MFCS

Cardinality

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1 **Cantor and Cardinality**

2 **Countability**

3 **Three Theorems**

Cardinality

Definition

The size of a set is called its cardinality.

Of course, this is not much of a definition. In the words of G. Cantor:

> Every aggregate M has a definite "power," which we also call its "cardinal number."
>
> . . . the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given.

WTF?

In other words, Cantor had some intuition, some vague idea, but absolutely no way to make this idea precise, at least in the modern sense. He was never interested in trying to axiomatize set theory.

He did manage, though, to prove a number of critical theorems.

So, there is life without axioms, but it has major pitfalls (Cantor did not realize, and could not possibly have realizede, that his quest to resolve the Continuum Hypothesis was doomed).

The Finite Case

First, the easy case: if the set in question is finite

$$A = \{a_1, a_1, a_2, \ldots, a_n\}$$

then our notion of cardinality is straightforward: it's just n. Of course, we assume that all the a_is are different.

A slightly more formal definition:

Definition

A set A is finite if there is a natural number n and a bijection $f : [n] \to A$.

As we have seen, the hard part here is define what a natural number is in the first place (formally as a set, not just an intuitive concept).

Big Question

How do we measure the size of sets like

$$\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C},\ \mathbb{N}\times\mathbb{N},\ \mathbb{N}\to\mathbb{N},\ \mathbb{R}\to\mathbb{R},\ldots$$

The cheap and useless answer is to say “they’re all infinite” and write stuff like $|\mathbb{N}| = \infty$.

We want much more precision. Ideally, we would like to organize all sets into a nice hierarchy according to size.

A Dirty Trick

To handle infinite sets and infinite cardinalities we really ought to generalize the natural numbers accordingly. Unfortunately, this requires an excursion into set theory and some clever ideas by von Neumann, so we will weasel around this.

Definition

For any set A, write $|A|$ for the cardinality of A.

This is mildly criminal, we have not explained what this cardinality is actually supposed to be. The notation $|A|$ looks scientific, but means squat at this point.

The Honest Approach

Recall our old project: we want to implement every mathematical object as a set.

Not really for actual use, but as a reference that settles all possible questions with absolute precision and rigor.

So, in this situation we should give a set-theoretic definition of cardinal numbers, thingies that start with the naturals, but then go into the transfinite.

People like Zermelo, Fraenkel and von Neumann figured out how to do this about a century ago, but it is too involved and technical for us. If you are interested, take a look at Ordinals.

Comparisons First

To avoid having to define cardinals, we focus instead on comparing sizes.

How do we compare the size of sets?

More precisely, we would like to pin down what it means that

$$|A| < |B| \quad \text{or} \quad |A| = |B| \quad \text{or} \quad |A| > |B|$$

No problem for finite sets, not so much for infinite sets like $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{R}$.

Here Goes . . .

Definition

Let A and B be two arbitrary sets.

$$|A| = |B| \iff \exists\, f \text{ bijection } (f : A \longleftrightarrow B)$$

$$|A| \leq |B| \iff \exists\, f \text{ injection } (f : A \longrightarrow B)$$

Sets with the same cardinality are called equipotent or equinumerous.
In symbols: $A \approx B$.

Exercise

Verify that at least for finite sets this all makes perfect sense.

Sanity Check I

We could also have defined

$$|B| \geq |A| \iff \exists\, f \text{ surjection}\,(f : B \longrightarrow A\,)$$

This makes sense since we already know that there is a surjection $B \to A$ iff there is an injection $A \to B$ (see the next slide in case you have forgotten).

So $|A| \leq |B|$ iff $|B| \geq |A|$, as any sane person would expect.

Recall: Injections versus Surjections

Lemma (AC)

Let A be non-empty.
There is an injection $f : A \to B$ if, and only if,
there is a surjection $g : B \to A$.

Proof.

Let f be the injection. Pick $a_0 \in A$ and get a surjection g by

$$g(b) = \begin{cases} a & \text{if } f(a) = b, \\ a_0 & \text{if } b \notin \operatorname{rng} f. \end{cases}$$

Assume g is the surjection. For each $a \in A$, there exists a $b \in B$ such that $g(b) = a$ by surjectivity. Pick one such b in the fiber $f^{-1}(a)$, say, b', and set $f(a) = b'$. This requires the axiom of choice in general.

