MFCS

Relations

Klaus Sutner
Carnegie Mellon University
Fall 2022

Beyond Functions

So far, we have introduced

sets   functions   induction/rectypes   naturals

as our basic mathematical objects.

This already covers quite a bit of ground, but there is one glaring omission: general relations between objects.

We can think of arbitrary relations as a generalization of functions: we give up on totality and single-valuedness and just focus on pairs (a, b) of objects.

Typical Examples

- divisibility relation on natural numbers,
- less-than relation on integers,
- greater-than relation on rational numbers,
- the “attends course” relation for students and courses,
- the “is prerequisite” relation for courses,
- the “is a parent of” relation for humans,
- the “terminates on input and produces output” relation for programs and inputs and outputs.

The number of things being related to each other could be any $k \geq 0$, but the most important case is $k = 2$; we will focus on this case. The definition is essentially the same as for functions.

Definition

A binary relation ρ from A to B is a triple $\rho = \langle R, A, B \rangle$ where $R \subseteq A \times B$.
A is the domain, B is the codomain, and R is the graph of ρ.

Just as for functions we write $\rho : A \to B$.

For binary relations it is often convenient to use infix notation $x \; \rho \; y$ rather than prefix notation $\rho(x, y)$ or the set-theoretic $(x, y) \in \rho$.

Endorelations

An important special case arises when the domain and codomain are the same.

Definition

An endorelation or a relation on A or a homogeneous relation is a binary relation with the same domain and codomain A. The domain is also called the carrier set or underlying set of the relation.

As a prime example for an endorelation consider equality $=$. Having the same domain and codomain is really essential here.

The concept of a relation is no doubt very useful, but does it really expand our vocabulary? Is it indispensable?

Logically, no. The reason is that we can express relations as Boolean-valued functions. Given a relation $\rho : A \to B$, consider the characteristic function

$$\chi_\rho : A \times B \to \mathbf{2} \qquad \chi_\rho(x, y) = [x \; \rho \; y]$$

Here we are using Knuth's bracket convention again: $[x \; \rho \; y] = 1$ if $x \; \rho \; y$ holds, and 0 otherwise.

We could replace ρ everywhere by χ_ρ without any loss of expressiveness.

Bullshit

Logically that is correct, but for humans, psychology and cognitive harmony is just as important as pure logic. Writing

$$\chi_<(2,5) = 1 \qquad \text{instead of} \qquad 2 < 5$$

leads to madness.

It took centuries to develop the notation system used in math today, we better not mess it up.

Do not waste brain cycles.

As long as domains and codomains are reasonably small, one can often get mileage out of drawing little pictures. Some properties of relations become much clearer in the picture than in a more abstract set-theoretic discussion.

For example, let's fix the carrier set $[10]$ and define the endorelation ρ as follows:

$$x \; \rho \; y \quad :\Leftrightarrow \quad (y+1) \mid x \; \vee \; (x+5) \mid y$$

There, done.

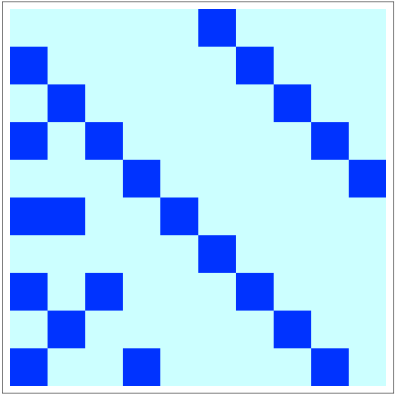
By definition, we can think of the relation as being given by a list of pairs. Voila:

$$(1,6),(2,1),(2,7),(3,2),(3,8),(4,1),(4,3),(4,9),(5,4),(5,10),(6,1),$$
$$(6,2),(6,5),(7,6),(8,1),(8,3),(8,7),(9,2),(9,8),(10,1),(10,4),(10,9)$$

