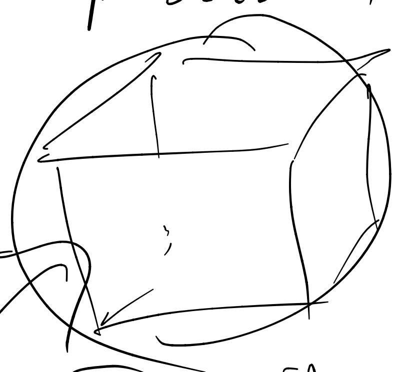
Lecture 10 : Probabilistic Method

12% of earth's surface is water rest is land.

Claim: No matter how the land & water are distributed, it is possible to inscribe a cube inside the earth s.t all 8 vertices of the cube are on land.



Idea: Inscribe a cube randomly (rotate it randomly)

W_i = event that i^{th} vertex/corner of cube is under water.

Want none of W_i to happen

$$\Pr[W_i] = 0.12$$

$$\Pr[\cup W_i] \le \sum \Pr[W_i] \le 8 \times 0.12 = 0.96$$

$\hookrightarrow$ union bound

$$\Pr[\cap \overline{W_i}] = 1 - \Pr[\cup W_i] > 0 \quad (< 1)$$

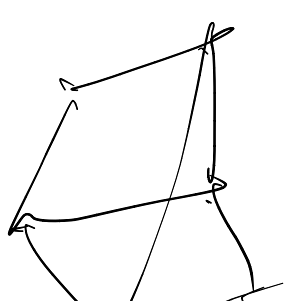
There is a positive prob that none of the corners are on water

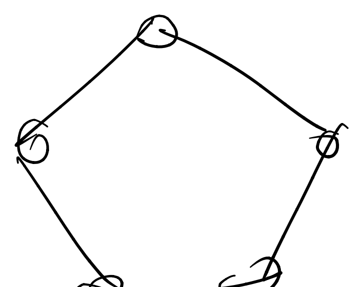
$\Rightarrow$ There exists a positioning for which none of the corners are on water.

Ramsey Number

Puzzle: In any group of six people, there are either mutual acquaintances or three mutual non-acquaintances

Equivalently, any graph on six vertices has either a clique of size 3 or an indep set of size 3





5 vertex gh has no 3-clique or indep set of size 3.

$R(k)$ = minimum n s.t. every n-vertex graph has either a k-clique or an indep. set of size k

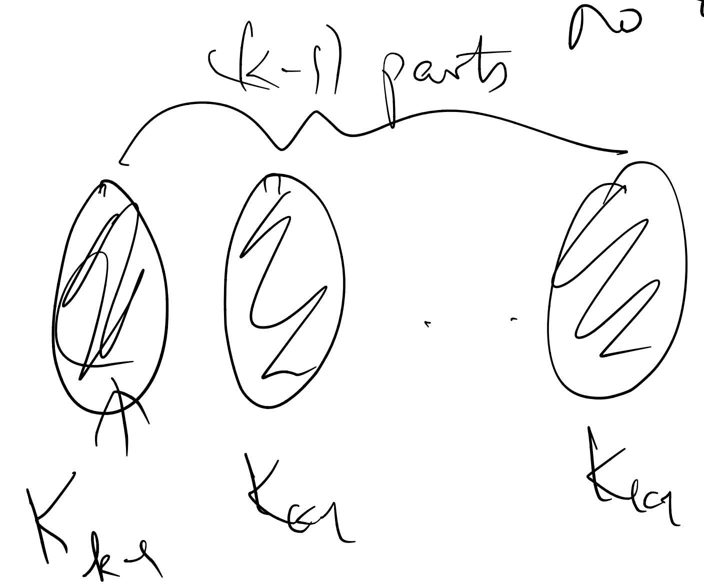
$$R(k) \leq \binom{2k-2}{k-1} \leq \frac{4^k}{\sqrt{k}}$$ (pf by induction)

$R(4) = 18$

$R(5)$ is unknown $\in [43, 49]$

Lower bounds on $R(k)$

$R(k) > N \iff$ Existence of a gh on N vertices with no k-clique or k-ind. set



$R(k) \geq \Omega(k^2)$

Conjecture: $R(k) \leq O(k^2)$

<u>Erdős</u> (1947) $R(k) \geq \Omega(k \cdot 2^{k/2})$

$\exists$ graph on n vertices with no clique or indep set of size $\approx 2\log n$

How to construct such a graph?

<u>Answer</u>: No one knows!

