Optimal Good-Aspect-Ratio Coarsening for Unstructured Meshes

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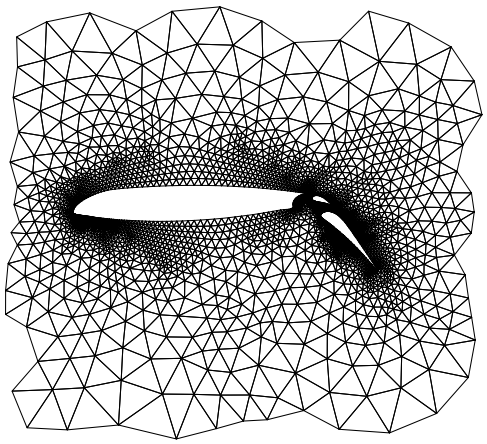
Mesh Properties

Mesh: a discretization of a geometric domain into simple elements (2D, triangles)

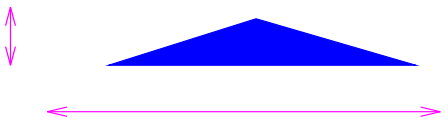


Barth's Mesh

Important Mesh properties:

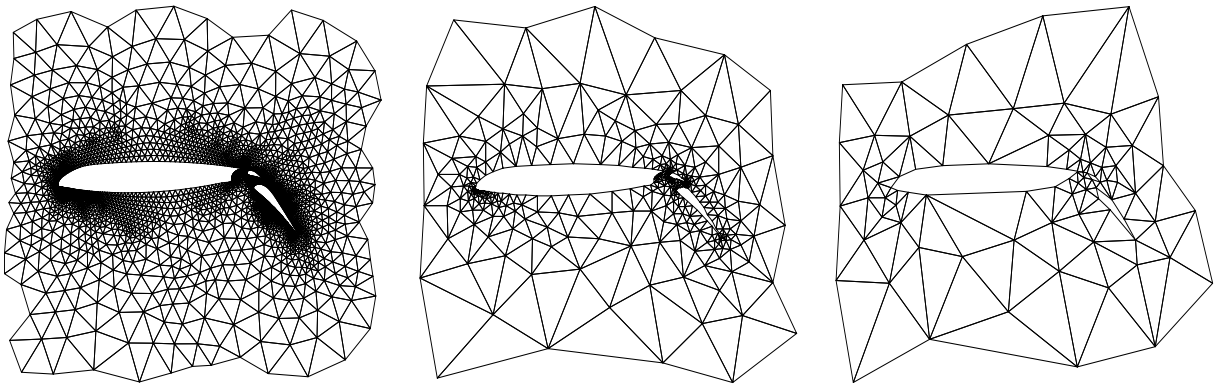
- **Element size**
- **Element shape:** aspect ratio of element

$$\frac{\text{Longest Dimension}}{\text{Shortest Dimension}} < A$$



Mesh Coarsening

Mesh coarsening: given an initial mesh, automatically produce a sequence of respectively coarser meshes with the desired qualities (to be specified).



Barth's Mesh Coarsened

Motivation: Numerical Methods

PROBLEM FORMULATION

- Describe the geometric domain
- Formulate Partial Differential Equation

MESH GENERATION

-Discretize the geometric domain

APPROXIMATION METHOD

Finite Element, Finite Difference
Control Volume

LINEAR SYSTEM
Ax = b

- Direct method
- Iterative method — multigrid

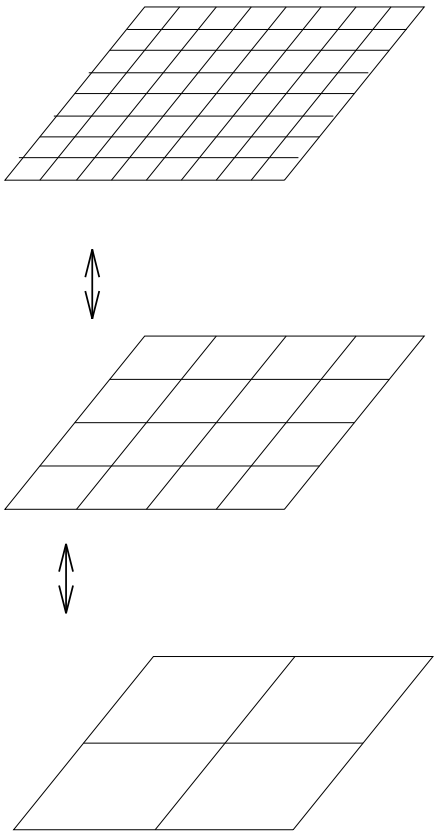
MESH COARSENING

Motivation: Multigrid Method

- First formulated for grids (Brandt77)
- Linear equation $A^i x^i = b^i$ for each grid in coarsening sequence
- Rather than solving only

$$A^0 x^0 = b^0$$

  - iterative method at each level
  - transfer intermediate solutions between levels



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What Makes the Grid Sequence Good?

...and what properties lend themselves to unstructured meshes?

