(1)

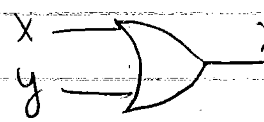
(Tue 3/26)

Last time:

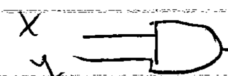
* Graph entropy vs $\log \chi(G)$
* Applications of Graph entropy:
    1) Covering of K_n
    2) k-perfect hash families.

Today: Application in circuit complexity.

* monotone Boolean formula $f(x_1, \ldots, x_n) : \{0,1\}^n \to \{0,1\}$ is a rooted binary tree with variables at leaves and and/or gates at internal nodes.



$x \vee y = 0$ iff $x=y=0$

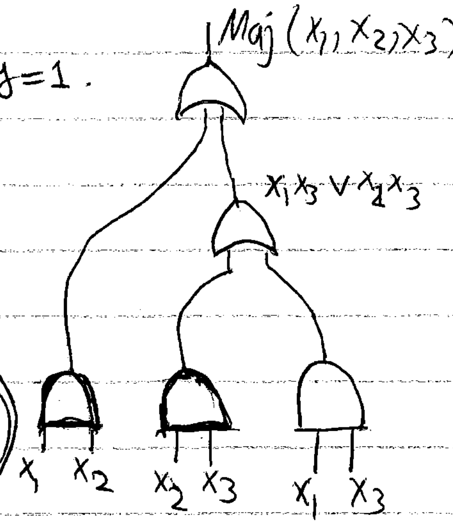


$x \wedge y = 1$ iff $x=y=1$.

example: Majority of x_1, x_2, x_3:

$$\begin{cases} 1 & \text{if } \#\text{true inputs} \geq 2 \\ 0 & \text{else.} \end{cases}$$

$$Maj(x_1,x_2,x_3) = (x_1 \wedge x_2) \vee ((x_2 \wedge x_3) \vee (x_1 \wedge x_3))$$



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Equivalent way of thinking about Boolean functions:

$f : 2^{[n]} \longrightarrow \{0,1\}$.

* **Def:** f is monotone if $\forall S \subseteq T \subseteq [n]$,

$$f(S) \le f(T).$$

e.g., OR, AND, Majority.

* **Clear:** monotone formula computes a monotone function.

* Conversely, If $f : 2^{[n]} \to \{0,1\}$ (not constant) is monotone, it can be represented by a monotone formula.

Def: $(f)_i \triangleq \{ S \subseteq [n], |S| = i, f(S) = 1, \forall T \subsetneq S, f(T) = 0 \}$.
($i \in [n]$)

[The sets in $(f)_i$ are min-terms of size i]

Observe: [monotone] $f(W) = 1$ iff $\exists S \in \bigcup_i (f)_i$, $S \subseteq W$.
(Proof: easy)

$$\Rightarrow f(W) = \bigvee_{S \in \bigcup_i (f)_i} \bigwedge_{j \in S} x_j \quad \text{(DNF representation)}$$

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* ~~Size~~ monotone formula φ,

$$\text{size}(\varphi) = \cancel{\#\text{gates in } \varphi}$$

$\#$vertices (gate+leaf) in the tree for φ.

* $f : 2^{[n]} \to \{0,1\}$ monotone,

$$\text{size}(f) = \min_{\substack{\varphi \\ \varphi \text{ computes } f}} \text{size}(\varphi).$$

* Threshold functions: [Generalize and, or, maj]

$k \in \{0, \ldots, n\}$, $Th_k^n : 2^{[n]} \to \{0,1\}$,

$$Th_k^n(S) = \begin{cases} 1 & \text{if } |S| \geq k, \\ 0 & \text{else.} \end{cases}$$

* Ex: $Th_1^n = OR$, $Th_n^n = AND$, $Th_{n/2}^n = Maj_n$.

$\text{Size}(\cancel{OR}\ Th_1^n) = \text{Size}(Th_n^n) = 2n - 1$ (obvious)

Valiant (1984): $\text{Size}(Th_k^n) = O(n^{5.3})$.
(prob. construction)

* <u>Question</u>: Lower bounds?

