

Homework 5

15-359/659 PnC II

2026 Fall

Due 12:50 pm on October 2, 2026

1 Practice Dirichlet form

Recall the following example we gave in lecture. A student chooses between two lunch spots, K-station (K) and Revolution Noodle (R). Each day, the student switches to the other lunch spot with probability p and stays at the same lunch spot with probability $1 - p$. Assume $0 < p \leq 1/2$. The transition probability matrix is

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix},$$

where the states are ordered as (K, R) .

- (a) Find the stationary distribution π of this Markov chain.
- (b) Consider a function $\phi : \{K, R\} \rightarrow \mathbb{R}$ with $\phi(K) = a$ and $\phi(R) = b$. Write ϕ as a column vector. Then compute $\mathbb{E}_\pi[\phi]$ and $\mathbf{Var}_\pi[\phi]$.
- (c) Compute the Dirichlet form $\mathcal{E}_P(\phi, \phi)$ using the definition $\mathcal{E}_P(\phi, \phi) = \langle \phi, L\phi \rangle_\pi$.
- (d) Compute the Dirichlet form $\mathcal{E}_P(\phi, \phi)$ using the alternative expression:

$$\mathcal{E}_P(\phi, \phi) = \frac{1}{2} \sum_{x, y \in \{K, R\}} \pi(x) P(x, y) (\phi(x) - \phi(y))^2.$$

- (e) Compare your expressions for $\mathcal{E}_P(\phi, \phi)$ and $\mathbf{Var}_\pi[\phi]$ and determine the largest Poincaré constant α for this chain. How does α change as p increases?

2 Time-reversed chain

Consider an ergodic, lazy Markov chain on the state space $\Omega = \{0, 1, \dots, M-1\}$. Let P be its transition probability matrix and π be the stationary distribution. Recall that its time-reversed chain is a Markov chain whose transition probability matrix is

$$P^*(x, y) = \frac{\pi(y)}{\pi(x)} P(y, x), \quad \forall x, y \in \Omega.$$

- (a) Prove that π is also a stationary distribution of the reversed chain.
- (b) Prove that the symmetric Dirichlet forms of P and P^* are the same, i.e., prove that

$$\mathcal{E}_P(\phi, \phi) = \mathcal{E}_{P^*}(\phi, \phi), \quad \forall \phi: \Omega \rightarrow \mathbb{R}.$$

- (c) Prove that if the Markov chain P is reversible, then $P^* = P$.