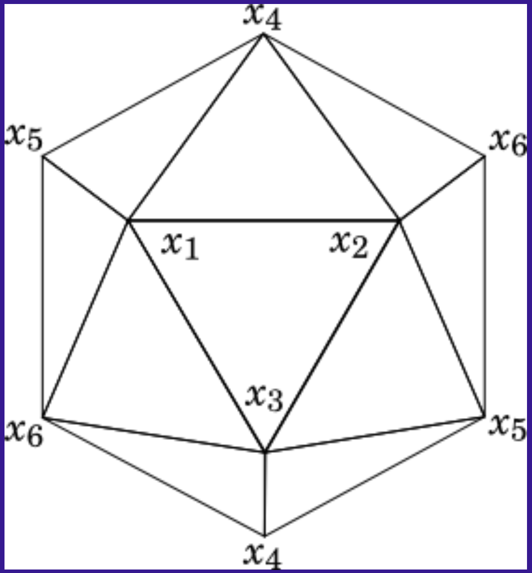
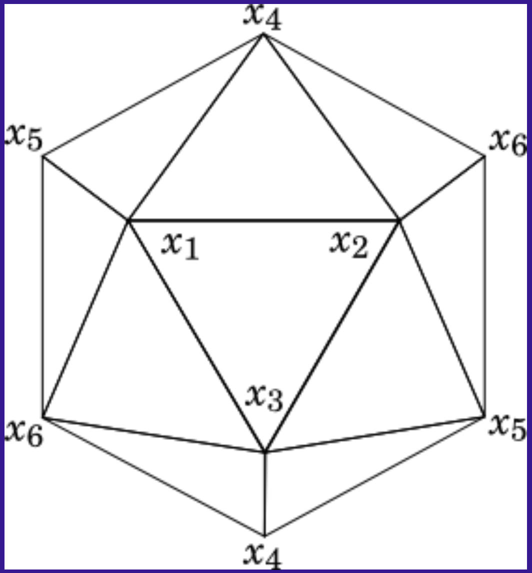


$\mathrm{DT}[\mathrm{HI}] = 6$

X_1	X_2	X_3	X_4	X_5	X_6	**HI**
1	1	?	1	1	1	?

DT vs Parity DT

$$\mathrm{HI}(x_1, x_2, \ldots, x_6) = \begin{cases} 1 & \text{if \#1's in } x \text{ is } 1, 2, 6 \\ -1 & \text{if \#1's in } x \text{ is } 0, 4, 5 \\ ? & \text{if \#1's in } x \text{ is } 3 \end{cases}$$



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x_1	x_2	x_3	x_4	x_5	x_6	**HI**
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DT vs Parity DT

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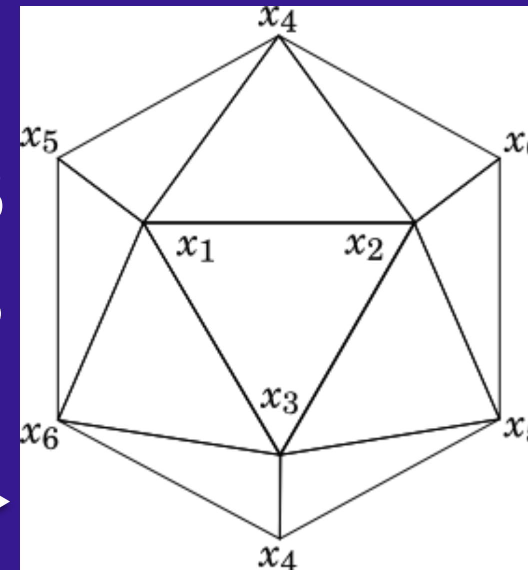
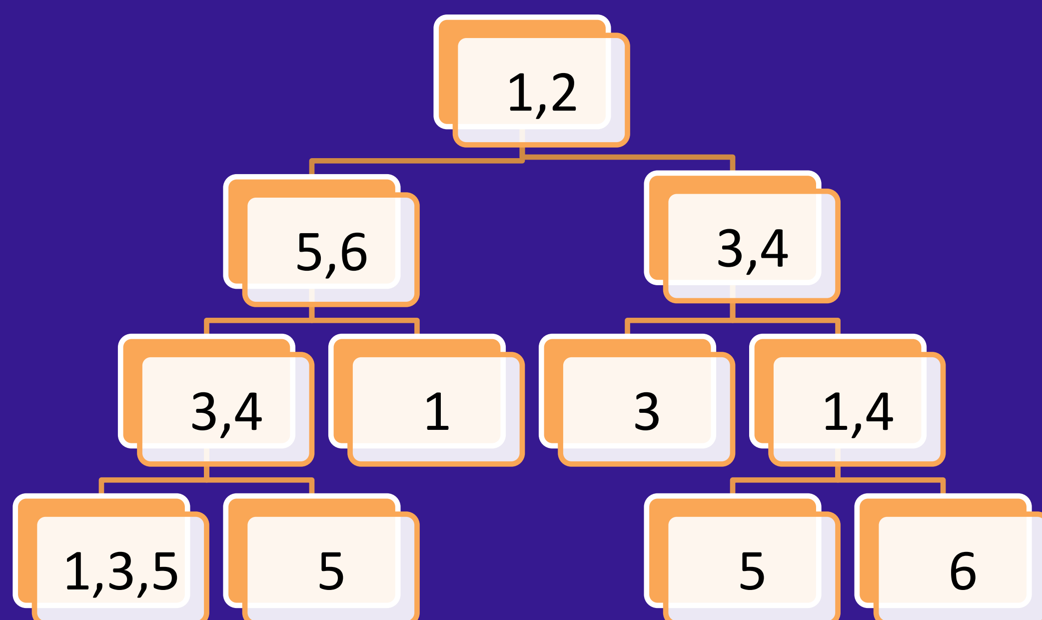


$\mathrm{HI}(x)=1$ iff one triangle in the graph has all vertices 1

$$\mathrm{DT}[\mathrm{HI}]=6$$

$$\mathrm{DT}^{\oplus}[\mathrm{HI}]=4$$



Parity Decision Tree

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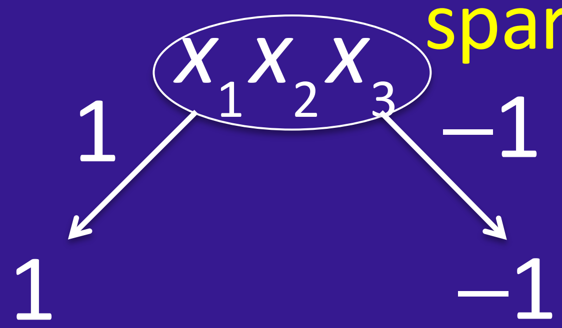
PARITY$_3$ $\quad f(x) = x_1 x_2 x_3$ $\quad$ SORT $\quad f(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$

