Capacity of Binary Deletion Channel

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Mitzenmacher (2008), "A survey of results for deletion channels and related synchronization channels"

Dec 3 2014

Definition

A *binary deletion channel with deletion probability* p takes a binary string and deletes each bit independently with probability p.

1011011001010000
1011110100

- We care about this channel's capacity
- One instance of deletion-insertion codes (where each bit gets replaced with some distribution of run lengths)

Upper bound

- less information than binary erasure channel
- Thus, capacity at most $1 - p$.
- Slightly better bounds are known.
- Substantially better bounds exist only for low p

Random Codes

Idea: pick codewords randomly. When receiving, if message could come from just one codeword, decode to that; else, fail.

- Works poorly on the deletion channel.
- Gives positive rate only if $p < 1/2$.
- Main idea: if X is a string of length $< n/2$, then w.h.p. X is a substring of a random codeword.

Markov Chain construction

Better idea: generate codewords using a Markov chain.

- There is a fixed probability γ that the ith bit is the same as the $i+1$th bit.
- Decode same way as before.
- Received codeword satisfies an analogous Markov chain, making analysis easier
- Gives positive rate.
- More generally, can have length of runs of 0s and 1s follow some distribution.

Jigsaw decoding

Better idea: generate codeword randomly, having run lengths follow some distribution

- split received word into blocks, see what each block came from:

sent	001100	1	010000	1011001
received	000	1	000	111

- will get some distribution of block lengths, for each length k, some distribution of sequences it could have come from
- for each pair (k, s) of a length and sequence it comes from (e. g. (3,001100)), number of times it happens is close to expected value.

Jigsaw decoding

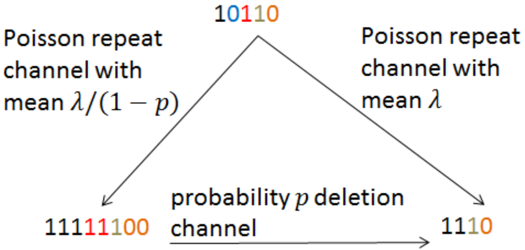
received	000	1	000	111
sent 1	001100	1	010000	1011001
sent 2	0010	110	01010110	1011001
sent 3	01010110	110	0010	1011001

- Look at all possibilities for how many times each pair happens
- For each, look at all possible ways the word you receive could be made
- if rate of original code below some value, unlikely to get any collisions

Poisson Repeat Channel

Different channel: independently repeat each bit of codeword based on Poisson distribution

- High-rate codes exist over this channel using jigsaw idea
- Take such a code C of length $n(1-p)/\lambda$ for the Poisson repeat channel with mean λ.
- send each $c \in C$ through Poisson repeat channel with mean $\lambda/(1-p)$to get a new codebook C' with words of length n
- Send through deletion channel. Each $c \in C$ is sent through Poisson channel with mean λ.



Poisson Repeat Channel

- Decode using the Poisson decoder w.h.p. there are few collisions in C'.
- At $\lambda = 1.79$, yields capacity bound of $0.1185(1-p)$ (about $(1-p)/9$).
- Best known for large p.

Maximum Likelihood Decoding

"Optimal" Idea: When receiving a codeword, pick the most likely codeword from the codebook.

- Would give better capacity, but difficult to analyze.
- Jigsaw gives better bounds because it is closer to maximum likelihood.

Open Questions

- Efficient algorithms for the deletion channel (especially p large)
- Improve bounds using the methods discussed.
- Find better approximations of maximum likelihood decoding.
- look at other deletion-insertion channels
    - sticky channels