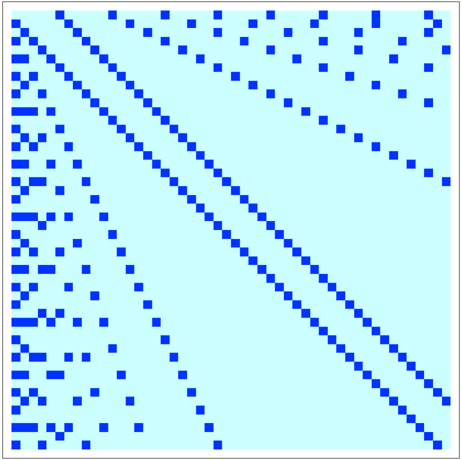
This is next to being useless, a human can barely read this.

A Square Plot

Since we are dealing with carrier set $[10]$, we can plot the characteristic function as a pretty 10×10 matrix (the top left corner is $(1,1)$):



Domain $[50]$



$$x \; \rho \; y \quad :\Leftrightarrow \quad (y{+}1) \mid x \;\vee\; (x{+}5) \mid y$$

Another often excellent way to visualize relations is to think of them as diretec graphs.

Definition (Directed Graphs)

A directed graph (or digraph) is a structure $G = \langle V, E \rangle$ where

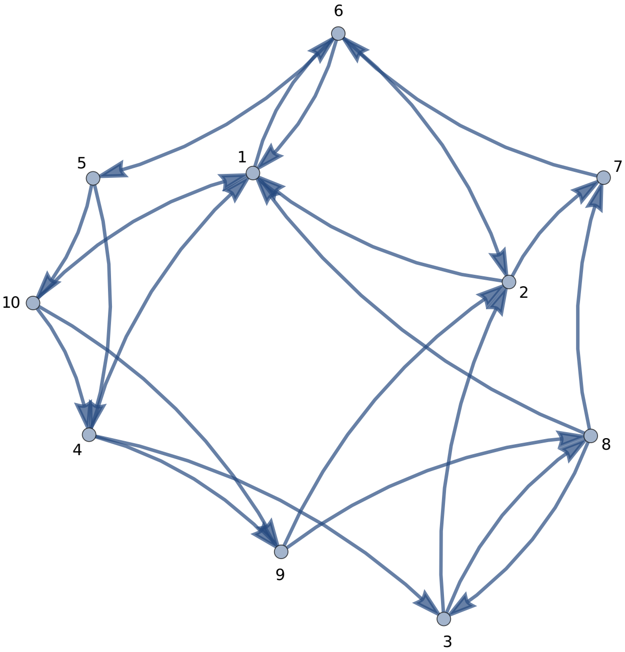
- V is a set of vertices (or nodes, points)
- $E \subseteq V \times V$ is a set of edges (or arcs, lines)

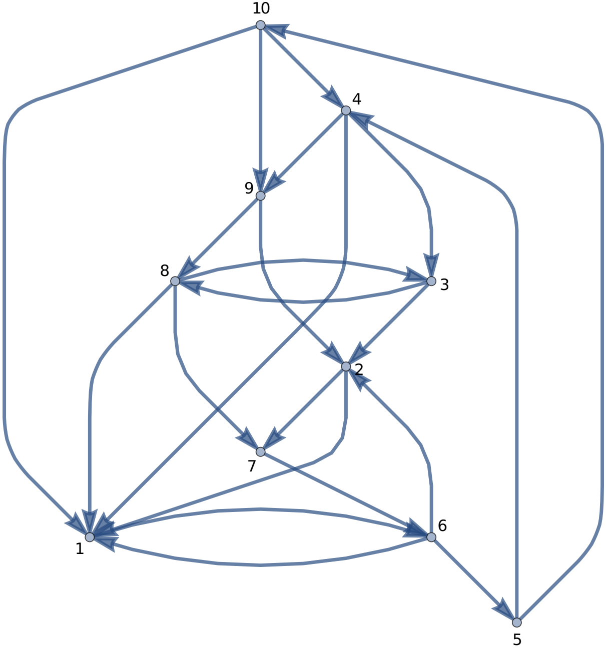
Given an endorelation ρ on A, we can build a corresponding digraph with vertex set A and edges $x \to y$ iff $x \; \rho \; y$.