- **Well-shaped elements**
  - Convergence of iterative method
  - Numerical error of discrete solution
- **Uniform coarsening**
  - Local similarity is important for transfer of solutions among neighboring meshes
  - Size reduction: important for work efficiency
- **Nesting of elements and nodes**
  - Simplifies theory behind the multigrid

Mesh Coarsening

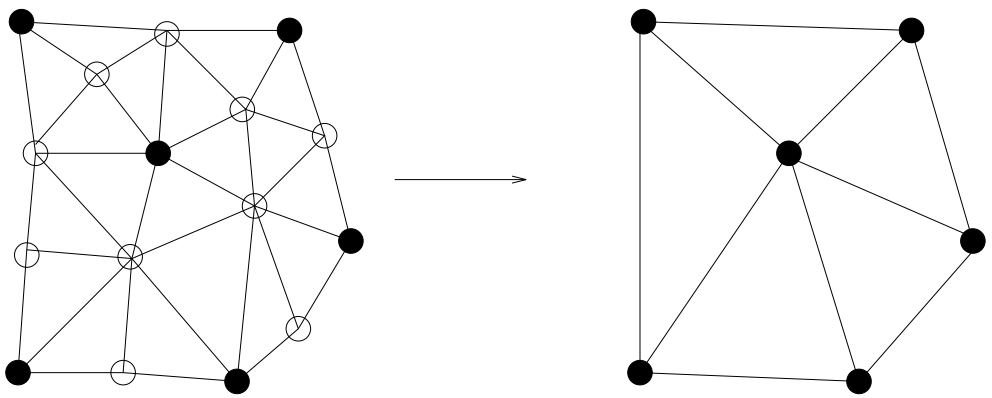
The problem: given an initial mesh, automatically produce a sequence of respectively coarser meshes, to be used e.g. in multigrid computations

Desired Properties :

1. **Well-shaped elements:** all meshes are of uniformly bounded good aspect ratio.
2. **Local approximation:** a coarse mesh element intersects a bounded by constant number of elements of the preceding finer mesh
3. **Node-nesting:** the node set of a mesh in the sequence is a subset of the preceding finer mesh node set

Previous Approaches

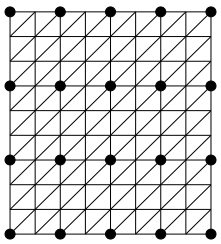
- Work by: Bank and Xu, Guillard, Mavriplis and Venkatakrishnan, Olliver-Gooch and others
- Popular method: Maximal Independent Set
  - retain an MIS of the mesh
  - re-triangulate the MIS
  - repeat



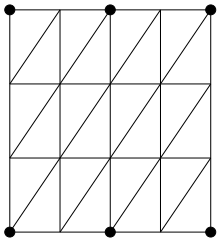
- Tarjan & Matheson(94): Guaranteed size reduction to $\frac{1}{3}n$ for triangular meshes
- No element quality guarantees in above

MIS: Example of Failure (1)

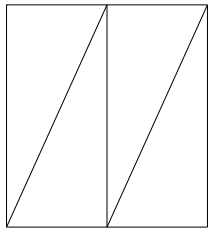
Uniform Meshes



E/e = 1



E/e = 3/2



E/e = 9/4

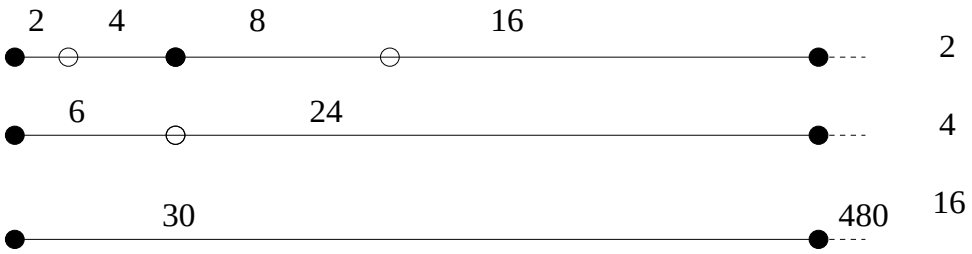
- Worst case example - repeated degradation
- In practice, randomized MIS performs well:
  - Experiments ran over million nodes grids
  - Smallest angle 18° - good aspect ratio

Our coarsening algorithm produces for this case

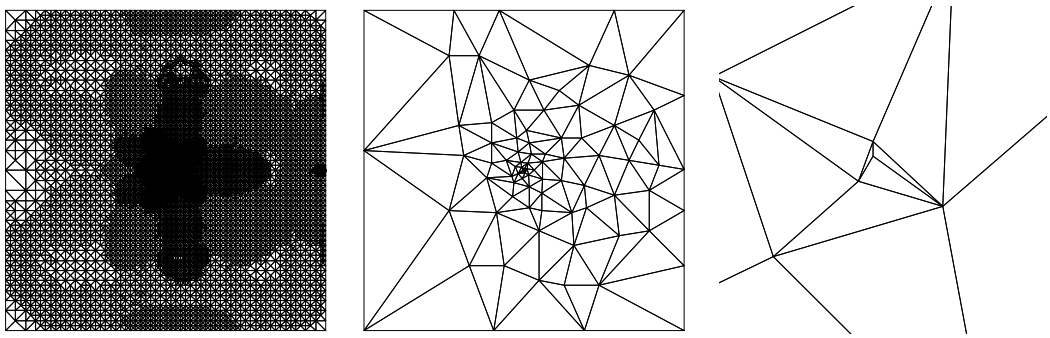
- *Guaranteed* good aspect ratio
- Geometric size reduction

MIS: Example of Failure (2)

Unstructured Graded Meshes



- we show:
  - it is *theoretically impossible* to repeatedly half this mesh, maintaining good aspect ratio
  - our algorithm reduces as much as possible
- in practice: small angles quickly form, e.g. 3°



Core of Coarsening Algorithm

Why does the MIS method fail?