□

Sanity Check II

At the very least, “same-cardinality” should be an equivalence relation.

- reflexive: $I_A : A \longleftrightarrow A$
- symmetric: $f : A \longleftrightarrow B$ yields $f^{-1} : B \longleftrightarrow A$
- transitive: $f : A \longleftrightarrow B$ and $g : B \longleftrightarrow C$ yields $g \circ f : A \longleftrightarrow C$

So far, so good. We can organize all sets into equivalence classes according the existence of bijections between them.

Sanity Check III

Likewise, “at-most-same-cardinality” is a pre-order: it is reflexive and transitive. But it catastrophically fails to be a partial order: $|A| \leq |B|$ and $|B| \leq |A|$ does not at all imply that $A = B$.

Another important property is comparability: we really want for two arbitrary sets A and B that their cardinalities are related:

$$|A| \leq |B| \quad \text{or} \quad |B| \leq |A|$$

This works fine, too, as long as we have sufficiently strong axioms of set theory. As usual, we need the Axiom of Choice.

Important Cases

For us, the most interesting applications will be to find bijections

- $f : [n] \longleftrightarrow X$
- $f : \mathbb{N} \longleftrightarrow X$
- $f : \mathfrak{P}(\mathbb{N}) \longleftrightarrow X$

corresponding to finite, mildly infinite, and badly infinite (size of the continuum).

Constructing such bijections by hand can be difficult, which is part of the reason Cantor's result came as such a surprise to many. And engendered fierce resistance in some quarters.

1 Cantor and Cardinality

2 **Countability**

3 Three Theorems

Countability

Definition

Let A be a set.

- A is countable if there is a surjection $f : \mathbb{N} \to A$ or $A = \emptyset$.
- A is uncountable if it fails to be countable.

So a non-empty set is countable if it can be listed just like $\mathbb{N}$.

$$a_0, a_1, a_2, \dots, a_n, a_{n+1}, \dots$$

If A is finite, there will be lots of repetitions, but for infinite A all the a_i can be chosen to be distinct.

Notation: Countably infinite sets are also called denumerable.

Integers are Countable

One might think that $\mathbb{Z}$ is infinitely larger than $\mathbb{N}$, but that's not true. We can easily list $\mathbb{Z}$ like so

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

Exercise

Construct more bijections $\mathbb{N} \leftrightarrow \mathbb{Z}$*.*

Exercise

After reading the last section in these slides, show that there are uncountably many such bijections.

Cartesian Products

Theorem (Cantor)

$\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.

Proof. Here is a pairing function $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

$$\pi(x, y) = 2^x(2y + 1) - 1$$

It is easy to check that π is indeed a bijection. □

Exercise

Come up with other pairing functions.

Really?

It is easy to get jaded about a result like this, but it's really far from clear: $\mathbb{N} \times \mathbb{N}$ contains infinitely many copies of $\mathbb{N}$, so how could it possibly have the same size?

Even more disconcerting is the fact that

$$\mathbb{N} \approx \mathbb{N}^{100^{100^{100}}}$$

One needs to consider something like $\mathbb{N} \to \mathbf{2}$ to get away from countability (this is basically the same as $\mathfrak{P}(\mathbb{N})$).

Rationals are Countable

Every rational number can be written uniquely in the form a/b where $a \in \mathbb{Z}$, $b \in \mathbb{N}_+$ and a and b are coprime. In other words, there is an injection

$$\mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{N}$$

But we already now how to map $\mathbb{Z}$ into $\mathbb{N}$, so we get an injection

$$\mathbb{Q} \longrightarrow \mathbb{N} \times \mathbb{N}$$

Using our pairing function, we get an injection

$$\mathbb{Q} \longrightarrow \mathbb{N}$$

Theorem (Cantor)

The rationals are countable.

Algebraic Numbers

A real number is said to be algebraic if it is the root of a polynomial with integer coefficients. These numbers have a finite description and are relatively easy to handle. Incidentally, the algebraic numbers form a nice field between $\mathbb{Q}$ and $\mathbb{R}$.

Theorem (Cantor)

The set of algebraic numbers is countable.

Cantor also showed that the reals are uncountable (see below), so this means that the algebraic numbers are few and far between. Most reals fail to be algebraic.