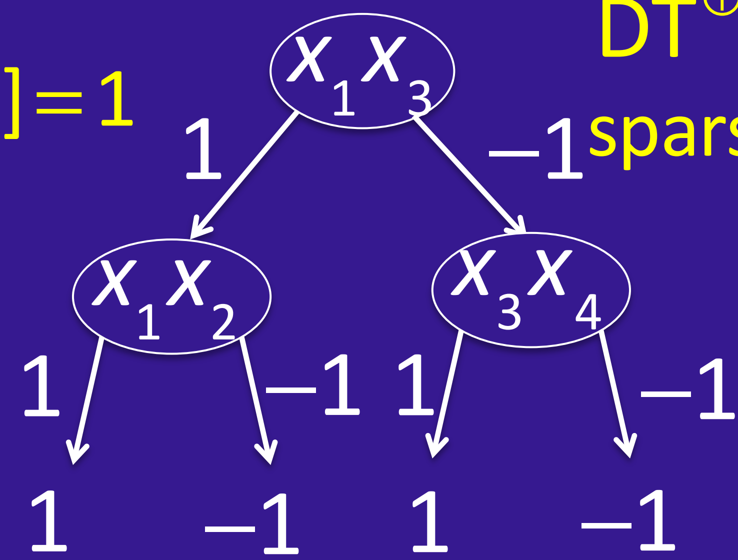
$\mathrm{DT}^{\oplus}[f] = 1$

$\mathrm{sparsity}[\hat{f}] = 1$



$\mathrm{DT}^{\oplus}[f] = 2$

$\mathrm{sparsity}[\hat{f}] = 4$



PDT vs Sparsity

sparsity$[\hat{f}]$ small $\not\longrightarrow$ DT[f] small

PDT vs Sparsity

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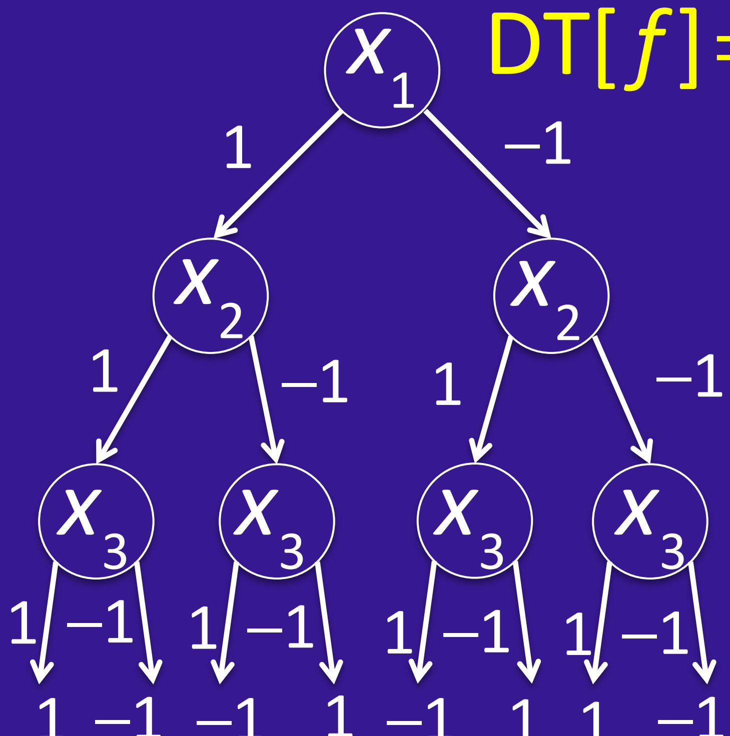

PDT vs Sparsity

$\text{sparsity}[\hat{f}]$ small $\xrightarrow{\times}$ DT[f] small

$\xrightarrow{?}$ $DT^{\oplus}[f]$ small

PDT vs Sparsity

sparsity$[\hat{f}]$ small $\xrightarrow{\times}$ DT[f] small

$\longrightarrow$ $DT^{\oplus}[f]$ small

$$DT^{\oplus}[f] \leq \text{sparsity}[\hat{f}]$$

PDT vs Sparsity

sparsity$[\hat{f}]$ small $\not\longrightarrow$ DT[f] small

$\longrightarrow$ $\mathrm{DT}^{\oplus}$[f] small

$\mathrm{DT}^{\oplus}[f] \le \mathrm{sparsity}[\hat{f}]$

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These two quantities ($\mathrm{DT}^{\oplus}$ and sparsity$[\hat{f}]$) were linked in the papers of [MO09] and [ZS10], both of which posed the following question:

Given a sparse Boolean function, must it have a short parity decision tree?

Outline

- Preliminaries
- Background
- Parity kill number
- Function composition
- Main Result
- Future Illustration

Log rank conjecture

Log rank conjecture

$x \in \{-1,1\}^n$

Alice

(･A･)

$y \in \{-1,1\}^n$

Bob

(˙‿˙)

Log rank conjecture

$x \in \{-1,1\}^n$ $\quad g(x,y) \quad$ $y \in \{-1,1\}^n$

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(･A･)?

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(｡･‿･｡)?

Log rank conjecture

$x \in \{-1,1\}^n$ $\qquad g(x,y) \qquad$ $y \in \{-1,1\}^n$

Alice $\qquad\qquad\qquad\qquad$ Bob

(*`^´)=3 ⟶ (・‿・)

Log rank conjecture

$x \in \{-1,1\}^n$

Alice

$g(x,y)$

$y \in \{-1,1\}^n$

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(･A･)

Σ(=ω= ;)

Log rank conjecture

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ヽ(`Д´)ノ $\qquad\qquad$ (・ ‿ ・)

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$x \in \{-1,1\}^n$

$g(x,y)$

$y \in \{-1,1\}^n$

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Bob

(·A·)

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…

Log rank conjecture

$x \in \{-1,1\}^n$

$g(x,y)$

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Alice

Bob

…

(＾v＾)

ヾ(*´∀｀*)ﾉ

Log rank conjecture

$x \in \{-1,1\}^n$ $g(x,y)$ $y \in \{-1,1\}^n$

Alice Bob

…

(＾v＾) ヾ(*´∀｀*)ノ

Deterministic communication complexity:

Minimum number of bits exchanged between Alice and Bob in the worst case

Log rank conjecture

$x \in \{-1,1\}^n$ $\quad g(x,y) = \text{parity}(x,y)$ $\quad y \in \{-1,1\}^n$

Alice $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Bob

Deterministic communication complexity:
Minimum number of bits exchanged between Alice and Bob in the worst case

Log rank conjecture

$x \in \{-1,1\}^n$ $g(x,y) = \text{parity}(x,y)$ $y \in \{-1,1\}^n$

Alice Bob

$$\xrightarrow{\text{parity}(x)}$$

Deterministic communication complexity:
Minimum number of bits exchanged between Alice and Bob in the worst case

Log rank conjecture

$x \in \{-1,1\}^n$ $g(x,y) = \text{parity}(x,y)$ $y \in \{-1,1\}^n$

Alice Bob

$\text{parity}(x)$

answer

Deterministic communication complexity:
Minimum number of bits exchanged between Alice and Bob in the worst case

Log rank conjecture

- Log rank conjecture: $D[g] = O(\log^c(\mathrm{rank}[M_g]))$?

Deterministic communication complexity

Input matrix

Log rank conjecture on XOR function

- Log rank conjecture: $D[g] = O(\log^c(\mathrm{rank}[M_g]))$?
- XOR function:

$$g(x,y) = f(x \oplus y) = f(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$$

Log rank conjecture on XOR function

- Log rank conjecture: $D[g] = O(\log^c(\text{rank}[M_g]))$?
- XOR function:

$$g(x, y) = f(x \oplus y)$$

$$\text{rank}[M_g] = \text{sparsity}[\hat{f}]$$

$$D[g] \le 2\text{DT}^{\oplus}[f]$$

Log rank conjecture on XOR function

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$$D[g] \le 2\text{DT}^{\oplus}[f]$$

$$\text{DT}^{\oplus}[f] = O(\log^c(\text{sparsity}[\hat{f}]))?$$

Trivial bounds

- Upper bound:

$$\mathrm{DT}^{\oplus}[f] \leq \mathrm{sparsity}[\hat{f}]$$

- Lower bound:

Trivial bounds

- Upper bound:

$$\mathrm{DT}^{\oplus}[f] \leq \mathrm{sparsity}[\hat{f}]$$

- Lower bound:

$$\mathrm{DT}^{\oplus}[f] \geq \frac{1}{2}\log(\mathrm{sparsity}[\hat{f}])$$

Trivial bounds

- Upper bound:

$$\mathrm{DT}^{\oplus}[f] \leq \mathrm{sparsity}[\hat{f}]$$

- Lower bound:

$$\mathrm{sparsity}[\hat{f}] \leq 4^{\mathrm{DT}^{\oplus}[f]}$$

$$\mathrm{DT}^{\oplus}[f] \geq \frac{1}{2}\log(\mathrm{sparsity}[\hat{f}])$$

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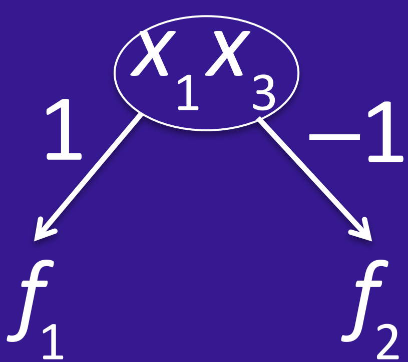

$$f(x) = \frac{1+x_1x_3}{2} f_1(x) + \frac{1-x_1x_3}{2} f_2(x)$$

Trivial bounds

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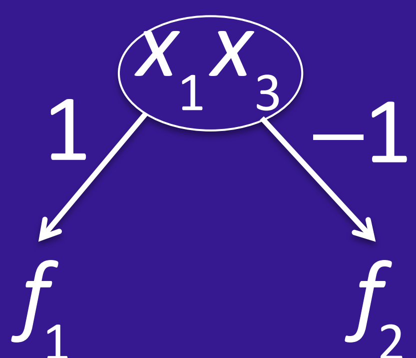

$$f(x) = \frac{1+x_1x_3}{2} f_1(x) + \frac{1-x_1x_3}{2} f_2(x)$$

$$s(f) \le 2s(f_1) + 2s(f_2)$$

Trivial bounds

- Upper bound:

$$\mathrm{DT}^{\oplus}[f] \leq \mathrm{sparsity}[\hat{f}]$$

- Lower bound:

