Name: ____________________ Andrew ID: ____________________

15-451/651 Algorithm Design & Analysis, Spring 2026

Quiz 5 Solutions

Problem: Using the max-flow/min-cut theorem, or any other method of your choice, prove Hall's theorem for the case of perfect matchings:

Theorem 1 *Let $G = (L \cup R, E)$ be a finite bipartite graph, where L and R are two vertex sets of the same size (so $|L| = |R|$), and for $S \subseteq L$, let $N(S)$ be the neighbors of the vertices in S, i.e.,*

$$N(S) = \{\, r \in R : \textit{there exists } \ell \in S \textit{ with } \ell r \in E \,\}.$$

If G has no perfect matching, then there exists a subset $S \subseteq L$ such that $|N(S)| < |S|$. Here, $|\cdot|$ denotes the cardinality of the set, so $|N(S)|$ is the number of neighbors of S, and $|S|$ is the number of vertices in S.

Solution. We start with the graph construction.

Graph construction. Add a super source s and connect it to all vertices in L. Add a super sink t and connect it to all vertices in R. Give all these edges capacity 1. All edges originally in G will have capacity ∞. Notice that the s-t maximum flow in G is equal to the maximum size of a perfect matching in G.

Constructing S. We know that there is a s-t cut of size less than n by the maxflow/mincut theorem. Because the edges between L, R have infinite capacity, all edges in the cut should be only touching s and t. Let $A \subseteq L$ be the set of vertices such that the edge to s was cut, and define $B \subseteq R$ to be the set of vertices such that the edge to t was cut. Define $S = L \setminus A$.

Proving that $|N(S)| < |S|$. Take any neighbor $y \in N(S)$, so $y \in R$ and y is adjacent to some $x \in S$. Since $x \in S = L \setminus A$, the edge $s \to x$ was not cut. Since y is adjacent to x via an infinite capactity edge, that edge cannot be cut either. Therefore, the only way to separate s from t along the path $s \to x \to y \to t$ is by cutting the edge $y \to t$, which means $y \in B$. Therefore every neighbor of S is in B

To conclude, note that

$$|S| = n - |A| > |B| \geq |N(S)|,$$

where $n - |A| > |B|$ because $|A| + |B| < n$.