Graph entropy approach by Newman, Ragde, Wigderson.

$$\text{size}(Th_2^n) \geq 2\lceil n \log n \rceil - 1.$$

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$\star$ <u>Upper bound:</u> $\left(f = Th_2^n\right)$

$$\begin{cases} (f)_2 = \{ S \subseteq [n],\ |S| = 2 \} \\ i \neq 2,\quad (f)_i = \phi \end{cases}$$

$$\Rightarrow \quad size(f) = O(n^2).$$

~~Better~~ <u>Divide-Conquer ~~appr~~ construction:</u>

$$\underbrace{\overset{x}{\boxed{}}}_{\lceil n/2 \rceil}\underbrace{\overset{y}{\boxed{}}}_{\lfloor n/2 \rfloor}$$

$$Th_2^n(x,y) = Th_2^{\lceil n/2 \rceil}(x) \vee Th_2^{\lfloor n/2 \rfloor}(y) \vee \left(Th_1^{\lceil n/2 \rceil}(x) \wedge Th_1^{\lfloor n/2 \rfloor}(y) \right).$$

$$\Rightarrow \quad S_n \leq 2\, S_{n-1} + O(n) \Rightarrow S_n = O(n \log n).$$

$$\left[\left(\text{more carefully: } S_n \leq \left(2n + \lceil \log n \rceil + 1\right) \lceil \log n \rceil \right] \right.$$

It's possible to refine the argument, and get

$$S_n \leq 2n \lceil \log n \rceil - 1. \quad (\text{Exercise})$$

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Lower bound: $\text{Size}(Th_2^n) \geq 2\lceil n \log n \rceil - 1$.

$\varphi :=$ Smallest formula computing f.

for a gate f in φ, define G_f as a graph with $V(G_0) = [n]$ and edges defined by $(f)_2$.

$\cancel{G(\text{root}) = G}$

$$\Rightarrow \begin{cases} G_{\text{root}} = G_{Th_2^n} = K_n \\ G_{\text{leaf}} = G_{x_i} = \text{empty graph} \end{cases}$$

[graphs involve on the interior vertices until we reach the complete graph from the empty graph].

* Claim: If $f = g \vee h$, then $G_f \subseteq G_g \cup G_h$.

<u>Proof:</u> $e \in E(G_f) \Rightarrow g(e) \vee h(e) = 1$

Say $g(e) = 1$, $e = \{i, j\}$

~~workout~~ $\Rightarrow g(\{i\}) = g(\{j\}) = 0$, since otherwise,

$f(\{i\}) = 1$ or $f(\{j\}) = 1$ (contradiction with $e \in (f)_2$)

$\Rightarrow e \in G_g$.

$\square$

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For <u>and</u>, $G_{g\wedge h} \subseteq G_g \cup G_h$ is wrong.

Example: $g \wedge h = x_1 \wedge x_2$,
$\{1,2\} \in E(G_{g\wedge h})$ but
$\{1,2\} \notin E(G_g)$, $\{1,2\} \notin E(G_h)$

Question: $G_{g\wedge h} \subseteq G_g \cup G_h \cup$?

Suppose $e \in G_{g\wedge h}$ but $\left(e \notin G_g,\ e \notin G_h \right)$.

$e = \{i, j\}$. Def $f', g', h' : \{0,1\}^2 \to \{0,1\}$
restriction of f, g, h on e.
(other coordinates $= 0$)

$$\Rightarrow \begin{cases} f'(x_i, x_j) = x_i \wedge x_j . \\ g'(x_i, x_j) \neq \quad '' \\ h'(x_i, x_j) \neq \quad '' \\ f' = g' \wedge h' \Longrightarrow g', h' \neq \text{const}. \\ \text{also } g'(x_i, x_j) \neq x_i \vee x_j \quad (\text{inspect}) \\ \qquad h'(x_i, x_j) \neq x_i \vee x_j \end{cases}$$

$$\Rightarrow \text{possible cases: } \begin{cases} g' = x_i,\ h' = x_j \ \checkmark \\ g' = x_j,\ h' = x_i \ \checkmark \end{cases}$$

$$\Rightarrow e \in \underbrace{\left((g)_1 - (h)_1 \right) \times \left((h)_1 - (g)_1 \right)}_{*}$$