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$$\mathrm{DT}^{\oplus}[f] \geq \frac{1}{2}\log(\mathrm{sparsity}[\hat{f}])$$

$$\begin{tikzcd}
 & x_1x_3 \arrow[dl, "1"'] \arrow[dr, "-1"] & \\
f_1 & & f_2
\end{tikzcd}$$

$$f(x) = \frac{1+x_1x_3}{2} f_1(x) + \frac{1-x_1x_3}{2} f_2(x)$$

$$s(f) \leq 2s(f_1) + 2s(f_2) \leq 2\cdot 4^{\mathrm{DT}^{\oplus}[f_1]} + 2\cdot 4^{\mathrm{DT}^{\oplus}[f_2]}$$

Trivial bounds

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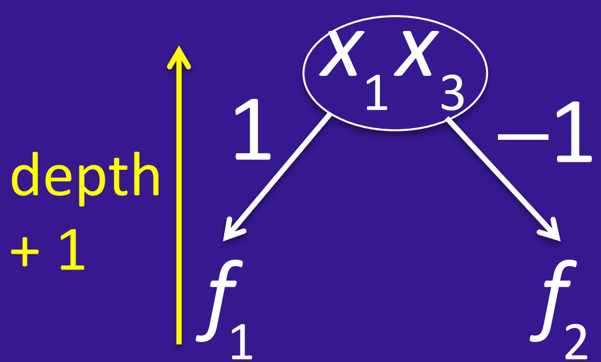

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$$\leq 2\cdot 4^{\mathrm{DT}^{\oplus}[f]-1} + 2\cdot 4^{\mathrm{DT}^{\oplus}[f]-1} = 4^{\mathrm{DT}^{\oplus}[f]}$$

Previous work on upper bound

$$\mathrm{DT}^{\oplus}[f] \leq O\left(\sqrt{\mathrm{sparsity}[\hat{f}]} \cdot \log(\mathrm{sparsity}[\hat{f}])\right)$$

[TWXZ13][STV14]

Conjecture on upper bound

For every Boolean function $f:\{-1,1\}^n \to \{-1,1\}$

$$\mathrm{DT}^{\oplus}[f] \leq O(\log(\mathrm{sparsity}[\hat{f}]))$$

Conjecture on upper bound

For every Boolean function $f:\{-1,1\}^n \to \{-1,1\}$

$$\xcancel{\mathrm{DT}^{\oplus}[f] \le O(\log(\mathrm{sparsity}[\hat{f}]))}$$

Our result on lower bound

For infinitely many n, there exist a Boolean function $f:\{-1,1\}^n \rightarrow \{-1,1\}$ satisfying

$$\mathrm{DT}^{\oplus}[f] \geq \Omega(\log^{1.63}(\mathrm{sparsity}[\hat{f}]))$$

$\log_3 6 \approx 1.63$

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Kill Number

Kill number (minimum certificate complexity) $\text{Kill}[f]$:

The minimum number of input bits one has to fix to force the function to be a constant.

The length of shortest path in any decision tree of the function

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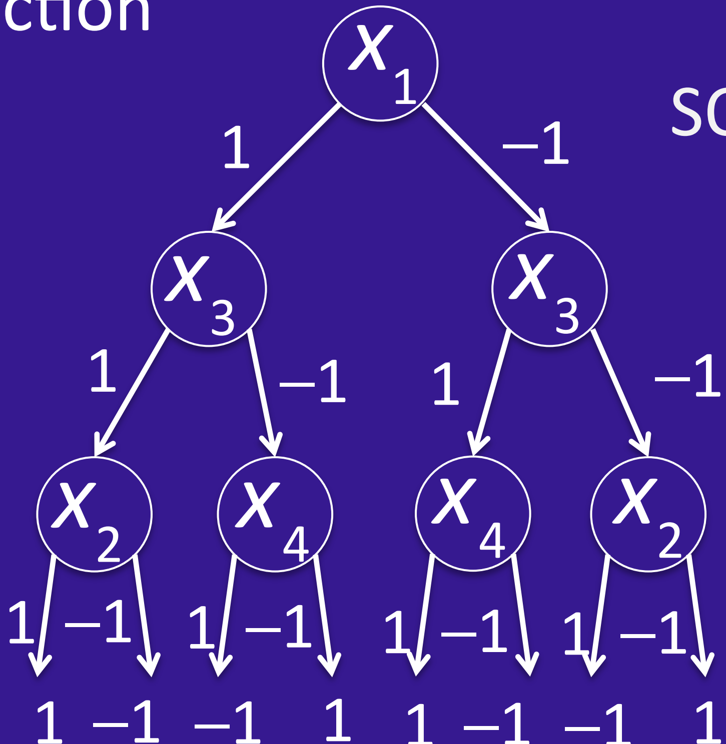

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$\mathrm{DT}[f]=3$

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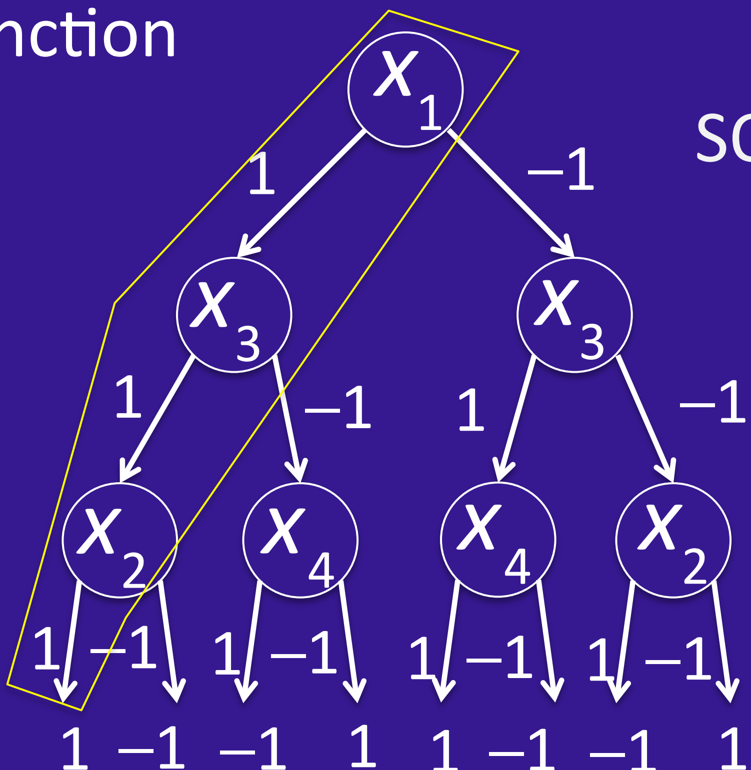

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$\mathrm{DT}[f]=3$

$x_1=1$

$x_3=1$

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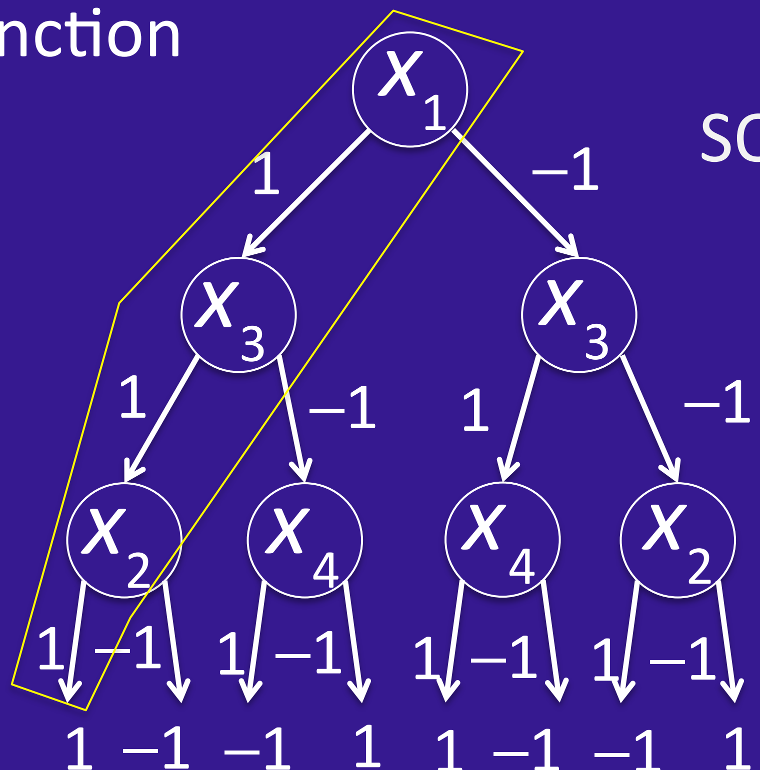

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$\mathrm{DT}[f] = 3$

$x_1 = 1$

$x_3 = 1$

$x_2 = 1$

$\mathrm{Kill}[f] = 3$

Kill Number

Kill number (minimum certificate complexity) $\mathrm{Kill}[f]$:

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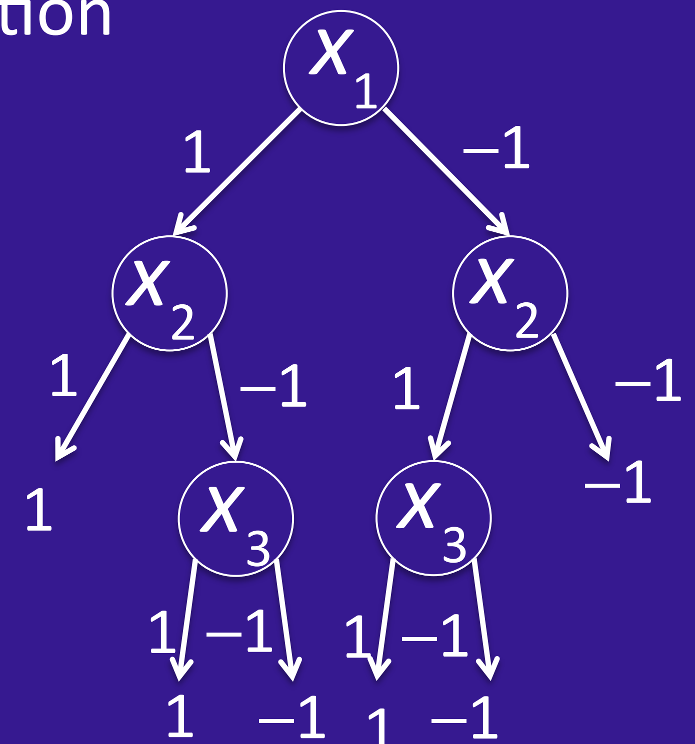

$\mathrm{Maj}_3(x_1, x_2, x_3)$

$\mathrm{DT}[f] = 3$

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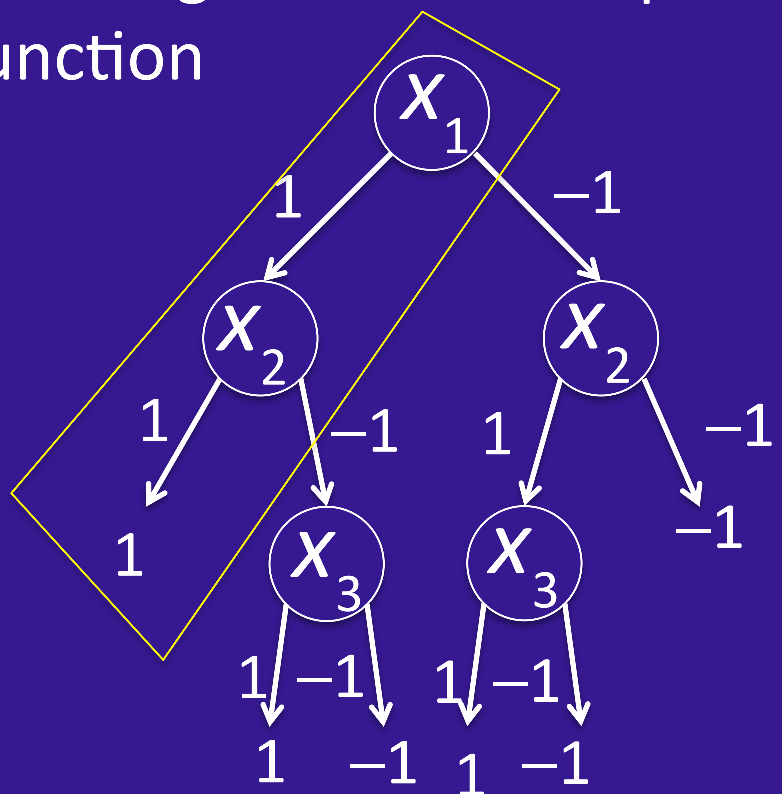

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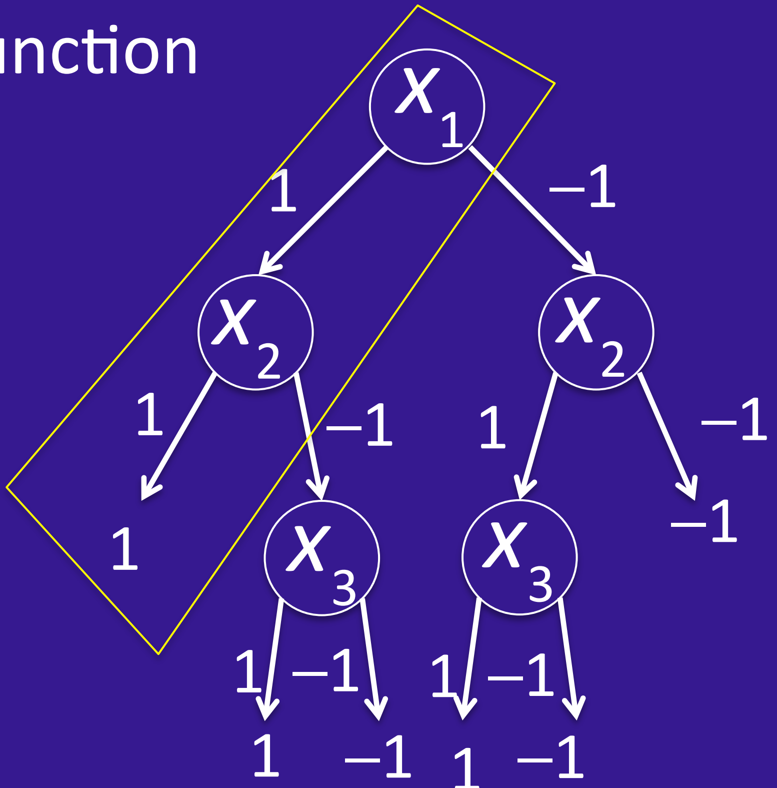
The length of shortest path in any decision tree of the function



$\mathrm{Maj}_3(x_1, x_2, x_3)$

$\mathrm{DT}[f] = 3$

$x_1 = 1$

$x_2 = 1$

$\mathrm{Kill}[f] = 2$

Kill Number

Kill number (minimum certificate complexity) $\text{Kill}[f]$:

The minimum number of input bits one has to fix to force the function to be a constant.

The length of shortest path in any decision tree of the function

$$\text{Kill}[f] \leq \text{DT}[f]$$

Parity kill number

Parity Kill Number $\mathrm{Kill}^{\oplus}[f]$

The minimum number of parity on input variables one has to fix to force the function to be a constant.

The length of shortest path in any parity decision tree of the function

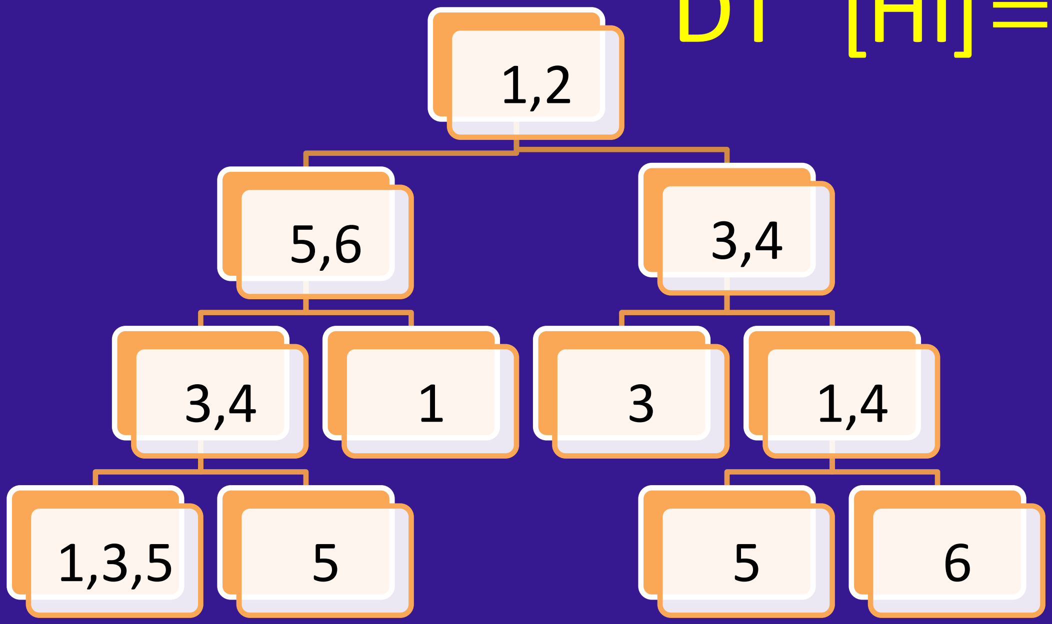
Parity kill number

Parity Kill Number $\mathrm{Kill}^{\oplus}[f]$

The minimum number of parity on input variables one has to fix to force the function to be a constant.