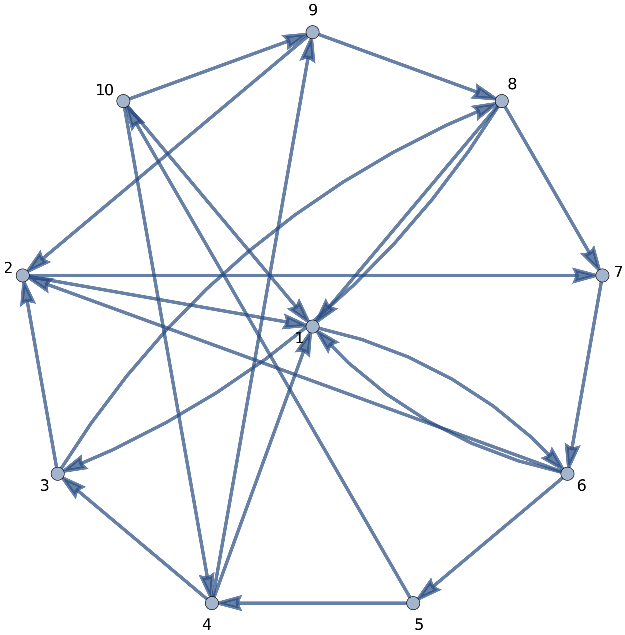
One might point out that digraphs and binary endorelations are "really" exactly the same thing, it's nothing more than a change in terminology. Logically they are utterly the same. Why bother?

Again, it's psychology that matters. Graph theory is an independent field of study, and historically has produced different results than the study of binary relations. Thinking about a problem just the right way is crucial in math and TCS.

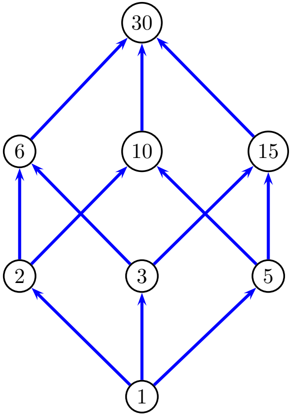
Another more mercenary difference is that a **lot** of work has gone into graph rendering algorithms, sometimes they do a fantastic job of laying out a graph. Use any help you can get.


6
5
1
7
10
2
4
8
9
3

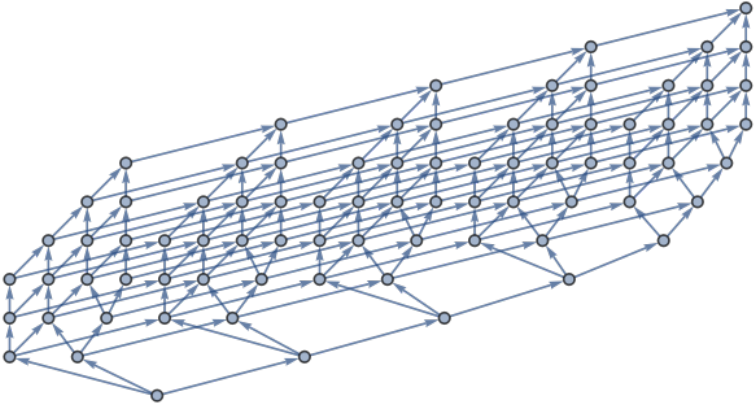

10
4
9
8
3
2
7
6
1
5


9
10
8
2
7
1
3
6
4
5

Our example relation is slightly chaotic, so the pictures may not be overwhelming. For simpler relations one can do better, in particular if the layout is constructed painstakingly by hand.



Here are the divisors of 148176.