Def: $T_{g,h}$:= induced subgraph of $*$ (bipartite)

$\Rightarrow G_{g\wedge h} \subseteq G_g \cup G_h \cup T_{g,h}$. $\square$

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$$\Rightarrow \begin{cases} H(G_{g\vee h}) \le H(G_g) + H(G_h) \\ H(G_{g\wedge h}) \le H(G_g) + H(G_h) + 1 \\ H(G_{x_i}) = 0, \quad H(G_{Th_2^n}) = \log n \end{cases}$$

$\Rightarrow$ any monotone formula for Th_2^n must have $\geq \lceil \log n \rceil$ AND gates.
[(this is tight)]

Improving the bound on $H(T_{g,h})$:

$V(T_{g,h}) = [n]$ but edges can only be between vertices in $((g)_1 - (h)_1) \cup ((h)_1 - (g)_1)$

$= (g)_1 \triangle (h)_1 \Rightarrow$ by disjoint union property: $H(T_{gh}) \le \dfrac{|(g)_1 \triangle (h)_1|}{n}$.

* Define potential function

$$\mu(f) \triangleq H(G_f) + \frac{|(f)_1|}{n}$$

NB: $\mu(x_i) = \frac{1}{n}$, $\mu(Th_{\frac{n}{2}}^{n}) = \log n$.

Claim: For both $f = g \vee h$ and $f = g \wedge h$,

$$\mu(f) \leq \mu(g) + \mu(h).$$

* OR gates: $f = g \vee h$.

Obviously, $(f)_1 = \{ i \mid f(\{i\}) = 1 \}$

$= (g)_1 \cup (h)_1$

[N.B. no gate computes a constant function].

$$\Rightarrow \mu(f) = H(G_f) + \frac{|(f)_1|}{n}$$

$$\leq H(G_g) + H(G_h) + \frac{|(g)_1| + |(h)_1|}{n}$$

$$\leq \mu(g) + \mu(h).$$

⑨

* <u>AND</u> gates: $f = g \wedge h$

$$(f)_1 = (g)_1 \cap (h)_1$$

$$\Rightarrow \mu(f) = H(G_f) + \frac{|(f)_1|}{n}$$

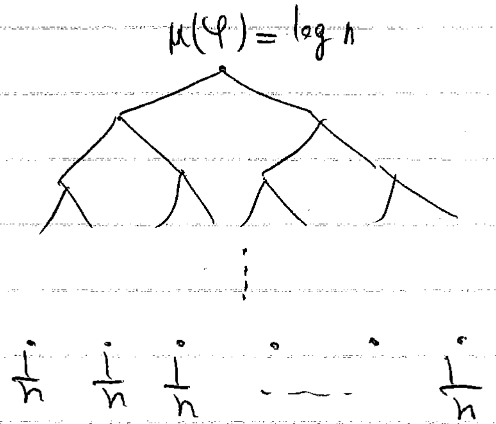
$$\leq H(G_g) + H(G_h) + \underbrace{H(T_{g,h})}_{\leq \frac{|(g)_1 \Delta (h)_1|}{n}} + \frac{|(g)_1 \cap (h)_1|}{n}$$

$$\leq H(G_g) + H(G_h) + \frac{|(g)_1| + |(h)_1|}{n}$$

$$\leq \mu(g) + \mu(h).$$

$\square$

$\Rightarrow$



$\Rightarrow$ # leaves $\geq \lceil n \log n \rceil$

leaves

$\Rightarrow$ Need $\lceil n \log n \rceil - 1$ internal nodes (gates)

$\Rightarrow$ in total $2\lceil n \log n \rceil - 1$ nodes are required.

$\square$