The length of shortest path in any parity decision tree of the function

$$\mathrm{DT}^{\oplus}[\mathrm{HI}] = 4$$



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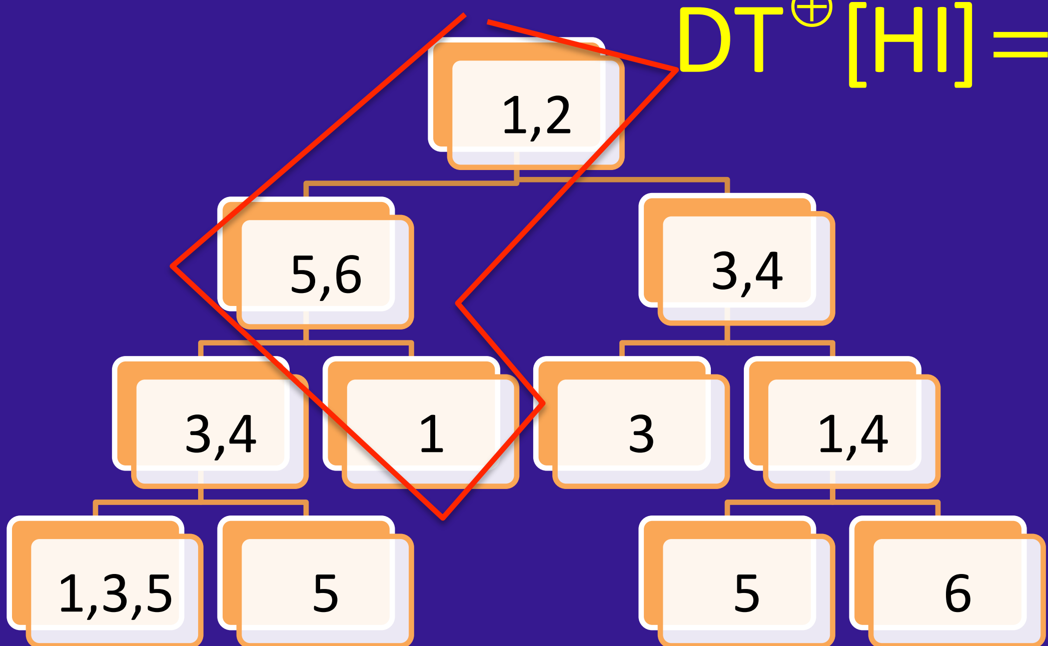


$$x_1 x_2 = 1$$

$$x_5 x_6 = -1$$

$$x_1 = 1$$

Parity kill number

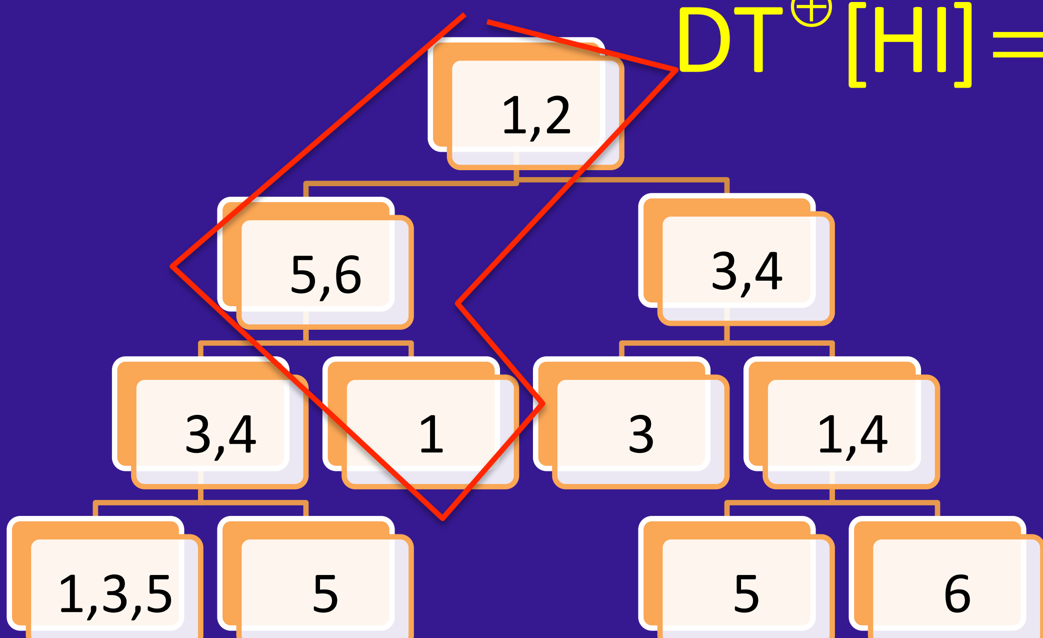
Parity Kill Number $\mathrm{Kill}^{\oplus}[f]$

The minimum number of parity on input variables one has to fix to force the function to be a constant.

The length of shortest path in any parity decision tree of the function

$\mathrm{DT}^{\oplus}[\mathrm{HI}] = 4$

$\mathrm{Kill}^{\oplus}[\mathrm{HI}] = 3$



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Parity kill number

Parity Kill Number $\mathrm{Kill}^{\oplus}[f]$

The minimum number of parity on input variables one has to fix to force the function to be a constant.

The length of shortest path in any parity decision tree of the function

$$\mathrm{Kill}^{\oplus}[f] \le \mathrm{DT}^{\oplus}[f] \le \mathrm{Kill}^{\oplus}[f] \cdot \log(\mathrm{sparsity}[\hat{f}])$$

Outline

- Preliminaries
- Background
- Parity kill number
- Function composition
- Main Result
- Future Illustration

Composition of Boolean function

$$f:\{-1,1\}^n \to \{-1,1\}$$
$$g:\{-1,1\}^m \to \{-1,1\}$$

Composition of Boolean function

$$f:\{-1,1\}^n \to \{-1,1\}$$
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$$f\circ g:\{-1,1\}^{mn} \to \{-1,1\}$$
$$f\circ g(y) = f(g(y^{(1)}),g(y^{(2)}),\ldots,g(y^{(n)}))$$
$$y^{(i)} \in \{-1,1\}^m$$

Composition of Boolean function

$$f:\{-1,1\}^n \to \{-1,1\}$$

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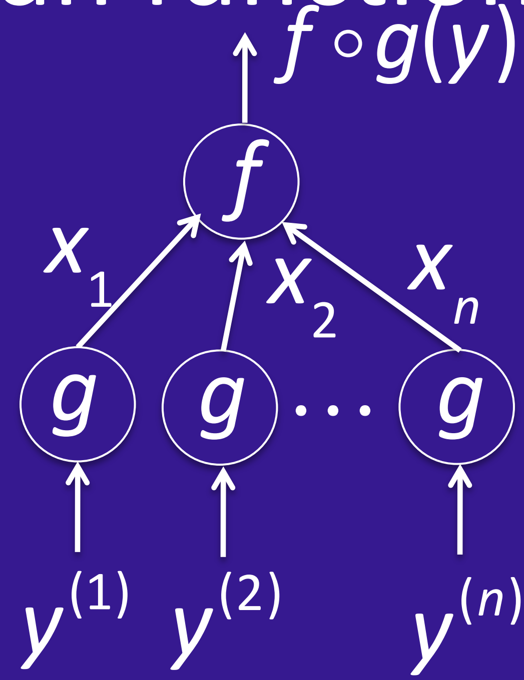

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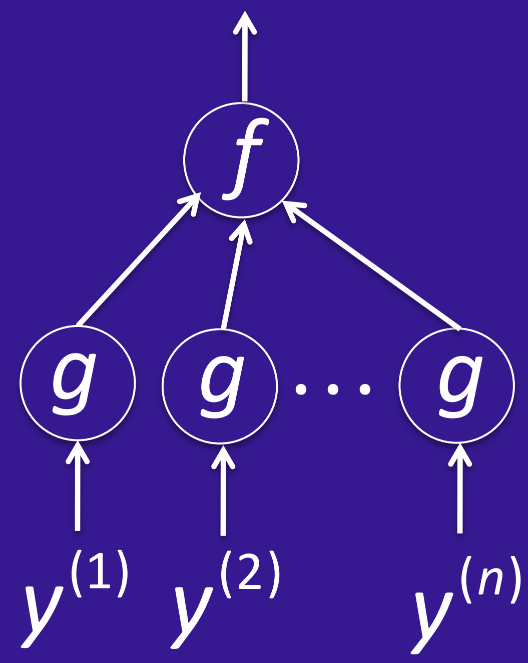