Exercise

Figure out what the vertices are as much as possible.

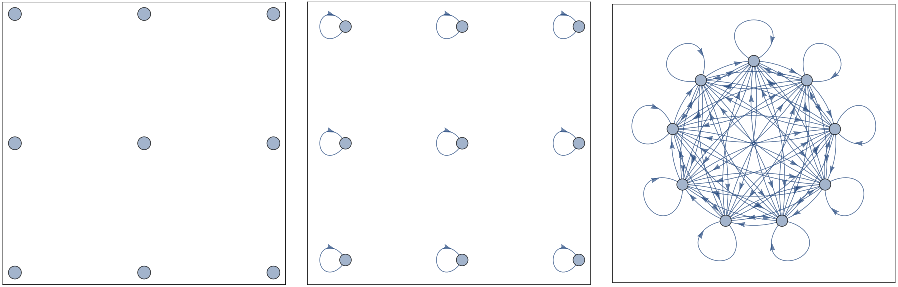
Here are a few particularly important endorelations.

Definition

- $I_A = \{ (x,x) \mid x \in A \}$, the identity or diagonal relation.
- $U_A = A \times A$, the universal relation.
- $\emptyset_A = \emptyset$, the empty relation.

Exercise

What would the pictures for these relations look like in the matrix and graph plots from above?



Experience shows that there is a fairly short list of basic properties of relations that can be combined to define important special types of relations (orders, equivalence relations). Let ρ be a binary relation on A.

Definition

property	$\forall x, y, z \in A$
reflexive	$x \; \rho \; x$
irreflexive	$\neg(x \; \rho \; x)$
symmetric	$x \; \rho \; y \Rightarrow y \; \rho \; x$
asymmetric	$\neg(x \; \rho \; y \wedge y \; \rho \; x)$
antisymmetric	$x \; \rho \; y \wedge y \; \rho \; x \Rightarrow x = y$
transitive	$x \; \rho \; y \wedge y \; \rho \; z \Rightarrow x \; \rho \; z$

Comments

Irreflexive is **not** the negation of reflexive.

The distinction between asymmetric and antisymmetric makes sense: $\neg(x < y \wedge y < x)$ but $x \leq y \wedge y \leq x \Rightarrow x = y$.

Transitive means that whenever we have a chain of related elements

$$x_0 \;\rho\; x_1 \;\rho\; x_2 \;\rho\; \ldots \;\rho\; x_{n-1} \;\rho\; x_n$$

we can conclude that $x_0 \;\rho\; x_n$. Algorithmically this is far and away the most important property.

Examples

- equal-to, subset-of and divides are reflexive
- less-than, proper-subset-of and parent-of are irreflexive
- equal-to and relatively-prime are symmetric
- less-than and parent-of are asymmetric
- less-than-or-equal, subset-of and divides are antisymmetric
- equal-to, subset-of, divides and ancestor-of are transitive
- parent-of and relatively-prime are not transitive

Note that there are corresponding decision problems: how do we check whether are relation is, say, transitive?

At least for finite carrier sets one would like to have efficient algorithms.

For endorelations ρ and σ on some set A we can use propositional logic to construct new relations.

$$
\begin{aligned}
x(\neg\rho)y &\iff \neg(x\ \rho\ y) \\
x(\rho \wedge \sigma)y &\iff x\ \rho\ y \wedge x\ \sigma\ y \\
x(\rho \vee \sigma)y &\iff x\ \rho\ y \vee x\ \sigma\ y \\
x(\rho \oplus \sigma)y &\iff x\ \rho\ y \oplus x\ \sigma\ y
\end{aligned}
$$

$\neg\rho$ is often written $\bar{\rho}$.

These operations translated directly into the corresponding set operations on the graphs of the relations, nothing really new here.

What would the matrix and digraph pictures look like?

Definition

Let $\rho : A \to B$ be a relation. The converse of ρ is a relation from B to A defined by

$$x \, \rho^c \, y \iff y \, \rho \, x.$$

Thus the domain/codomain of ρ^c is the codomain/domain of ρ.