A Helpful lemma

Lemma (AC)

The union of a countable family of countable sets is countable.

Proof.

Let A_n, $n \in \mathbb{N}$, be the family of sets and let $A = \bigcup A_n$. We may safely assume that $A_n \neq \emptyset$ for all n. Hence there are surjections $f_n : \mathbb{N} \to A_n$. Then the map

$$F : \mathbb{N} \times \mathbb{N} \to A \qquad F(n, m) = f_n(m)$$

is a surjection, done.

□

Subsets

Lemma

Any subset of a countable sets is countable.

Proof.

Let A countable and $B \subseteq A$; we may safely assume B is infinite.

Let $f : \mathbb{N} \to A$ be a surjection.

Here is a convoluted recursive definition of a partial function $g : \mathbb{N} \to \mathbb{N}$

$$g(n) = f\big(\min\big(\, j \in \mathbb{N} \mid f(j) \in B \land \forall\, i \, < n(f(j) \neq g(i))\,\big)\big)$$

Whatever g does, it is clear from the definition that the range of g is a subset of B.

We claim it is equal to B.

Come again . . .

First, $g(0)$ is just $f(j_0) \in B$ where j_0 is minimal such that $f(j_0) \in B$.

But then $g(1)$ is $f(j_1) \in B$ such that $j_1 > j_0$ is minimal and $f(j_0) \neq f(j_1)$.

And so on and so forth. We are picking elements in B according to the order produced by f.

Since B is infinite, we never run out of js, so there is always a minimal one and g is really defined on all of $\mathbb{N}$.

Note that g is an injection by its definition. Make sure to figure out how to prove it is also surjective.

□

Another Angle

Here is another way to look at this. Suppose we are in the interesting case when B is infinite.

Define an order $\prec$ on B as follows:

$$b \prec b' \iff \min f^{-1}(b) < \min f^{-1}(b')$$

This order allows us to list B as

$$b_0, b_1, b_2, \ldots, b_n, \ldots$$

So B is countable.

Words are Countable

Question:
How many words over a finite alphabet are there?

Again, the easiest way to see that there are only countably many words is to arrange them into an ω-sequence

$$w_0, w_1, w_2, \ldots, w_n, w_{n+1}, \ldots$$

The standard way of doing this is to use length-lex order. E.g., for alphabet $\{a, b, c\}$ we get:

$$\varepsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, \ldots$$

Note that standard lexicographic order does not work in this case:

$$b > ab > aab > aaab > \ldots > a^n b > a^{n+1} b > \ldots$$

Depressing Consequence

Theorem

There are only countably many computable functions.

Proof.

A computable function can be described by a program, a word over some alphabet. There are only countably many programs, so there are only countably many computable functions.

□

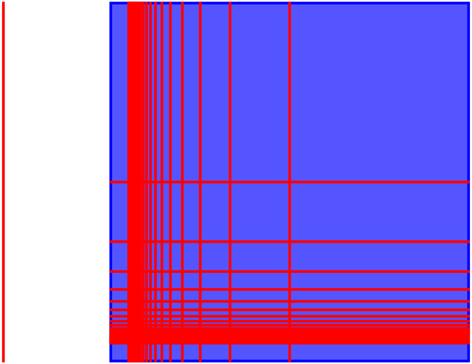
At least in mathematics[†] we encounter uncountably many objects; for example the set of reals is irreparably uncountable. So most reals are not computable—whatever computations we perform, they all take place in a tiny subspace of actual Euclidean space, causing endless problems. Rcall the lecture on the Traveling Salesperson Problem.

[†] Hard question: how about physics?

Bizarre Results

Lemma

The unit interval $[0,1] \subseteq \mathbb{R}$ *has the same size as unit square* $[0,1]^2 \subseteq \mathbb{R}^2$*.*



Intuition?

This is rather counter-intuitive and raised quite a few eyebrows. Cantor himself wrote in a letter to Dedekind[†]:

> Je le vois, mais je ne le crois pas ...

The problem is that one naturally searches for "nice" functions (simple definition, continuous, differentiable, etc.), not for bizarre constructs that come out of the abyss of set theory.

[†] I see it, but I can't believe it.

Aside: Reals

At this point you might ask: what on earth are the reals? If we are supposed to show the existence of a bijection $[0,1] \leftrightarrow [0,1]^2$ we better have a solid definition. "Numberline" won't cut it.