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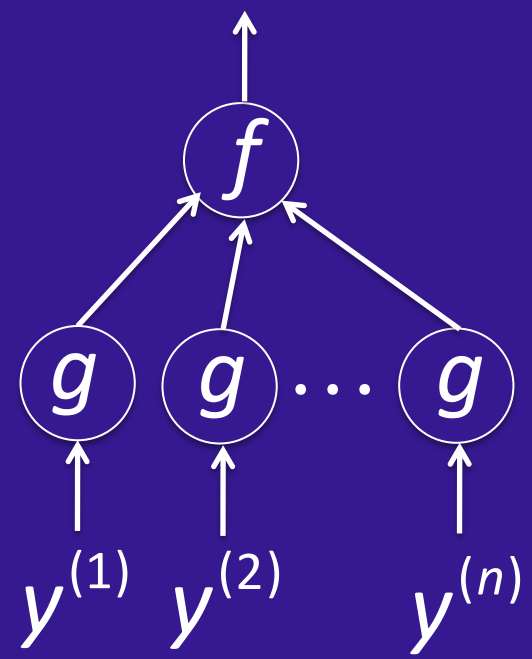
$f\circ g(y) = f(g(y^{(1)}),g(y^{(2)}),\ldots,g(y^{(n)}))$

$y^{(i)} \in \{-1,1\}^m$

$f^{\circ k} = \underbrace{f\circ f\circ\cdots\circ f}_{k}$

Composition of Boolean function

$$\text{parity}_n^{\circ 2} = \text{parity}_{n^2}$$



Composition of Boolean function

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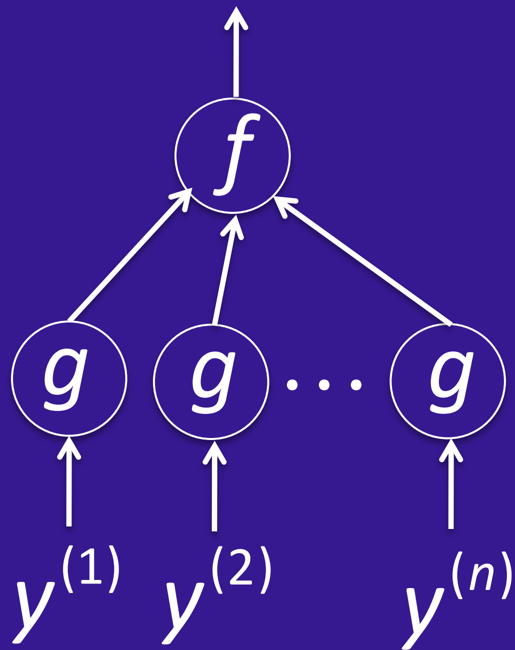

Composition of Boolean function

$$\mathrm{SORT}(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$$

Composition of Boolean function

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$$\begin{aligned}\mathrm{SORT}^{\circ 2}(x_1,\ldots,x_{16}) = {} & \frac{1}{2}\mathrm{SORT}(x_1,\ldots,x_4)\mathrm{SORT}(x_5,\ldots,x_8) + \frac{1}{2}\mathrm{SORT}(x_5,\ldots,x_8)\mathrm{SORT}(x_9,\ldots,x_{12}) \\ & + \frac{1}{2}\mathrm{SORT}(x_9,\ldots,x_{12})\mathrm{SORT}(x_{13},\ldots,x_{16}) - \frac{1}{2}\mathrm{SORT}(x_1,\ldots,x_4)\mathrm{SORT}(x_{13},\ldots,x_{16})\end{aligned}$$

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$$\deg[\mathrm{SORT}] = 2$$

$$\mathrm{SORT}^{\circ 2}(x_1,\ldots,x_{16}) = \frac{1}{2}\mathrm{SORT}(x_1,\ldots,x_4)\mathrm{SORT}(x_5,\ldots,x_8) + \frac{1}{2}\mathrm{SORT}(x_5,\ldots,x_8)\mathrm{SORT}(x_9,\ldots,x_{12})$$

$$+ \frac{1}{2}\mathrm{SORT}(x_9,\ldots,x_{12})\mathrm{SORT}(x_{13},\ldots,x_{16}) - \frac{1}{2}\mathrm{SORT}(x_1,\ldots,x_4)\mathrm{SORT}(x_{13},\ldots,x_{16})$$

Composition of Boolean function

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$$\deg[\mathrm{SORT}^{\circ 2}] = 4$$

Composition of Boolean function

$$\mathrm{SORT}(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$$ sparsity[SORT] = 4

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Composition of Boolean function

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$$\mathrm{sparsity}[\mathrm{SORT}] = 4$$

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$$\mathrm{sparsity}[\mathrm{SORT}^{\circ 2}] = 64$$

Composition of Boolean function

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$$\deg[f^{\circ 2}] = \deg[f]^2$$

Composition of Boolean function

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$$\deg[f^{\circ 2}] = \deg[f]^2$$

$$\mathrm{sparsity}[f^{\circ 2}] \approx \mathrm{sparsity}[\hat{f}]^{\deg[f]}$$

Properties of Boolean function Composition by previous work

$$\deg[f \circ g] = \deg[f] \cdot \deg[g] \qquad \deg[f^{\circ k}] = \deg[f]^k$$

$$\mathrm{DT}[f \circ g] = \mathrm{DT}[f] \cdot \mathrm{DT}[g] \qquad \mathrm{DT}[f^{\circ k}] = \mathrm{DT}[f]^k$$

$$\mathrm{Kill}[f \circ g] \geq \mathrm{Kill}[f] \cdot \mathrm{Kill}[g] \qquad \mathrm{Kill}[f^{\circ k}] \geq \mathrm{Kill}[f]^k$$

[Tal13]

Properties of Boolean function Composition by previous work

$\deg[f \circ g] = \deg[f] \cdot \deg[g]$

$\deg[f^{\circ k}] = \deg[f]^k$

$\mathrm{DT}[f \circ g] = \mathrm{DT}[f] \cdot \mathrm{DT}[g]$

$\mathrm{DT}[f^{\circ k}] = \mathrm{DT}[f]^k$

$\mathrm{Kill}[f \circ g] \geq \mathrm{Kill}[f] \cdot \mathrm{Kill}[g]$

$\mathrm{Kill}[f^{\circ k}] \geq \mathrm{Kill}[f]^k$

[Tal13]

$\log \mathrm{sparsity}[f^{\circ k}] = \Theta(\deg[f]^k)$

Properties of Boolean function Composition by previous work

$\deg[f \circ g] = \deg[f] \cdot \deg[g]$

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$\mathrm{DT}[f^{\circ k}] = \mathrm{DT}[f]^k$

$\mathrm{Kill}[f \circ g] \geq \mathrm{Kill}[f] \cdot \mathrm{Kill}[g]$

$\mathrm{Kill}[f^{\circ k}] \geq \mathrm{Kill}[f]^k$

[Tal13]

$\log \mathrm{sparsity}[f^{\circ k}] = \Theta(\deg[f]^k)$

What about $\mathrm{DT}^{\oplus}$ and $\mathrm{Kill}^{\oplus}$?

Outline

- Preliminaries
- Background
- Parity kill number
- Function composition
- Main Result
- Future Direction

Composition theorem for parity kill number

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

Composition theorem for parity kill number

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \mathrm{Kill}^{\oplus}[f^{\circ(k-1)} \circ f]$$

Composition theorem for parity kill number

Thm1:

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$$\begin{aligned}\text{Kill}^{\oplus}[f^{\circ k}] &= \text{Kill}^{\oplus}[f^{\circ(k-1)} \circ f] \\ &\geq \text{Kill}[f^{\circ(k-1)}] + \text{Kill}^{\oplus}[f^{\circ(k-1)}] \qquad (\text{Kill}^{\oplus}[f] \geq 2)\end{aligned}$$

Composition theorem for parity kill number

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$$\begin{aligned}\mathrm{Kill}^{\oplus}[f^{\circ k}] &= \mathrm{Kill}^{\oplus}[f^{\circ(k-1)} \circ f] \\ &\geq \mathrm{Kill}[f^{\circ(k-1)}] + \mathrm{Kill}^{\oplus}[f^{\circ(k-1)}] \qquad (\mathrm{Kill}^{\oplus}[f] \geq 2) \\ &\geq \mathrm{Kill}[f]^{(k-1)} \qquad (\mathrm{Kill}[f^{\circ k}] \geq \mathrm{Kill}[f]^{k})\end{aligned}$$

