15-496/859X: Computer Science Theory for the Information Age — Spring 2012
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Problem Set 2

Due by 6 pm, Friday, February 17

Instructions

- You should think about *each* problem by yourself for *at least 30 minutes* before choosing to collaborate with others.
- You are allowed to collaborate with fellow students taking the class in solving the problems (in groups of at most 3 people for each problem). In fact this is encouraged so that you interact with and learn from each other. However, *you must write up your solutions on your own.* If you collaborate in solving problems, you should clearly acknowledge your collaborators for each problem.
- Reference to any external material besides the course text and material covered in lecture is not allowed. In particular, you are not allowed to search for answers or hints on the web. You are encouraged to contact the instructors or the TA for a possible hint if you feel stuck on a problem and require some assistance.
- Solutions typeset in LaTeX are preferred.
- Feel free to email the instructors or the TA if you have any questions or would like any clarifications about the problems.
- You are urged to start work on the problem set early.

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1. (10 points) Let A be an $n \times d$ matrix (for $n \geq d$) with orthogonal columns $\mathbf{w_1}.\mathbf{w_2}, \ldots, \mathbf{w_d}$ of positive lengths $\ell_1, \ell_2, \ldots, \ell_d$ no two of which are equal. What is the Singular Value Decomposition of A? Justify your answer.

2. (10 points) Let A be a square $n \times n$ matrix whose rows are orthonormal. Prove that the columns of A must also be orthonormal. Is this necessarily true for $m \times n$ matrices when $m < n$?

3. (15 points) Suppose you are given an overdetermined system of linear equations $Ax = b$ where A is an $m \times n$ matrix with $m > n$, and $b \in \mathbb{R}^m$, which we would like to solve for $x \in \mathbb{R}^n$. That is, we have more equations (m) than variables (n), so there may be no solution to the system. Thus a natural goal is to find a "solution" x that minimizes the error $\|Ax - b\|_2$.

   (a) First, let us assume that A has rank n. Show that the x that minimizes $\|Ax - b\|_2$ is unique and given by $(A^T A)^{-1} A^T b$.

   (b) Now consider the general case when the rank r of A may be less than n. Let $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$ be a SVD of A with nonzero singular values $\sigma_1, \sigma_2, \ldots, \sigma_r$. Prove that

   $$x^* = \sum_{i=1}^{r} \frac{\langle b, u_i \rangle}{\sigma_i} v_i$$

   satisfies $\|Ax^* - b\|_2 = \min_{x \in \mathbb{R}^n} \|Ax - b\|_2$.

4. (20 points) In this problem, we will explore a different way to decompose matrices (focusing on square matrices for concreteness). Along the way we will encounter the very important notion of positive semidefiniteness.

   (a) A real symmetric matrix $M \in \mathbb{R}^{n\times n}$ is said to be positive semidefinite (psd) if all its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are non-negative. What are the singular values of such a positive semidefinite matrix M?

   (b) Prove that a real symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only if $x^T Mx \geq 0$ for all $x \in \mathbb{R}^n$.

   (c) Prove that if a real symmetric matrix M is positive semidefinite, then so is VMV^T for any $n \times n$ matrix V.

   (d) Prove that every $n \times n$ matrix A can be decomposed as $A = WP$ for $n \times n$ matrices W, P satisfying the conditions (i) $W^T W = WW^T = I$, and (ii) P is real symmetric and positive semidefinite.

5. (15 points) Let $G = (V, E)$ be a d-regular simple, undirected graph[1] with $V = \{1, 2, \ldots, n\}$. Define the $n \times n$ matrix L as follows:

$$L_{ij} = \begin{cases} d & \text{if } i = j \\ -1 & (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

   (a) By definition L is a real, symmetric matrix. Prove that L is positive semidefinite. (Hint: Expand the quadratic form $x^T Lx$ for $x \in \mathbb{R}^n$.)

   (b) What is the smallest eigenvalue of L? Is L invertible?

   (c) Prove that $\|L\|_2 \leq 2d$. When is equality attained?

[1] This means there are no loops or multiple edges, and each vertex is adjacent to exactly d other vertices