Without going into the weeds, think of a real as consisting of an integer part plus a fractional part. This is often (and poorly) written

$$x = \lfloor x \rfloor + \{x\}$$

The integer part is no problem, the fractional part is represented by an infinite sequence of, say, decimal digits:

$$r = .r_1 r_2 \ldots r_n \ldots \rightsquigarrow \sum_{n \geq 1} r_n \, 10^{-n}$$

where $r_i \in \{0, 1, \ldots, 9\}$ so that $0 \leq r \leq 1$.

Getting Serious?

This “definition” is lousy at best, it is based on a fairly abritrary representation rather than an attempty to address the actual object.

We will forego the opportunity to inflict cognitive pain on the student body and not worry about a precise definition in set theory.

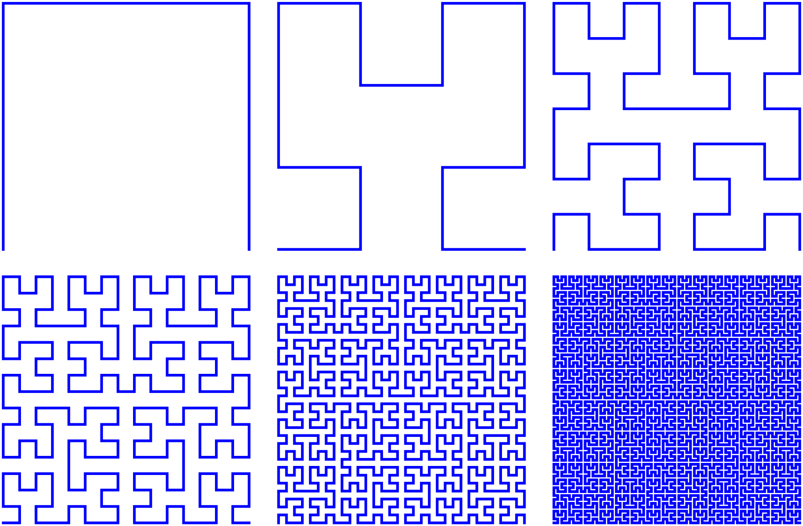
If you really want to know, look up

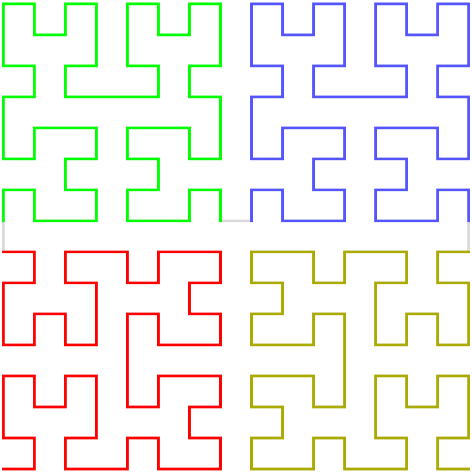
- Dedekind cuts,
- Cauchy sequences.

Exercise

Try to come up with a bijection based on this fractional part idea.

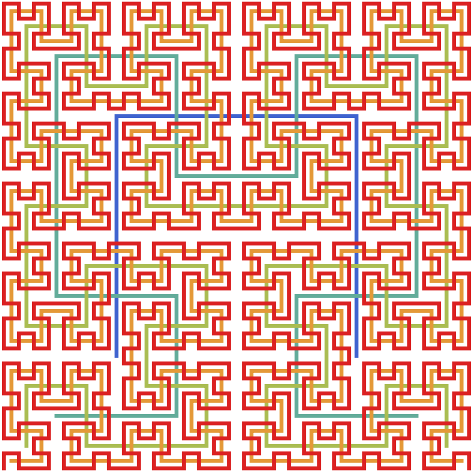
Hilbert’s Approach





This produces a sequence of curves $[0,1] \to [0,1]^2$ that fill the unit square in the limit. This is much more natural than Cantor's original approach.

Overlay



Just Notation

In the context of cardinality, it is standard usage to write

$$\aleph_0$$

for the first infinite cardinal. Again, we will not get involved with the tempting problem to really define what $\aleph_0$ is in terms of set theory.

$\aleph_0$ is the cardinality of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, the algebraic numbers, the set of words over any (countable) alphabet, the set of decidable relations and computable functions, the collection of hereditarily finite sets, . . .