Composition theorem for parity kill number

Thm1:

$$\text{Kill}^{\oplus}[f \circ g] \geq \text{Kill}[f] + \text{Kill}^{\oplus}[f] \quad \text{when} \quad \text{Kill}^{\oplus}[g] \geq 2$$

$$\begin{aligned}
\text{Kill}^{\oplus}[f^{\circ k}] &= \text{Kill}^{\oplus}[f^{\circ(k-1)} \circ f] \\
&\geq \text{Kill}[f^{\circ(k-1)}] + \text{Kill}^{\oplus}[f^{\circ(k-1)}] \qquad (\text{Kill}^{\oplus}[f] \geq 2) \\
&= \Omega(\text{Kill}[f]^{k})
\end{aligned}$$

Composition theorem for parity kill number

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

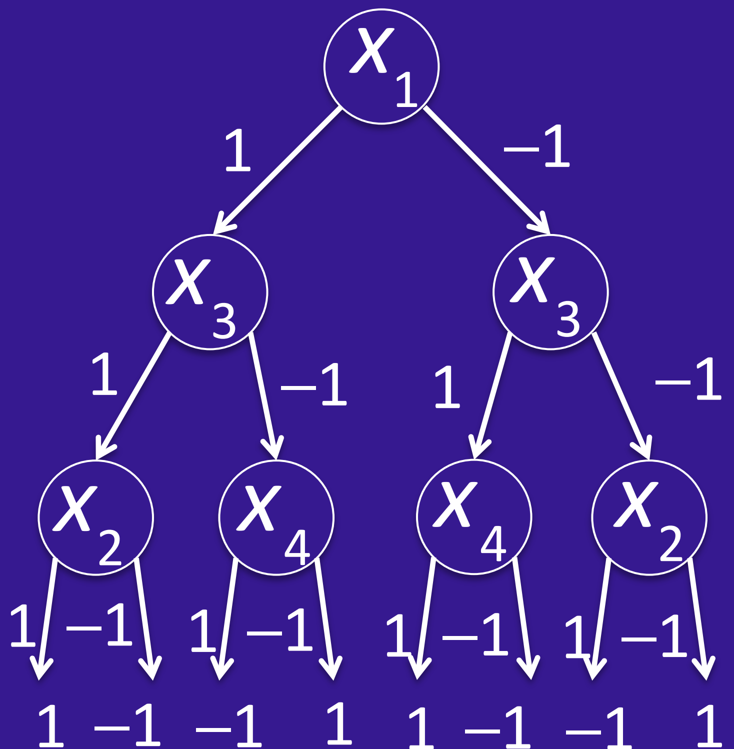
Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^{k}) \qquad (f \text{ is not a parity})$$

Composition theorem for parity kill number

Thm2:

$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^k)$ (f is not a parity)



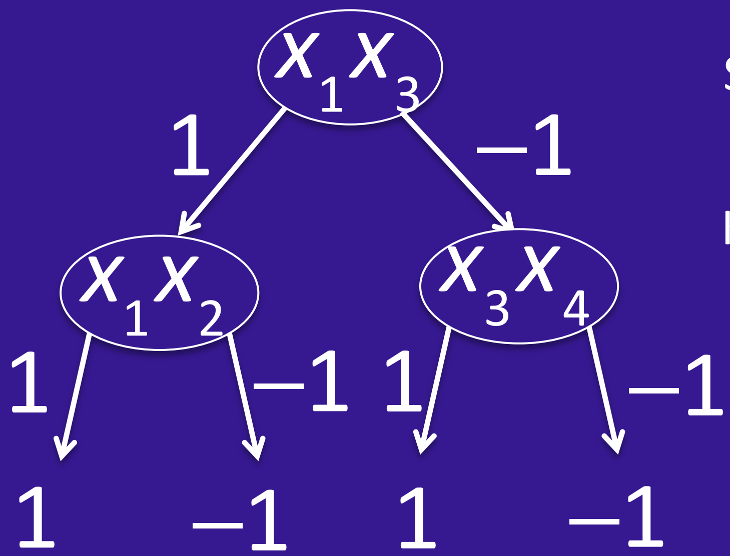
$$\mathrm{SORT}(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$$

$\mathrm{Kill}[\mathrm{SORT}] = 3$

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^k) \qquad (f \text{ is not a parity})$$



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Composition theorem for parity kill number

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$\mathrm{Kill}[\mathrm{SORT}] = 3 \qquad \mathrm{Kill}^{\oplus}[\mathrm{SORT}] = 2$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{SORT}^{\circ k}] = \Omega(3^k)$$

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^k) \qquad (f \text{ is not a parity})$$

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$$\mathrm{Kill}[\mathrm{SORT}] = 3 \qquad \mathrm{Kill}^{\oplus}[\mathrm{SORT}] = 2$$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{SORT}^{\circ k}] = \Omega(3^k)$$

$$\log \mathrm{sparsity}[\mathrm{SORT}^{\circ k}] = \Theta(\deg[\mathrm{SORT}]^k) = \Theta(2^k)$$

Composition theorem for parity kill number

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$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{SORT}^{\circ k}] = \Omega(3^k)$$

$$\log \mathrm{sparsity}[\mathrm{SORT}^{\circ k}] = \Theta(\deg[\mathrm{SORT}]^k) = \Theta(2^k)$$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{ok}] = \Omega(\log^{\log_2 3}(\mathrm{sparsity}[\mathrm{SORT}^{ok}]))$$

Composition theorem for parity kill number

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

$$\begin{aligned}
\mathrm{Kill}^{\oplus}[f^{\circ k}] &= \mathrm{Kill}^{\oplus}[f^{\circ(k-1)} \circ f] \\
&\geq \mathrm{Kill}[f^{\circ(k-1)}] + \mathrm{Kill}^{\oplus}[f^{\circ(k-1)}] \qquad (\mathrm{Kill}^{\oplus}[f] \geq 2) \\
&\geq \mathrm{Kill}[f]^{(k-1)} \qquad (\mathrm{Kill}[f^{\circ k}] \geq \mathrm{Kill}[f]^{k})
\end{aligned}$$

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^{k}) \qquad (f \text{ is not a parity})$$

$$\mathrm{SORT}(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$$

$$\mathrm{Kill}[\mathrm{SORT}] = 3 \qquad \mathrm{Kill}^{\oplus}[\mathrm{SORT}] = 2$$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{SORT}^{\circ k}] = \Omega(3^{k})$$

$$\log \mathrm{sparsity}[\mathrm{SORT}^{\circ k}] = \Theta(\deg[\mathrm{SORT}]^{k}) = \Theta(2^{k})$$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\mathrm{o}k}] = \frac{3^{k}+1}{2}$$

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^k) \qquad (f \text{ is not a parity})$$

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$\mathrm{Kill}[\mathrm{SORT}] = 3 \qquad \mathrm{Kill}^{\oplus}[\mathrm{SORT}] = 2$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{SORT}^{\circ k}] = \Omega(\mathrm{Kill}[SORT]^k) = \Omega(3^k)$$

$$\log \mathrm{sparsity}[\mathrm{SORT}^{\circ k}] = \Theta(\deg[\mathrm{SORT}]^k) = \Theta(2^k)$$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\mathrm{ok}}] = \Omega(\log^{\log_2 3}(\mathrm{sparsity}[\mathrm{SORT}^{\mathrm{ok}}]))$$

Composition theorem for parity kill number

Thm2:

$$\text{Kill}^{\oplus}[f^{\circ k}] = \Omega(\text{Kill}[f]^k) \qquad (f \text{ is not a parity})$$

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$$\text{Kill}[\text{SORT}] = 3 \qquad \text{Kill}^{\oplus}[\text{SORT}] = 2$$

$$\text{DT}^{\oplus}[\text{SORT}^{\circ k}] \geq \text{Kill}^{\oplus}[\text{SORT}^{\circ k}] = \Omega(\boxed{\text{Kill}[\mathit{SORT}]}^k) = \Omega(3^k)$$

$$\log \text{sparsity}[\text{SORT}^{\circ k}] = \Theta(\boxed{\deg[\text{SORT}]}^k) = \Theta(2^k)$$

$$\text{DT}^{\oplus}[\text{SORT}^{\text{o}k}] = \Omega(\log^{1.58}(\text{sparsity}[\text{SORT}^{\text{o}k}]))$$

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^k) \qquad (f \text{ is not a parity})$$

$$\begin{aligned}\mathrm{HI}(x) = \frac{1}{4}(-\sum_i x_i &+ x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 \\ &+ x_1x_4x_5 + x_1x_5x_6 + x_2x_3x_5 + x_2x_4x_6 \\ &+ x_2x_5x_6 + x_3x_4x_5 + x_3x_4x_6)\end{aligned}$$