It's a major open question

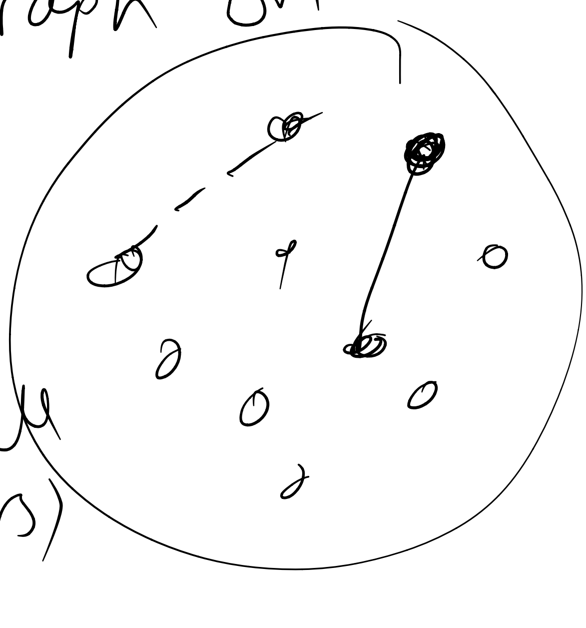
But.. they exist (in fact in abundance)

<u>Proof</u>: Birth of the probabilistic method

$n = 2^{k/2}$ (for simplicity)

Pick a random graph on n vertices

For each $\{u, v\}$ include edge (u, v) with prob $1/2$ (independently for all pairs)



Each possible labeled graph is sampled with prob. $\frac{1}{2^{\binom{n}{2}}}$

Let's count the number of k-cliques & the number of ind sets of size k.

$S \subseteq V$, $|S| = k$ $G = (V, E)$ ↳ random gh

$X_S \triangleq$ indicator random var that S induces a clique or indep set

$$N = \sum_{\substack{S \subseteq V \\ |S| = k}} X_S$$

$N = \#$ of "bad" subsets that violate the Ramsey property

$$E[X_S] = Pr[X_S = 1]$$
$$= Pr[S \text{ induces clique}] + Pr[S \text{ induces indep set}]$$
$$= 2^{-\binom{k}{2}} + 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

$$\mathbb{E}[N] = \mathbb{E}\Big[\sum_{\substack{S \in V \\ |S| = k}} X_S\Big]$$

$$= \sum_S \mathbb{E}[X_S]$$

$$= \binom{n}{k} 2^{1 - \frac{k(k-1)}{2}}$$

Recall:

$n = 2^{k/2}$

$$\le \frac{n^k}{k!} 2^{1 - \frac{k(k-1)}{2}}$$

$$= \frac{\cancel{2^{k^2/2}} \cdot 2^{1 - \cancel{\frac{k^2}{2}} + \frac{k}{2}}}{k!}$$

≈ 0 for large k $$= \frac{2^{\frac{k}{2}+1}}{k!} < 1 \text{ for } k \ge 4.$$

$\exists\, G$ for which $N(G) \le \mathbb{E}[N] < 1$

$N(G) \in \mathbb{N}$

$\Rightarrow N(G) = 0$

$N = \#$ bad configurations

Want object with $N=0$

$\mathbb{E}[N] < 1$ (often $\ll 1$)

$\Pr[N>0] = \Pr[N \geq 1] \underset{\text{(Markov inequality)}}{\leq} \mathbb{E}[N] < 1$

$\Rightarrow \Pr[N=0] > 0$

Comments:

- Random graphs a very active & rich area
- Prob. method also used in another pioneering work by Claude Shannon (1948) to prove the existence of good error-correction schemes

Ramsey graph constructions

Current record (2018)

Explicit graph on n vertices with no clique or ind set of size $(\log n)^{O(\log\log n)}$

Conje: optimal $\Theta(\log n)$

Motivation: Randomness extraction

Prove theorems in extremal combinatorics

$\mathcal{F}$ = family of subsets of $\{1, 2, \ldots, n\}$

$\mathcal{F}$ is an antichain if
no set in $\mathcal{F}$ is contained in
another set of $\mathcal{F}$.

$\mathcal{F} = \{ \{1,2\}, \{2,3\} \{1,3\} \}$

Q: ~~If~~ $\mathcal{F}$ is an antichain,
how big can $\mathcal{F}$ be?

Sperner's Theorem: $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

if $\mathcal{F}$ is an antichain

This is tight: Take $\mathcal{F}$ to contain
all subsets of size $\lfloor \frac{n}{2} \rfloor$

Take random ordering of elements
of Universe