- Cumulative degradation
  - **Solution:** capture all mesh information using "spacing functions" representation
- Coarsening too aggressively
  - **Solution:** optimally coarsen spacing function

Algorithm:

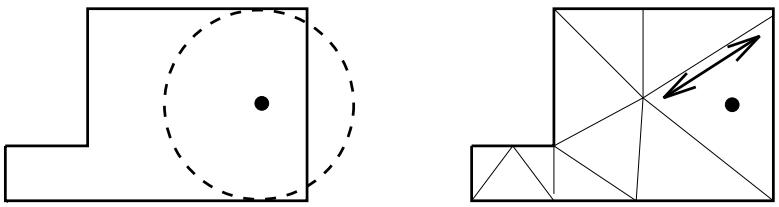
1. Recover spacing function of original mesh
2. Coarsen the spacing function "by a factor of C"
3. Prune away from original points until set is spaced by coarse spacing function
4. triangulate

Repeat with $C = 2, 4, 8, ...N$

Previously defined Mesh Functions

Edge Length (EL) function(Mitchell & Vavasis): Assigns to each point the length of the longest edge in its triangle.

Local Feature Size (LFS) (Ruppert,SODA92): Assigns to each point the radius of the smallest circle containing two disjoint boundary elements.



Used in analysis of mesh properties:

- EL measures local size of mesh elements
- LFS used to estimate the local size of the optimal mesh

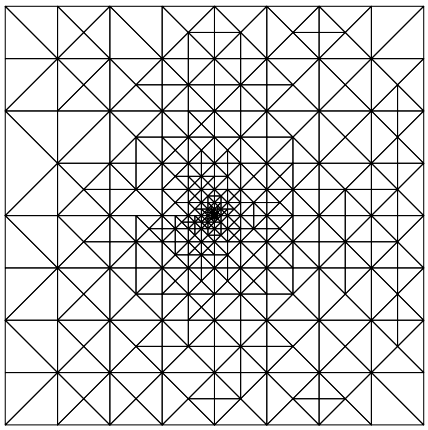
Mesh Spacing functions

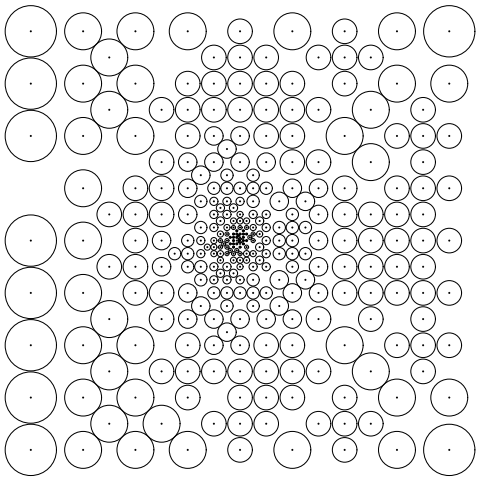
A point set is **well-spaced** by a spacing function f if:

- **Slowly Changing:** f is α-Lipschitz:

$$|f(x) - f(y)| \leq \alpha |x - y|$$

- **Sphere-Packing:**
  - f-spheres around points are disjoint,
  - gaps are bounded, i.e. $\int_B \frac{1}{f^2} < C$

