

Homework 6

15-359/659 PnC II

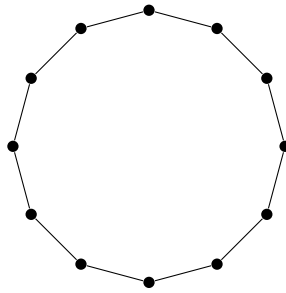
2026 Fall

Due 12:50 pm on October 9, 2026

Exercises 26.10, 26.16, 26.20, 26.24 in the textbook, plus the problem below.

Conductance of a Lazy Random Walk

Let $n \geq 4$ be even. Consider the lazy random walk on the cycle C_n with state space $\Omega = \{0, 1, \dots, n-1\}$. From each state x , the chain stays at x with probability $1/2$, moves clockwise to $x+1 \pmod{n}$ with probability $1/4$, and moves counterclockwise to $x-1 \pmod{n}$ with probability $1/4$. Let $\pi = (\pi(0), \pi(1), \dots, \pi(n-1))$ denote the stationary distribution.



From each vertex: stay with probability $1/2$,
move to each neighbor with probability $1/4$

Figure 1: Lazy random walk on the cycle C_n .

Recall that

$$Q(x, y) = \pi(x)P(x, y),$$

and, for $S \subseteq \Omega$,

$$Q(S, S^c) = \sum_{x \in S} \sum_{y \in S^c} Q(x, y).$$

For a set S with $0 < \pi(S) \leq 1/2$, the conductance of S is defined as

$$\Phi(S) = \frac{Q(S, S^c)}{\pi(S)},$$

and the conductance of the chain is defined as

$$\Phi = \min_{S: 0 < \pi(S) \leq 1/2} \Phi(S).$$

(a) Let S be any nonempty subset of Ω , with $|S| = k$ and $0 < \pi(S) \leq 1/2$. Prove that

$$Q(S, S^c) \geq \frac{1}{2n}.$$

(b) Find the conductance Φ of this Markov chain.

(c) Derive a lower bound on $t_{\text{mix}}(\varepsilon)$. Assume $0 < \varepsilon < 1/2$.

(d) Use Cheeger's inequality together with the Poincaré mixing-time bound from the previous lecture to upper bound $t_{\text{mix}}(\varepsilon)$. Assume $0 < \varepsilon < 1/2$.