Is There More?

As it turns out, there are many, many more cardinals floating around. And, thanks to Cantor, we have very pretty symbols to write them down:

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots, \aleph_\omega, \aleph_{\omega^\omega}, \aleph_{\varepsilon_0}, \aleph_{\aleph_1}, \ldots$$

Some find this headache inducing.

Fortunately, in practice, the most important ones are

$$\aleph_0 \quad \aleph_1 \quad 2^{\aleph_0}$$

$\aleph_1$ is the least cardinality bigger than $\aleph_0$, and $2^{\aleph_0}$ is the cardinality of $\mathfrak{P}(\mathbb{N})$ which turns out to be the cardinality of the continuum $\mathbb{R}$.

Cardinal Arithmetic

How does one compute with cardinals?

The good news is that addition and multiplication are trivial.

Lemma

Let λ and κ be two infinite cardinals. Then

$$\lambda + \kappa = \lambda \cdot \kappa = \max(\lambda, \kappa).$$

Intuitively, this means that the size of the union of two infinite sets is the same as the cardinality of the larger set. Ditto for Cartesian product.

The Swamp

Sadly, cardinal exponentiation is hugely complicated. For example, in ordinary set theory one cannot even determine the relative size of

$$2^{\aleph_0} \quad \text{versus} \quad \aleph_1$$

We will see in a moment that $2^{\aleph_0} \geq \aleph_1$, but equality is open. This is the famous Continuum Hypothesis and caused Cantor endless grief (some would say: drove him mad). As it turns out, in Zermelo-Fraenkel set theory, things could go either way and one can choose $2^{\aleph_0}$ to be just about anything one would like it to be.

1 Cantor and Cardinality

2 Countability

3 **Three Theorems**

Three Theorems

Theorem (Cantor-Schröder-Bernstein)

Suppose $f : A \to B$ *and* $g : B \to A$ *are injective.*
Then A *and* B *have the same cardinality.*

Theorem (Cantor I)

The set of real numbers, $\mathbb{R}$*, is not countable.*

Theorem (Cantor II)

For any set A*, the cardinality of* $\mathfrak{P}(A)$ *is greater than the cardinality of* A*.*

Say What?

Cantor-Schröder-Bernstein is really a sanity check.
We want for all cardinals κ and λ

$$\kappa \leq \lambda \quad \text{and} \quad \lambda \leq \kappa \quad \text{implies} \quad \kappa = \lambda$$

Cantor's first theorem shows that there are at least two levels of infinity, and that they play a role in calculus.

Cantor's second theorem shows that there are infinitely many levels of infinity:

$$|\mathbb{N}| < |\mathfrak{P}(\mathbb{N})| < |\mathfrak{P}^2(\mathbb{N})| < |\mathfrak{P}^3(\mathbb{N})| < \dots$$

CSB Proof

It suffices to establish the following claim.

Claim

Suppose $C \subseteq B \subseteq A$ and $A \approx C$. Then $A \approx B$.

To see why the claim suffices, consider injective functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$. Now set

$$\begin{aligned} A &= X \\ B &= g(Y) \\ C &= g(f(X)). \end{aligned}$$

Then certainly $X \approx A \approx C$, and $B \approx Y$. From the claim it follows that $A \approx B$, and, by transitivity, $X \approx Y$.

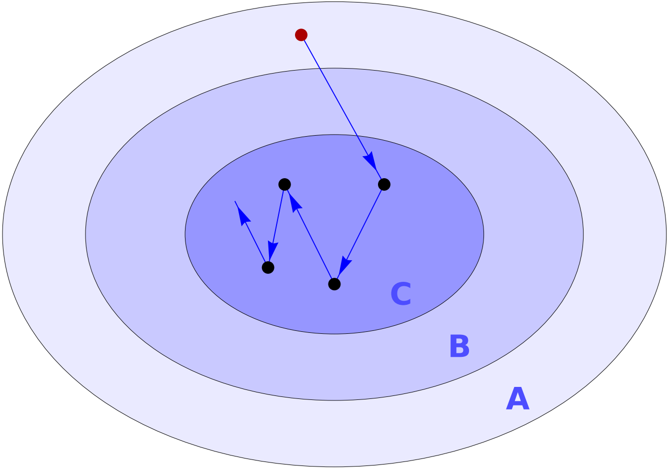
Intuition

Intuitively, one can motivate the following construction as follows. Think of the elements of A and B as places, and put a pebble on each place in A. We have to move all the pebbles in such a way that they wind up on all the places in B, without collisions.

