15-451/651 Algorithm Design & Analysis, Spring 2026

Oral Homework #3 Solutions

1 Q1

The variables are f_e per edge.

The objective function to minimize is

$$\sum_e f_e c_e$$

If a vertex is a conservation vertex, we need amount in equals to amount out, so

$$\sum_{v \to u} f_{v \to u} = \sum_{u \to w} f_{u \to w}$$

if u is a random walk vertex, we have that each edge out has half the flow of the amount in, so

$$f_{u \to w} = \frac{1}{2} \sum_{v \to u} f_{v \to u}$$

for each edge $u \to w$ leaving u.

The capacity constriants are $0 \le f \le \text{cap}$, but to get it into standard form, we introduce slacks $s \in \mathbf{R}^e$, and require

$$f + s = \text{cap}$$
$$f, s \ge 0$$

2 Q2

We can show that this is a zero-sum game, where player A picks from a set of paths and player B picks from a set of edges. The pay off is then 1 if player B wins (selected edge is on selected path), otherwise 0 (based on following explanations).

(by Richard)

The structure of zero-sum games states that all we have to do is to exhibit two mixed strategies for each player whose response as pure players are the same.

For A, find the max-flow (in the edge disjoint sense) from s to t, say it has value k. Then pick one of these paths with probability $1/k$.

Since the paths are edge disjoint, for any edge, the path hits it with probability $1/k$. So A can limit B's winning probability to at most $1/k$.

For B, take a minimum cut, which also has size k, and pick an edge on it at random.

Any path must cross this cut, so we hit the path with probability $1/k$. So B is able to win with probability $1/k$.

(by Yang)

Game value proof. We prove that game value is $1/\mu$ where μ is the value of the s-t maxflow/mincut. We will prove that the path player has a strategy which achieves this, and also prove that the edge player has a strategy achieving this. This will prove that the game value is exactly $1/\mu$.

For the path player, pick any set of μ edge-disjoint s-t paths, and the strategy is to play a random such path. Clearly, regardless of the edge player's strategy, the probability of winning is at most $1/\mu$ (because the paths are disjoint).

For the edge player, pick any s-t mincut of size μ, and the strategy is to play a random edge in the cut. Any s-t path must involve at least one of the cut edges, so the probability of winning is at least $1/\mu$. This proves that the game value is exactly $1/\mu$, as desired.

Direct LP Proof Define the payoff matrix A to be a $|\mathscr{P}| \times E$ matrix where each row corresponds to a path, and each column corresponds to an edge.

Specifically, $A_{P,e} = 1$ if and only if e is on the path P.

So the zero-sum game for the row player (A) becomes finding a distribution $x \in \mathbf{R}^{\mathscr{P}}$ that minimizes the max pay off any pure strategy of the column player (edge player, B).

As a linear program this is

$$\begin{aligned} \min \quad & \lambda \\ \text{subject to:} \quad & A^\top x \leq \lambda \cdot \mathbb{1} \\ & \mathbb{1}^\top x = 1 \\ & x \geq 0 \end{aligned}$$

Now notice that x can be directly interpretted as the path decomposition of a flow that routes 1 unit from s to t.

Also, for each edge e, the condition

$$\sum_{P : P \ni e} x_P \leq \lambda$$

means that the congestion on edge e is at most λ, so this linear program seeks to minimize the congestion of a flow that sends one unit from s to t.

This value is exactly $1/F$, where F is the max flow value, since the max-flow routes F units, with congestion at most 1.

3 Q3

First, we will write out the primal LP in standard form. This is similar to the max-flow/min-cut LP from lecture.

$$\min c^T x \text{ subject to } Ax = b$$

Where $c = \vec{0}$ and $x = (f_e, s_e)_{e \in E} \in \mathbb{R}^{2|E|}$ and f_e represents the flow on each edge e and s_e is the slack variable for edge e. Note, we set c to 0 since we are simply trying to check for feasibility, not the optimal solution.

In addition to the constraints $f_e \geq 0$ and $s_e \geq 0$ for all edges e, we have the constraints for flow conservation and capacity:

$$\sum_{(u,v) \in E} f_{(u,v)} - \sum_{(v,w) \in E} f_{(v,w)} = d_v$$

$$f_e + s_e = \text{cap}_e$$

Next, we write the dual LP in standard form:

We have variables y_v for each $v \in V$ and z_e for each $e \in E$.

Our objective function will be as follows:

$$\max_{y,z} \sum_{v \in V} d_v y_v + \sum_{e \in E} \text{cap}_e z_e$$

We also have the following constraints:

(i) **For each f_{uv}:** The variable f_{uv} appears in the conservation row for u (with coefficient -1), the conservation row for v (with coefficient $+1$), and the capacity row for edge uv (with coefficient $+1$). Since $c = 0$, the dual constraint is:

$$-y_u + y_v + z_{uv} \leq 0 \quad \Longrightarrow \quad z_{uv} \leq y_u - y_v.$$

(ii) **For each s_e:** The variable s_e appears only in the capacity row for e (with coefficient $+1$). The dual constraint is:

$$z_e \leq 0.$$

Since there are two upper bounds on z_e and we want to maximize our dual objective, we will set z_e to the smaller of the two upper bounds $z_e = \min(0, y_u - y_v)$.