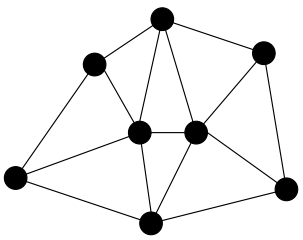


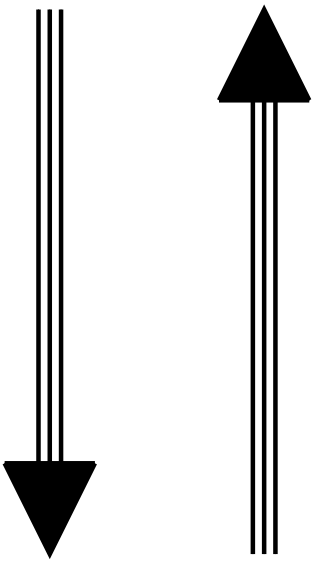


Spacing Functions ≡ Good Meshes

two dimensional mesh with good aspect ratio

α–multiple of Nearest Neighbor function

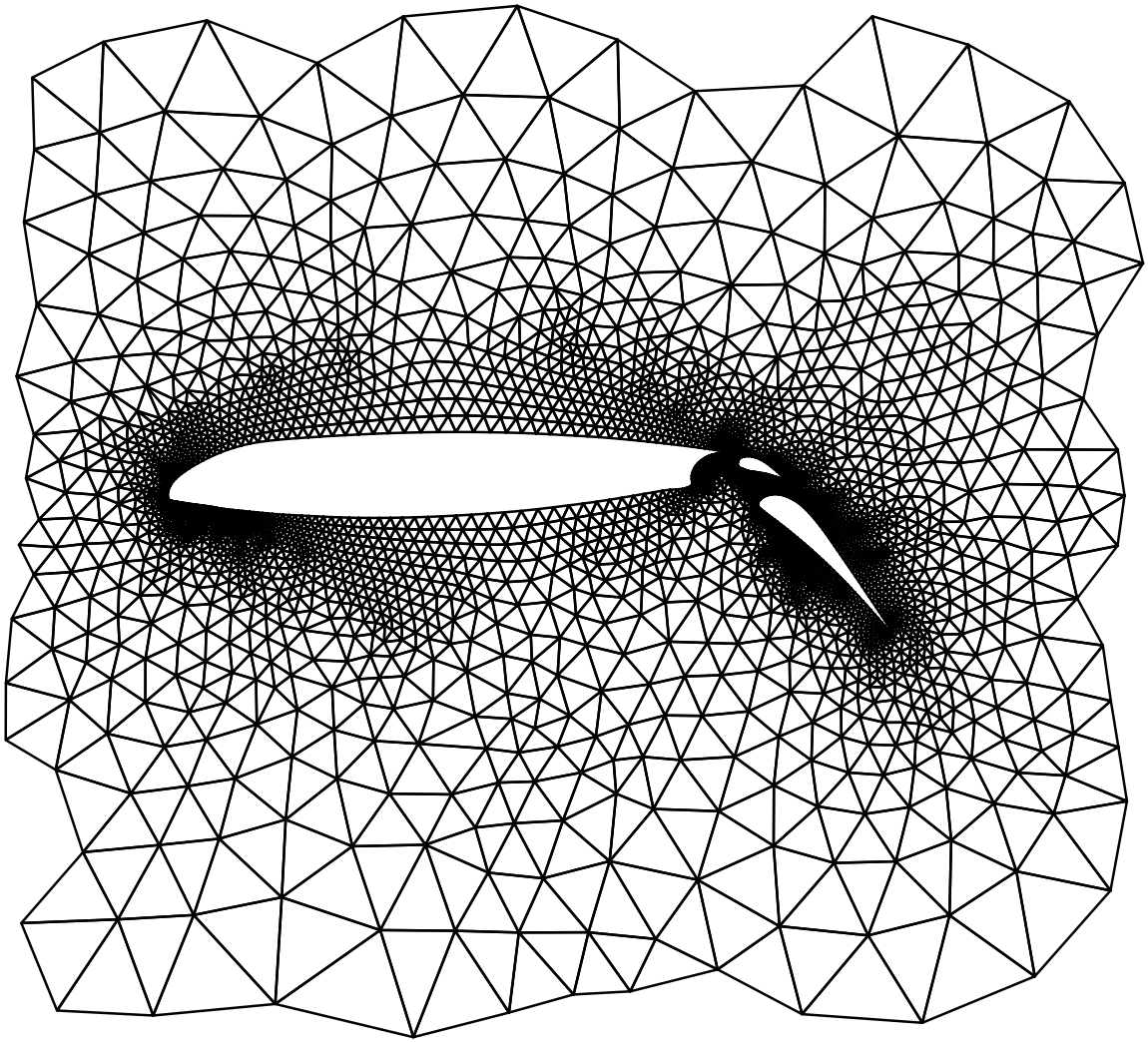




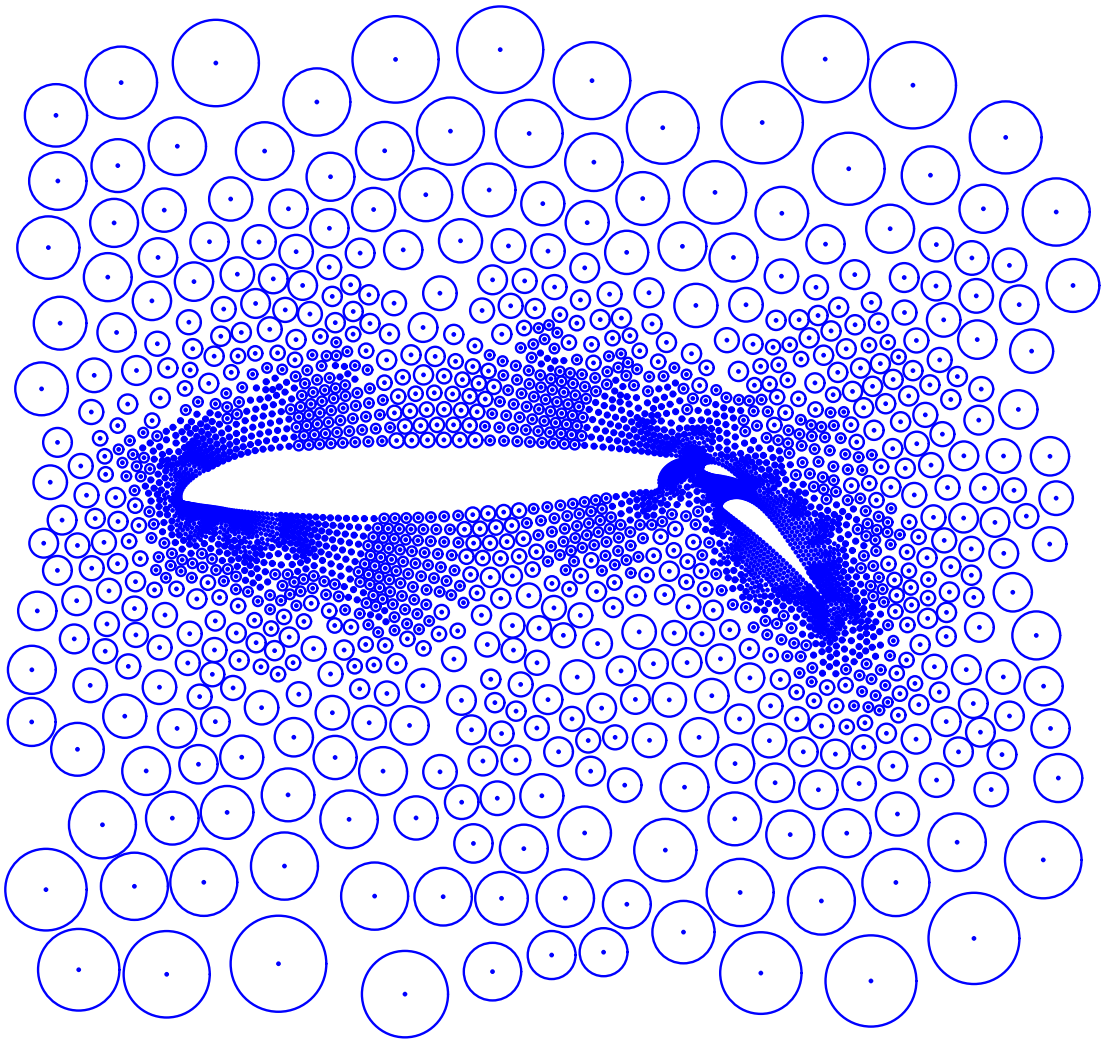
Delaunay triangulation of spheres' centers

