15-451/651 Algorithm Design & Analysis, Spring 2026

Oral Homework #3

Oral Homework #3, **Due: Week of Mar 23-27**

Guidelines:

1. This is an oral homework.
2. You will work in groups of size two or three.
3. A scheduling link will be sent later in the week, around Thursday or Friday.
4. You will be asked to present solutions to each of the three problems below. Groups can decide who presents which, but:
   - for groups of size 3, one person per problem.
   - for groups of size 2, one person presents 2, while the other presents 1.
5. Your presentations will be graded on correctness. There may be back and forth, where the TAs will help guide you through solutions, etc.
6. **You will have 30 total minutes to present 3 problems, one problem presented per group member.** Please make sure your presentations are concise to fit in this time limit.
7. Notes are allowed during presentations. However, the grading TAs will ask you questions.

Problems:

1. (*) Consider the following flow-based view of Markov Decision Processes:
   (a) Flows still need to be in the direction of the edge, and at most capacities.
   (b) There are now two types of vertices. The ones from max-flow / min-cost flows are now 'conservation vertices', where the flow in can be distributed among the edges out arbitrarily.
   (c) There also 'random walk' vertices, which have out degree 2: any flow that arrives at this vertex get distributed evenly among the two edges out.

   Again, we want to route one unit of flow from s to t, satisfying conservation at all other vertices, subject to minimum total cost on the edges, as well as capacity constraints on edges.

   Formulate this as a linear program in the standard form as shown in Lectures 15 and 16.

2. (**)

   Consider the following game played on the edges of a directed graph with two special vertices s (source) and t (sink). The following are played simultaneously (using a randomized strategy):
   (a) A picks a path from s to t
   (b) B picks an edge.

   B wins if the edge picked is on the path, otherwise A wins.

   Show that this is a 0-sum game, formulate this as a linear program, and relate the objective (optimal game value for both players) of this game to the s-t max-flow/min-cut in this graph.

   hint/clarification added on Tue Mar 24: The strategies that you find for both players should have very close connections with max-flow / min-cut.

3. (***) **Using linear programming duality**, prove the demand version of max-flow min-cut: in a directed capacitated graph $G = (V, E, \text{cap})$, a demand $d \in \mathbf{R}^V$ is not feasible if and only if there exist a cut such that the total capacity of the cut is less than the demand leaving it, aka there exists some $S \subseteq V$ such that

$$\sum_{u \notin S} d_u > \sum_{u \in S, v \notin S, u \to v \in E} \text{cap}_{u \to v}.$$

The version of LP duality that you should use is the infeasibility version, which says that the standard form $\min_{x \geq 0, Ax=b} c^\top x$ is infeasible if and only if the dual's objective value is unbounded.

You might also want to use (and show correctness) of the following subroutine for converting a fractional cut to an integral one:

(a) Define 'threshold cut' for some θ on a vector x as $S_{\leq \theta}(x, \theta) = \{u \in V | x_u \leq \theta\}$.

(b) Check the condition (total demand minus total capacity) for all threshold cuts.