$\mathrm{Kill}[\mathrm{HI}] = 4$

$\mathrm{Kill}^{\oplus}[\mathrm{HI}] = 3$

$\deg[\mathrm{HI}] = 3$

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^k) \qquad (f \text{ is not a parity})$$

$$\mathrm{HI}(x) = \frac{1}{4}(-\sum_i x_i + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 + x_1x_4x_5 + x_1x_5x_6 + x_2x_3x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_4x_6)$$

$\mathrm{Kill}[\mathrm{HI}] = 4$

$\mathrm{Kill}^{\oplus}[\mathrm{HI}] = 3$

$\deg[\mathrm{HI}] = 3$

$$\mathrm{DT}^{\oplus}[\mathrm{HI}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{HI}^{\circ k}] = \Omega(\mathrm{Kill}[\mathrm{HI}]^k) = \Omega(4^k)$$

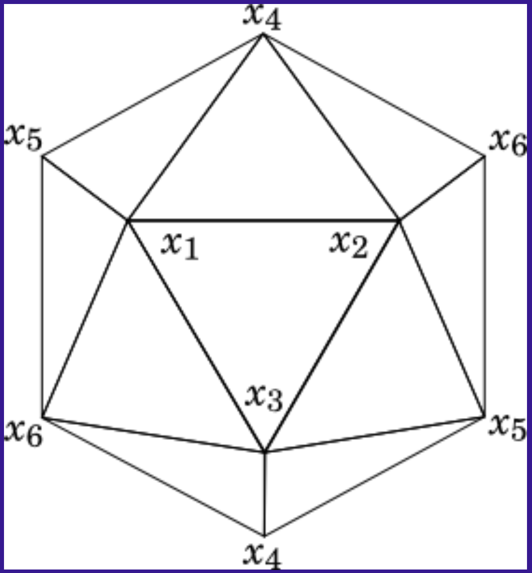
$$\log \mathrm{sparsity}[\mathrm{HI}^{\circ k}] = \Theta(\deg[\mathrm{HI}]^k) = \Theta(3^k)$$

$$\mathrm{DT}^{\oplus}[\mathrm{HI}^{\mathrm{o}k}] = \Omega(\log^{\log_3 4}(\mathrm{sparsity}[\mathrm{HI}^{\mathrm{o}k}]))$$

DT vs Parity DT

$$\mathrm{HI}(x_1, x_2, \ldots, x_6) = \begin{cases} 1 & \text{if \#1's in } x \text{ is } 1, 2, 6 \\ -1 & \text{if \#1's in } x \text{ is } 0, 4, 5 \\ ? & \text{if \#1's in } x \text{ is } 3 \end{cases}$$

$\mathrm{HI}(x) = 1$ iff one triangle in the graph has all vertices 1



$\mathrm{DT}[\mathrm{HI}] = 6$

x_1	x_2	x_3	x_4	x_5	x_6	HI
1	1	?	1	1	1	?

Composition theorem for parity kill number

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^{k}) \qquad (f \text{ is not a parity})$$

$$\begin{aligned}\mathrm{HI}(x) = \frac{1}{4}(&-\sum_i x_i + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 \\ &+ x_1x_4x_5 + x_1x_5x_6 + x_2x_3x_5 + x_2x_4x_6 \\ &+ x_2x_5x_6 + x_3x_4x_5 + x_3x_4x_6)\end{aligned}$$

$\mathrm{Kill}[\mathrm{HI}, 1^6] = 6$

$\deg[\mathrm{HI}] = 3$

$$\mathrm{DT}^{\oplus}[\mathrm{HI}^{\circ k}] \geq \mathrm{Kill}^{\oplus}[\mathrm{HI}^{\circ k}, 1^{6^k}] = \Omega(\mathrm{Kill}[\mathrm{HI}, 1^6]^k) = \Omega(6^k)$$

$$\log \mathrm{sparsity}[\mathrm{HI}^{\circ k}] = \Theta(\deg[\mathrm{HI}]^k) = \Theta(3^k)$$

$$\mathrm{DT}^{\oplus}[\mathrm{HI}^{\mathrm{o}k}] = \Omega(\log^{\log_3 6}(\mathrm{sparsity}[\mathrm{HI}^{\mathrm{o}k}]))$$

Our result

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^{k}) \qquad (f \text{ is not a parity})$$

Our result

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

Thm2:

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] = \Omega(\mathrm{Kill}[f]^{k}) \qquad (f \text{ is not a parity})$$

$$\mathrm{DT}^{\oplus}[\mathrm{SORT}^{\circ k}] = \Omega(\log^{\log_2 3}(\mathrm{sparsity}[\mathrm{SORT}^{\circ k}]))$$

$$\mathrm{DT}^{\oplus}[\mathrm{HI}^{\circ k}] = \Omega(\log^{\log_3 6}(\mathrm{sparsity}[\mathrm{HI}^{\circ k}]))$$

Illustration of Theorem 1

Thm1:

$$\mathrm{Kill}^{\oplus}[f\circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

$$y_1 y_2 y_3 = 1$$

$$y_4 y_5 = 1$$

$$y_6 = -1$$

$$y_1 y_4 y_7 = 1$$

$$\vdots$$

$$y_8 y_9 = -1$$

parity constraints that kill $f \circ g$

Illustration of main theorem

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

$$y_1 y_2 y_3 = 1$$
$$y_4 y_5 = 1$$
$$y_6 = -1$$
$$y_1 y_4 y_7 = 1$$
$$\vdots$$
$$y_8 y_9 = -1$$

parity constraints that kill $f \circ g$

$$\Rightarrow$$

$$x_1 x_2 = -1$$
$$x_3 x_5 = 1$$
$$\vdots$$
$$x_4 = 1$$

parity constraints that kill f

Illustration of main theorem

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

$$y_1 y_2 y_3 = 1$$
$$y_4 y_5 = 1$$
$$y_6 = -1$$
$$y_1 y_4 y_7 = 1$$
$$\vdots$$
$$y_8 y_9 = -1$$

parity constraints of $f \circ g$

$$\Rightarrow$$

$$\left.\begin{array}{l} x_1 x_2 = -1 \\ x_3 x_5 = 1 \\ \vdots \\ x_4 = 1 \end{array}\right\} \geq \mathrm{Kill}^{\oplus}[f]$$

parity constraints that kill f

Illustration of main theorem

Thm1:

$$\mathrm{Kill}^{\oplus}[f \circ g] \geq \mathrm{Kill}[f] + \mathrm{Kill}^{\oplus}[f] \quad \text{when} \quad \mathrm{Kill}^{\oplus}[g] \geq 2$$

$$\begin{array}{l} y_1 y_2 y_3 = 1 \\ y_4 y_5 = 1 \\ y_6 = -1 \\ y_1 y_4 y_7 = 1 \\ \vdots \\ y_8 y_9 = -1 \end{array}$$

parity constraints of $f \circ g$

$$\left.\begin{array}{l} x_1 x_2 = -1 \\ x_3 x_5 = 1 \\ \vdots \\ x_4 = 1 \end{array}\right\} \geq \mathrm{Kill}^{\oplus}[f]$$

parity constraints that kill f

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \geq \mathrm{Kill}[f]$$

Outline

- Preliminaries
- Background
- Parity kill number
- Function composition
- Main Result
- Future Direction

Future Direction

$$\boxed{\Omega(\log^{\log_3 6}(\text{sparsity}[\hat{f}])) \leq} \text{Kill}^{\oplus}[f] \leq O(\sqrt{\text{sparsity}[\hat{f}]})$$

$$\boxed{\Omega(\log^{\log_3 6}(\text{sparsity}[\hat{f}])) \leq} \text{DT}^{\oplus}[f] \leq O(\sqrt{\text{sparsity}[\hat{f}]} \cdot \log(\text{sparsity}[\hat{f}]))$$

Future Direction

$$\boxed{\Omega(\log^{\log_3 6}(\text{sparsity}[\hat{f}])) \leq} \text{Kill}^{\oplus}[f] \leq O(\sqrt{\text{sparsity}[\hat{f}]})$$

$$\boxed{\Omega(\log^{\log_3 6}(\text{sparsity}[\hat{f}])) \leq} \text{DT}^{\oplus}[f] \leq O(\sqrt{\text{sparsity}[\hat{f}]} \cdot \log(\text{sparsity}[\hat{f}]))$$

Can we do better?

Future Direction

$$\mathrm{Kill}^{\oplus}[f^{\circ k}] \geq \Omega[\mathrm{Kill}[f]^{k}] \quad (f \text{ is not a parity})$$

Future Direction

$\mathrm{Kill}^{\oplus}[f^{\circ k}] \geq \Omega[\mathrm{Kill}[f]^{k}]$ (f is not a parity)

What about $\mathrm{DT}^{\oplus}[f^{\circ k}]$ and $\mathrm{DT}[f]$?

Thank you