α– Lipschitz spacing function
sphere packing: disjoint spheres, bounded gaps

Coarsening Barth's Mesh



Recovering the Spacing Function



Coarsening the Spacing Function

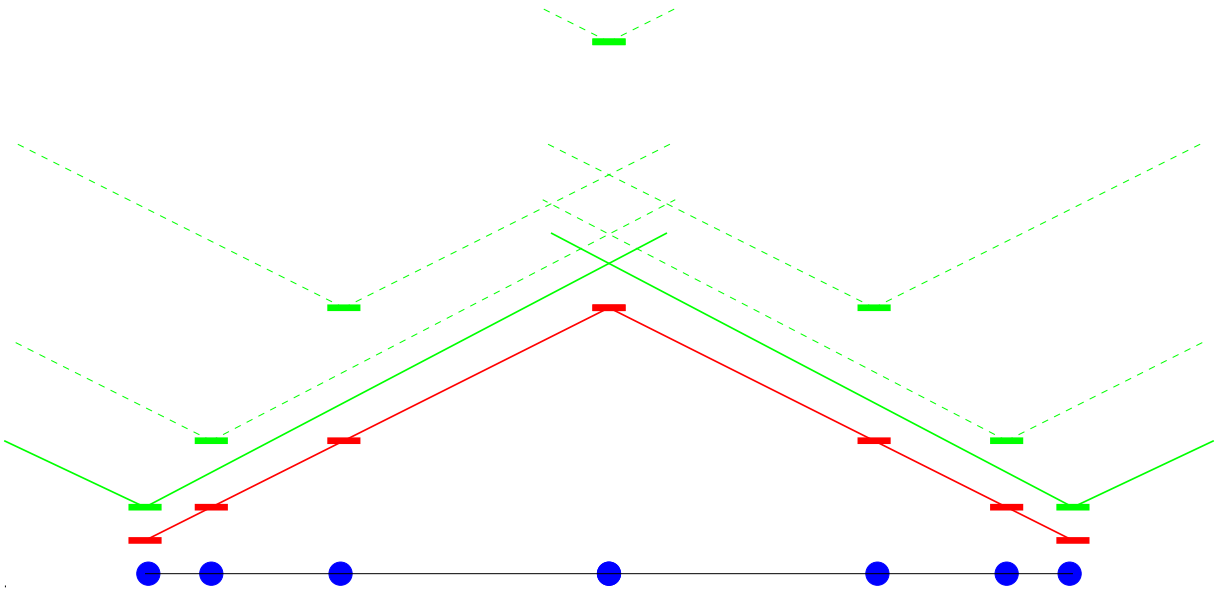
To coarsen function by a factor C:

- Point coarsening function:

$$f_p(x) = C f(p) + \alpha |x - p|$$

- Global coarsening function:

$$f_c(x) = \min_P f_p(x)$$



Optimality of Function Coarsening

Theorem: Any α-Lipschitz function g such that

$$g(p) \leq C f(p) \; \forall p \in P$$

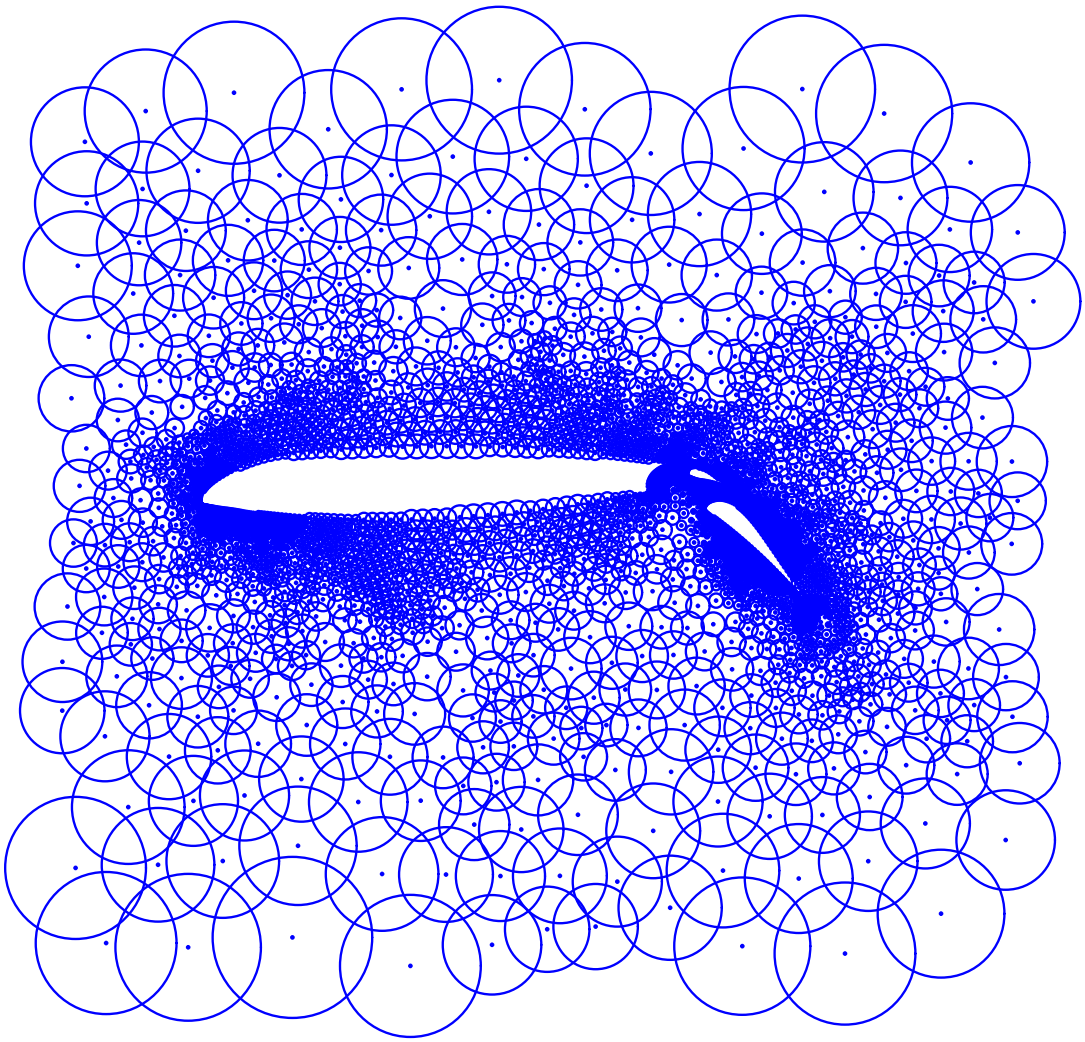
is smaller than f_C, i.e.

$$g(x) \leq f_C(x) \; \forall x \in \Omega$$

Corollary: Size Optimality
Any good aspect ratio mesh M, whose elements are at most C larger than the original mesh's elements, is not much smaller than the mesh M generated by our algorithm with coarsening factor C:

$$|M| \leq D|M'|$$

Coarsened Spacing Function



Pruning Points - the Conflict Graph

The coarser function f_C is no longer a sphere-packing of the original mesh point set P

- Form the **Conflict Graph (CG)** over P:

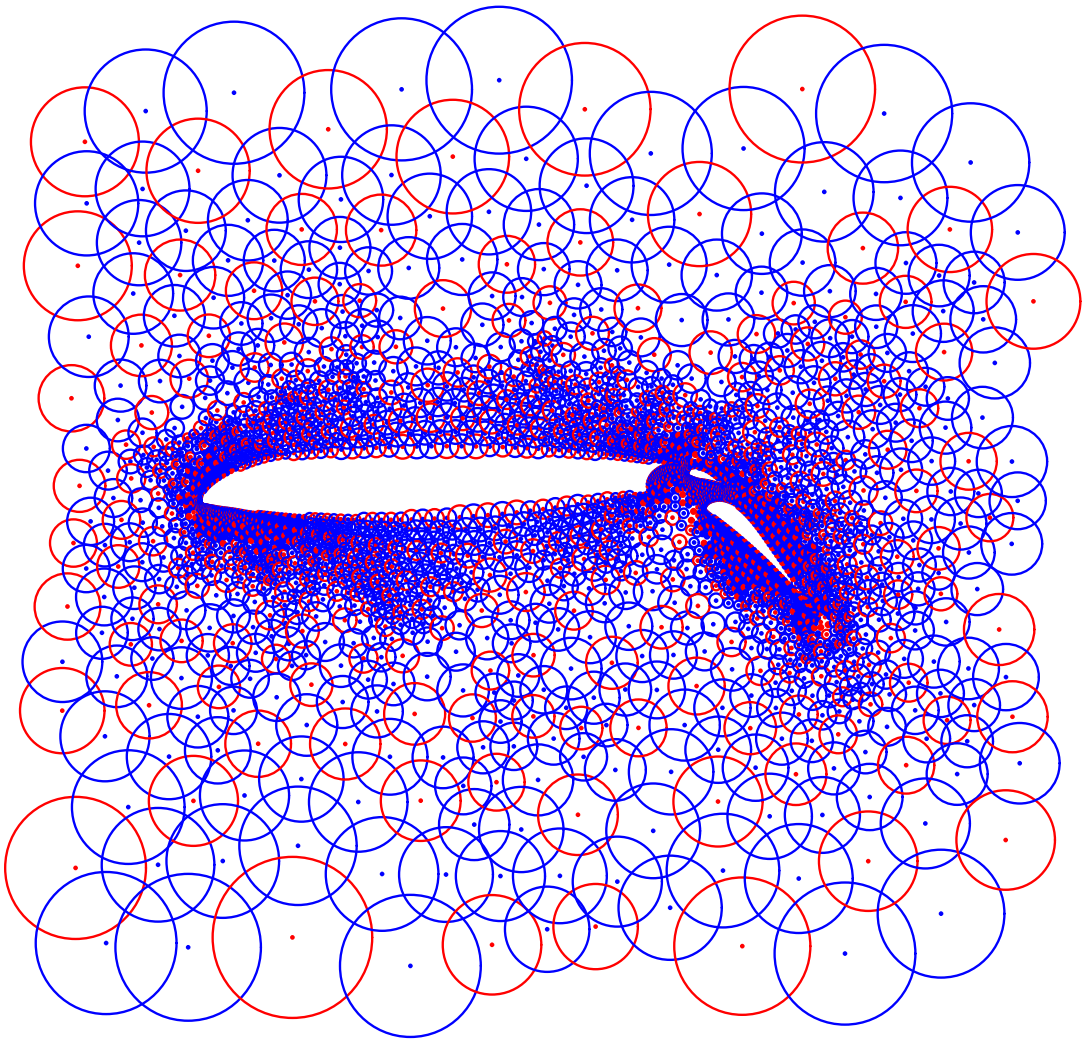
$$E = \{(p, q) | f_c(p) + f_c(q) > ||p - q||\}$$

- Let P_C be the MIS of the conflict graph. P_C is sphere-packed by f_C

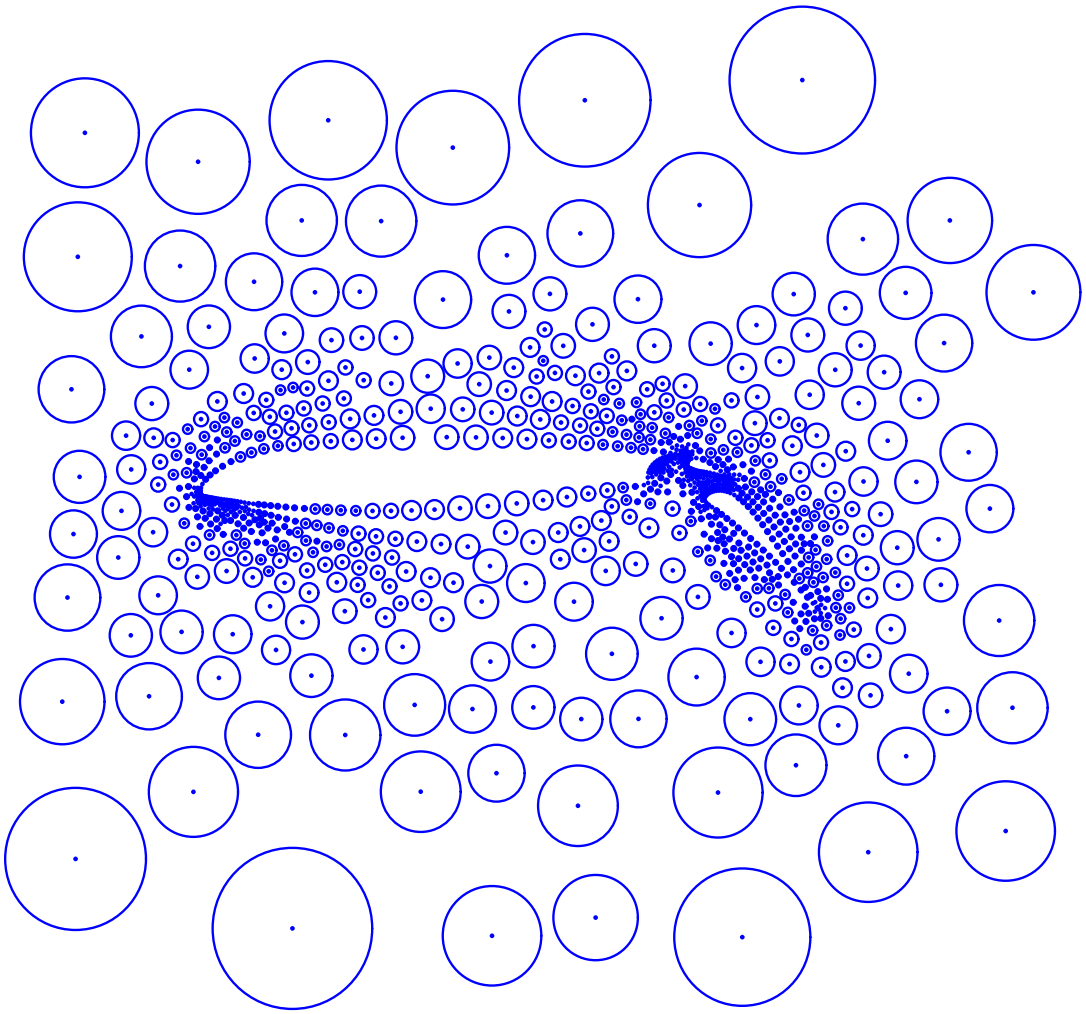
- **Theorem:** The gaps of P_C in terms of f_C are bounded by a constant depending only on the aspect ratio of the original mesh.