Unlike with functions there is no problem flipping a relation.

Note that the converse operation is an involution: $(\rho^c)^c = \rho$.

Clearly, ρ is symmetric iff $\rho^c = \rho$.

Mental Health Warning

Remember the rant about how the standard definition of functional composition, $f \circ g$, is backwards?

Well, we are now going to give the standard definition of relational composition, and the chickens will come home to roost.

☠ ☠ ☠ ☠ ☠ ☠ ☠

Definition

Suppose $\rho : A \to B$ and $\sigma : B \to C$ are relations. The (relational) composition of ρ and σ is defined to be the relation $\tau : A \to C$ where

$$x \;\tau\; y \iff \exists\, z \in B\,(x \;\rho\; z \wedge z \;\sigma\; y).$$

The intermediate element $z \in B$ is a witness for $x \;\tau\; y$.
In symbols:

$$\tau = \rho \bullet \sigma$$

Right, this is exactly the opposite direction of $f \circ g$.

WTF?

Historically, functional composition came first. Since Euler, we tend to write function application on the left, as in $f(a)$, so the definition of $f \circ g$ makes some amount of sense.

But when the relational calculus was invented in the 19th century, somehow people felt that composition should be written in the natural, diagrammatic order.

If relations and functions were living on different planets, this would not be an issue. But functions are a special case of a relations, so we have a rather terrible clash of notation.

No one ever said math *notation* was logical.

Examples

Let ρ be the "parent of" relation. Then $\rho \bullet \rho$ is "grandparent of."

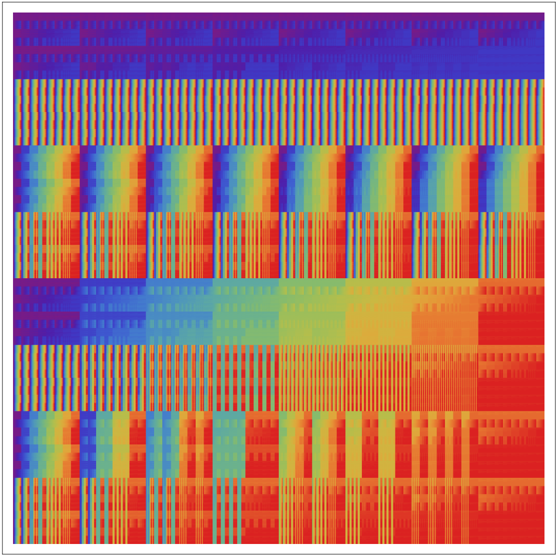
Let $<$ be the natural order on $\mathbb{N}$. Then $x(<\bullet<)y \iff y \geq x + 2$.

How about $<$ the natural order on $\mathbb{Q}$? How about $\mathbb{R}$?

In the plane $\mathbb{R}^2$, let ρ be "directly North of" and σ "directly East of." Then $\rho \bullet \sigma$ is "North-East of."

Let ρ be "x is a prime factor of y" on $\mathbb{N}_+$. What is $\rho \bullet \rho$?

Soothing Picture (Composing All Relations on [3])



Proposition

Show that $(\rho \bullet \sigma)^c = \sigma^c \bullet \rho^c$.

Proof.

$$\begin{aligned} x\,(\rho \bullet \sigma)^c\, y &\Leftrightarrow y\,(\rho \bullet \sigma)\,x \\ &\Leftrightarrow \exists\, z\,\big(y\rho z \wedge z\sigma x\big) \\ &\Leftrightarrow \exists\, z\,\big(z\rho^c y \wedge x\sigma^c z\big) \\ &\Leftrightarrow \exists\, z\,\big(x\sigma^c z \wedge z\rho^c y\big) \\ &\Leftrightarrow x\,(\sigma^c \bullet \rho^c)\,y \end{aligned}$$

□