More precisely, each place in B must be occupied by exactly one pebble after all the moving is finished. That produces a bijection $h : A \longleftrightarrow B$.

We have to move all pebbles in $x \in A - B$; say, pebble x goes to $h(x)$. Unfortunately, there is already a pebble in this position, so this one has to move to $h(h(x))$, displacing yet another pebble, and so forth. Any pebble unaffected by this process simply stays where it is.

Kicking the Can . . .



Here Goes . . .

Suppose we have a bijection $h : A \longleftrightarrow C$. We need to concoct a new bijection $H : A \longleftrightarrow B$. To this end, define the *displacement* set

$$D = \{\, h^n(x) \mid x \in A - B, n \geq 0 \,\}.$$

In other words, D is the union of all h-orbits of points in $A - B$.

Define a function H from A to A as follows:

$$H(x) = \begin{cases} h(x) & \text{if } x \in D, \\ x & \text{otherwise.} \end{cases}$$

Essentially, we are hiding the elements of $A - B$ in C, the rest we leave in place.

Mopping Up

Note that D consists of $A - B$ plus part of the range of h, a subset of C. So the range of H is certainly contained in B.

Now consider an arbitrary point $x \in B$. If $x \in D$, then for some z we have $h(z) = x$, so x is in the range of H. If $x \notin D$, then $H(x) = x$ and again in the range.

For injectivity suppose $H(x_1) = H(x_2)$. If both x_1 and x_2 are in D, or both are not in D, it follows that $x_1 = x_2$.

So suppose $x_1 \in D$ but $x_2 \notin D$. Then $H(x_1) = h(x_1) \in D$, but $H(x_2) = x_2 \notin D$, contradiction.

□

Application

It is easy to show that the open interval $(0,1) \subseteq \mathbb{R}$ has the same cardinality as all of $\mathbb{R}$:

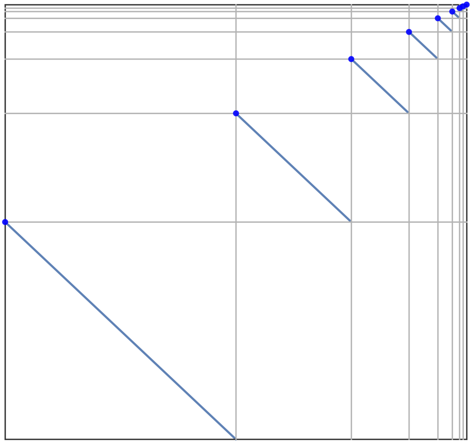
$$\begin{aligned} f &: (0,1) \to \mathbb{R} \\ f(x) &= \tan \pi(x - 1/2). \end{aligned}$$

How about the half-open interval $[0,1) \subseteq \mathbb{R}$?

This is trivial with Cantor-Schröder-Bernstein: all we need is 2 injections

$$\begin{aligned} f^{-1} &: \mathbb{R} \to (0,1) \subseteq [0,1) \\ \mathsf{Id} &: [0,1) \to \mathbb{R} \end{aligned}$$

Hard Labor



A direct bijection $[0,1] \to (0,1]$. Ponder deeply.

Cantor: Diagonalization

Warm-up:
The number of binary sequences of length n is larger than n.

Yes, yes, we can do this by counting, but ordinary counting does not work for infinite sets; we need a different approach.

We will prove something more constructive:
Given n binary sequences s_i, $i < n$, of length n, there is a binary sequence t of length n that differs from all of them.

Here goes: define the new sequence t by changing the diagonal sequence $s_i(i)$:

$$t(i) = 1 - s_i(i).$$

where $i < n$. Then t differs from all the s_i in at least one bit, so $t \neq s_i$ for all $i < n$.

