Carnegie Mellon University Instructor: Venkatesan Guruswami
15-252: More GITCS Term: Spring 2021

Problem Set 8
Due date: Saturday, April 3 (by midnight EST)

Instructions

- You are allowed to collaborate with one other student taking the class, or do it solo.
- Collaboration is defined as discussion of the lecture material and solution approaches to the problems. Please note that *you are not allowed to share any written material and you must write up solutions on your own*. You must clearly acknowledge your collaborator in the write-up of your solutions.
- Solutions must be submitted on gradescope. Typesetting in LaTeX is recommended but not required. If submitting handwritten work, please make sure it is a legible and polished final draft.
- You should not search for solutions on the web. More generally, you should try and solve the problems without consulting any reference material other than what we cover in class and any provided notes.
- Please start working on the problem set early. Though it is short, the problem(s) might take some time to solve.

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1. We considered LP program in the Inequality form $K = \{x \in \mathbb{R}^n \ : \ Ax \geq b\}$, and its dual LP $K_d = \{\lambda^T A = 0, \ \lambda^T b = 1, \ \lambda \geq 0\}$. However, sometimes it is useful to have different representations of LP.

   (a) Show how to transform to above representation of LP K to the Equation form

   $$\widetilde{K} = \begin{cases} Cy = d, \\ y \geq 0 \end{cases}$$

   for some matrix C and vector d. More precisely, you need to find a matrix C and a vector d such that for the polytope $\widetilde{K} = \{y \ : \ Cy = d, \ y \geq 0\}$ there is a matrix F for which it holds:

   - $y \in \widetilde{K} \Rightarrow Fy \in K$,
   - for any $x \in K$ there is $y \in \widetilde{K}$ such that $x = Fy$.

   These two conditions mean that the linear map F maps the polytope $\widetilde{K}$ to the polytope K, or equivalently $K = \{Fy \ : \ y \in \widetilde{K}\}$.

   (b) Write the dual for the LP in the Equation form from part (a) (i.e. the dual should be in terms of C and d).

2. Consider an LP optimization problem: maximize $c \cdot x$ over $x \in K$, where $K = \{x \in \mathbb{R}^n : Ax \geq b\}$. For this problem we will assume that $K \neq \emptyset$ and that K is *bounded*, i.e. including constraints $\{-B \leq x_i \leq B\}$ for some large number B doesn't change the set K.

   Recall that we called a vertex (or extreme point) of K to be any point $x^* \in K$ which is a unique solution to a system of n linearly independent *equalities* from K. However, here we will use another definition of a vertex: we say that $x^* \in K$ is a vertex of K if there do not exist two other points $x_1, x_2 \in K$ such that $x^* = \frac{1}{2}(x_1 + x_2)$. You can use the claim that these definitions are equivalent without a proof.

Prove that there exists an optimal solution x^* to the optimization problem (max $c{\cdot}x$ for $x \in K$) such that x^* is a vertex of K.

Hint: Consider the set Q of all optimal solutions of the optimization problem. Can Q be represented as an LP program?