$\Rightarrow$ the Delaunay triangulation of P_C is of aspect ratio bounded by a constant depending on the aspect ratio of the original mesh.

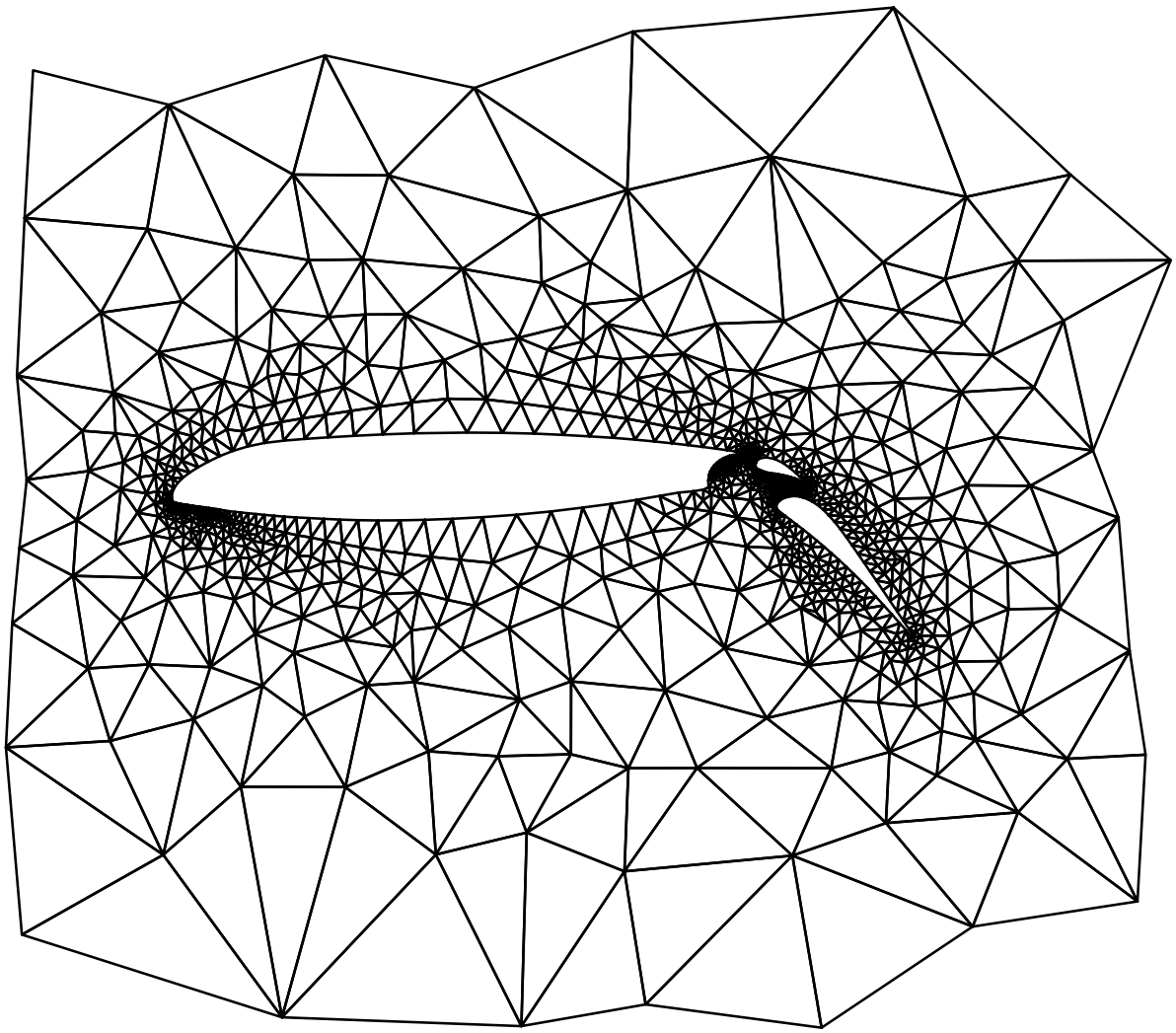
MIS of Conflict Graph



Resulting Point Set



Resulting Mesh



Coarsening Results

We presented the first mesh coarsening algorithm, generating a sequence with the following guarantees:

1. **Element Quality** Each mesh in the coarsening sequence is a good aspect-ratio mesh
2. **Local Approximation** Adjacent meshes in the sequence are similar
3. **Size Optimality** of the height and width of our coarsening sequence
4. **Node Nesting:** Can obtain node nesting by first creating coarser meshes, and forcing their nodes into the conflict graph of finer coarsenings.

Extensions - Future Work

- Randomized MIS works for quasi-uniform meshes
- A weighted version of Randomized MIS for unstructured meshes?
- Better understanding of coarsening the 1D boundaries
- Other applications:
  - Graphics and visualization
  - Time dependent PDEs