Flipping Bits

We get t by flipping each bit along the diagonal of a matrix. Hence the resulting sequence cannot be a row in the matrix.

$$
\begin{array}{ccccc}
\color{red}{s_0(0)} & s_0(1) & s_0(2) & \ldots & s_0(n-1) \\
s_1(0) & \color{red}{s_1(1)} & s_1(2) & \ldots & s_1(n-1) \\
s_2(0) & s_2(1) & \color{red}{s_2(2)} & \ldots & s_2(n-1) \\
\vdots & & & & \vdots \\
s_{n-1}(0) & s_{n-1}(1) & s_{n-1}(2) & \ldots & \color{red}{s_{n-1}(n-1)}
\end{array}
$$

In general, it does not matter how we change the element $s_i(i)$ in t, it just has to be different. With bits there is only one choice, of course.

The Key Insight

This also works for infinite sequences.

Simply replace $i < n$ by $i \in \mathbb{N}$ and everything works just fine.

Claim

There are uncountably many binary sequences: $\mathbb{N} \to \mathbf{2}$ is uncountable.

Note that $|\mathfrak{P}(\mathbb{N})| = |\mathbb{N} \to \mathbf{2}|$: a map $f : \mathbb{N} \to \mathbf{2}$ is just a bitvector (characteristic function) for a subset of $\mathbb{N}$. So we know that $\mathfrak{P}(\mathbb{N})$ is uncountable.

The Real Thing

To show that $\mathbb{R}$ is uncountable, it clearly suffices to show that the open interval $(0,1) \subseteq \mathbb{R}$ is uncountable. Assume we have an enumeration of $(0,1)$, i.e., a list

$$x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$$

that contains each real in $(0,1)$ exactly once. Since $0 < x_i < 1$, we have decimal expansions

$$x_i = 0.x_{i1}x_{i2}x_{i3}\ldots$$

This representation is potentially ambiguous, so let's agree that that there are no trailing infinite blocks of 9s: increment previous digit, and replace by 0s.

E.g., write 0.1235, not $0.1234999999\ldots$.

Contd.

Now define digits y_j by

$$y_j = \begin{cases} 3 & \text{if } x_{jj} = 2, \\ 2 & \text{otherwise.} \end{cases}$$

and let $y = \sum y_j \cdot 10^{-j}$.

Note that y has only decimal digits 2 and 3, and in particular no trailing 9s. Hence $0 < y < 1$.

Now suppose $y = x_i$ for some i.

Then x_i also has only decimal digits 2 and 3, and we must have $x_{ij} = y_j$ for all j, clearly contradicting the construction: $x_{ii} \neq y_i$.

□

Cantor II

The next task is to show that the cardinality of $\mathfrak{P}(A)$ is strictly greater than the cardinality of A.

There is a trivial injection from A to $\mathfrak{P}(A)$: $a \mapsto \{a\}$, so $|A| \leq |\mathfrak{P}(A)|$.

So suppose there is a surjection $f : A \to \mathfrak{P}(A)$. Think of element a as a "name" for the set $f(a)$ and define a set

$$B = \{\, a \in A \mid a \notin f(a) \,\} \subseteq A.$$

Since f is surjective, we must have $B = f(b)$ for some $b \in A$. But then $b \in B$ implies $b \notin B$, and conversely; contradiction.

Cantor vs. Russell

Note that Cantor's construction is very similar to Russell's paradox, surprisingly Cantor never made the transition.

$$R = \{\, x \mid x \notin x \,\}$$

$$B = \{\, a \in A \mid a \notin f(a) \,\}$$

The existence of R is self-contradictory, but can be proven from the axioms developed by G. Frege in the late 1800s. We all hope that no such set can be constructed in Zermelo-Fraenkel set theory.

But there is nothing wrong with B, it just shows that f cannot be surjective. The existence of B is easy to prove in Zermelo-Fraenkel set theory; it's just comprehension.

Diagonalization, the Idea

Cantor was interested solely in set theory (actually: to use set theory to explain classical mathematics).

However, the idea to construct a provably new object by starting with a list of objects, and then modifying object x_i in position i carries over to other areas.

In particular the infamous undecidability of the **Halting Problem** uses the exact same idea.

Similar methods are also hugely important in complexity theory.

Quoi?

This is really amazing: diagonalization in set theory deals with cardinality, a concept that appears to have absolutely nothing to do with computability. In fact, when diagonalization was introduced by Cantor in 1891, no concept of computability even existed; it would take another 40 years for that development.

And yet, diagonalization is a critical tool in computability and complexity.