

Homework 4

15-359/659 PnC II

2026 Fall

Due 12:50 pm on September 25, 2026

1 Maximal coupling

Let p and q be two pmfs on a discrete set Ω . During the lecture, we proved that for any coupling (X, Y) with $X \sim p$ and $Y \sim q$,

$$d_{\text{TV}}(p, q) \leq \mathbb{P}(X \neq Y).$$

In this problem, we will prove that there exists a coupling called the *maximal coupling* that achieves equality. Recall that

$$d_{\text{TV}}(p, q) = \frac{1}{2} \sum_{x \in \Omega} |p(x) - q(x)|.$$

(a) Define

$$f(x) = \min\{p(x), q(x)\}, \quad x \in \Omega.$$

Prove that

$$\sum_{x \in \Omega} f(x) = 1 - d_{\text{TV}}(p, q).$$

(b) We consider the following coupling of X and Y . Let $A = \{x \in \Omega: p(x) \geq q(x)\}$.

- With probability $1 - d_{\text{TV}}(p, q)$, we draw $X = Y$ from the pmf $\frac{f(x)}{1 - d_{\text{TV}}(p, q)}$.
- With probability $d_{\text{TV}}(p, q)$, we draw X and Y independently following the pmfs below:

$$X \sim \begin{cases} \frac{p(x) - f(x)}{d_{\text{TV}}(p, q)}, & x \in A \\ 0, & x \in A^c \end{cases}, \quad Y \sim \begin{cases} 0, & x \in A \\ \frac{q(x) - f(x)}{d_{\text{TV}}(p, q)}, & x \in A^c \end{cases}.$$

Prove that this is indeed a valid coupling, i.e., the X and Y generated here satisfy $X \sim p$ and $Y \sim q$. From that you can conclude that under this coupling, $d_{\text{TV}}(p, q) = \mathbb{P}(X \neq Y)$.

2 Bounding the mixing time for a simple random walk

Consider a random walk on the set $\mathcal{X} = \{0, 1, \dots, n-1\}$ for a positive integer n . At each time step, we walk to the right ($\text{mod } n$) with probability $1/4$, to the left ($\text{mod } n$) with probability $1/4$, and stay put with probability $1/2$; i.e., the random walk is a Markov chain with the following transition probability matrix:

$$P_{ij} = \begin{cases} 1/4, & \text{when } j = (i+1) \bmod n, \\ 1/2, & \text{when } j = i, \\ 1/4, & \text{when } j = (i-1) \bmod n. \end{cases} \quad (1)$$

Our goal is to bound the mixing time of this random walk and see how it is related to n .

Construction of two random walks. We construct two random walks, X_t and Y_t , in the following way. Let $X_0 = x$ and $Y_0 = y$. For every time step t , we flip a fair coin. If the coin comes up heads, then X_t moves to $(X_t + 1) \bmod n$ with probability $1/2$ and moves to $(X_t - 1) \bmod n$ with probability $1/2$, and Y_t does not move. If the coin comes up tails, then Y_t moves to $(Y_t + 1) \bmod n$ with probability $1/2$ and moves to $(Y_t - 1) \bmod n$ with probability $1/2$, and X_t does not move. As soon as (X_t) and (Y_t) hit the same location, say, at time τ , they move identically according to the transition matrix P in (1).

- (a) Prove that (X_t) and (Y_t) are both Markov chains with transition matrix P .
- (b) Let $Z_t = (X_t - Y_t) \bmod n$. Then (Z_t) is also a Markov chain. Draw the transition diagram of (Z_t) .
- (c) Prove that starting from $Z_0 = k \triangleq (x - y) \bmod n$, the expected time for Z_t to hit 0 is $k(n - k)$.
- (d) Recall that

$$d_t = \max_{x \in \Omega} d_{\text{TV}}(\pi_t^{(x)}, \pi),$$

where $\pi_t^{(x)}$ is the distribution of the state of a Markov chain with transition probability matrix P and start state x , and π is the stationary distribution of this chain. Prove that

$$d_t \leq \frac{n^2}{4t}.$$

- (e) Recall that

$$t_{\text{mix}}(\epsilon) \triangleq \min\{t: d_t \leq \epsilon\}, \tag{2}$$

and $t_{\text{mix}} = t_{\text{mix}}\left(\frac{1}{2e}\right)$. Prove that

$$t_{\text{mix}} \leq \frac{e}{2} n^2.$$

Remark. One can also prove a lower bound on t_{mix} that is $\Theta(n^2)$. So the $\Theta(n^2)$ dependence is tight.