Therefore, our objective becomes:

$$\max_y D(y) := \sum_{v \in V} d_v y_v + \sum_{e \in E} \text{cap}_e \min(0, y_u - y_v)$$

Since $c = 0$, the dual is unbounded if and only if there exists a y such that $D(y) > 0$ (you can scale y by α for any $\alpha > 0$ to increase the objective to ∞).

Now that we found the dual and primal, we want to show that the dual is unbounded $\iff$ $\exists$ a violating cut, so that we can combine this with the dual is unbounded $\iff$ the primal is infeasible to get the primal is infeasible (flow is infeasible) $\iff$ $\exists$ a violating cut

Dual is unbounded $\longleftarrow$ $\exists$ a violating cut:

Suppose there is a $S \subseteq V$ with: $\sum_{v \notin S} d_v > \sum_{u \in S, v \notin S, u \to v \in E} \text{cap}_{u \to v}$,

We define y_v to be an indicator that is 0 if $v \in S$ and 1 if $v \notin S$.

Therefore, for each edge (u, v), $\min(0, y_u - y_v) = \begin{cases} -1 & \text{if } u \in S,\ v \notin S, \\ 0 & \text{otherwise.} \end{cases}$

Therefore, the objective evaluates to:

$$D(y) = \sum_{v \notin S} d_v \cdot 1 + \sum_{v \in S} d_v \cdot 0 + \sum_{\substack{u \in S,\ v \notin S \\ u \to v \in E}} \text{cap}_{uv} \cdot (-1) = \sum_{v \notin S} d_v - \sum_{\substack{u \in S,\ v \notin S \\ u \to v \in E}} \text{cap}_{u \to v} > 0.$$

Since $D(y) > 0$, we can scale y to make the dual unbounded.

Dual is unbounded $\longrightarrow$ $\exists$ a violating cut:

Since the dual is unbounded, there exists a vector $y \in \mathbb{R}^V$ such that $D(y) > 0$. We need to show we can make an actual set $S \subseteq V$ from this. For a threshold θ, define $S_\theta = \{u \in V : y_u \leq \theta\}$

The corresponding "demand minus capacity" for this cut is:

$$\Delta(\theta) = \sum_{v \notin S_\theta} d_v - \sum_{\substack{u \in S_\theta,\ v \notin S_\theta \\ u \to v \in E}} \text{cap}_{u \to v}.$$

Claim: If $D(y) > 0$, then there exists some θ such that $\Delta(\theta) > 0$.

Proof. Without loss of generality, normalize y so that its values lie in $[0, 1]$ (we can shift and scale, which preserves $D(y) > 0$ by homogeneity and the fact that $\sum_v d_v = 0$ for a valid demand).

We integrate $\Delta(\theta)$ over $\theta \in [0,1)$ to compute the average value of $\Delta(\theta)$ over all values of θ.

Consider each vertex v: the vertex $v \notin S_\theta$ (i.e., $y_v > \theta$) for $\theta \in [0, y_v)$, contributing d_v over a range of length y_v.

Consider each edge $u \to v$: the edge crosses the cut (i.e., $u \in S_\theta$ and $v \notin S_\theta$, meaning $y_u \le \theta < y_v$) for $\theta \in [y_u, y_v)$ when $y_u < y_v$, contributing $-\text{cap}_{uv}$ over a range of length $\max(0, y_v - y_u)$.

Therefore:

$$\int_0^1 \Delta(\theta)\, d\theta = \sum_{v \in V} d_v\, y_v - \sum_{\substack{u \to v \in E \\ y_u < y_v}} \text{cap}_{u \to v}(y_v - y_u) = \sum_{v \in V} d_v\, y_v + \sum_{u \to v \in E} \text{cap}_{uv} \cdot \min(0, y_u - y_v) = D(y).$$

Since $D(y) > 0$, we have $\int_0^1 \Delta(\theta)\, d\theta > 0$. Therefore $\Delta(\theta) > 0$ for at least one value of θ, meaning the threshold cut S_θ satisfies:

$$\sum_{v \notin S_\theta} d_v > \sum_{\substack{u \in S_\theta,\, v \notin S_\theta \\ u \to v \in E}} \text{cap}_{u \to v}. \qquad \square$$

Therefore, we have shown that:

$$\text{Demand } d \text{ is infeasible} \quad \Longleftrightarrow \quad \exists S \subseteq V \text{ such that } \sum_{v \notin S} d_v > \sum_{\substack{u \in S,\, v \notin S \\ u \to v \in E}} \text{cap}_{u \to v}.$$