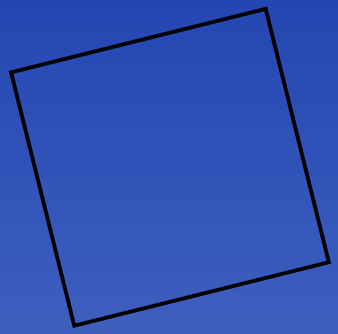
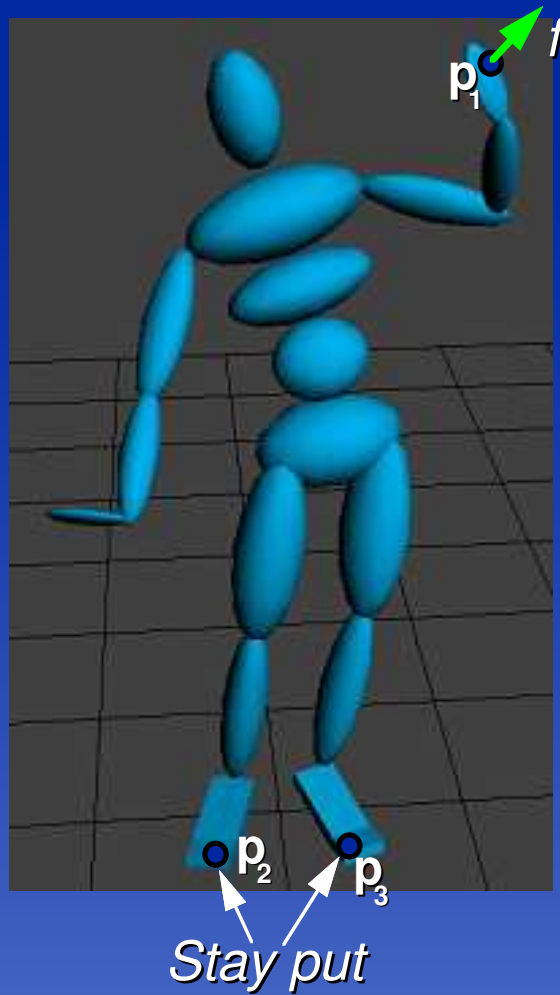
Inverse Kinematics



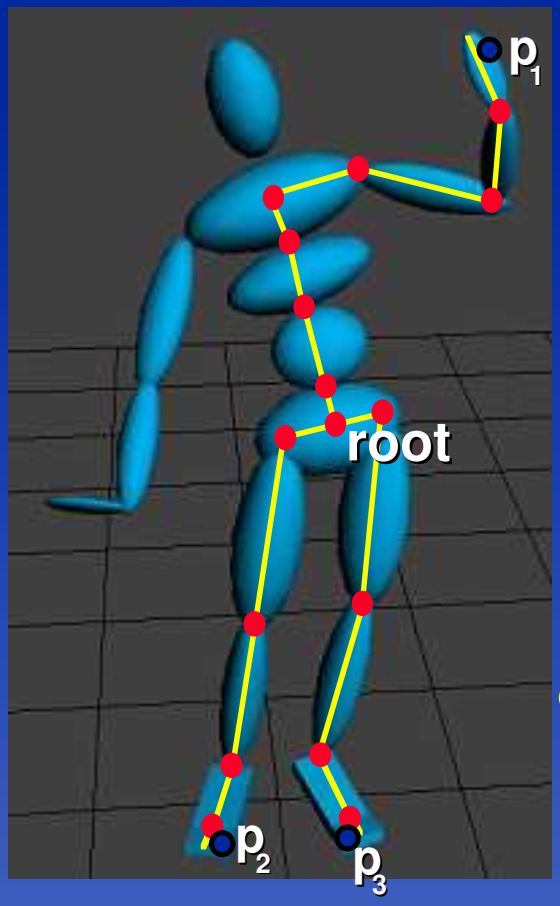
Define the Problem



Goals

- Drag in realtime with mouse
- Maintain multiple constraints
- Figure responds in “reasonable” ways

... Then Name the System!



Use standard transformation hierarchy

$$p_1 = T(x, y, z)R(\theta, \phi, \varphi)TR(\gamma)\,..\,TR(\sigma)\,p_b$$

$$q(t) =$$

$$p_1 = C(q(t))$$

Idea #1

Let

$p_m(t) =$ 3D mouse positioner at time t

$C(q(t)) =$ position of hand at time t

... then solve for q such that

$$C(q(t)) = p_m(t)$$

Unfortunately...

$C(q(t)) = p_m(t)$ contains rotations, therefore nonlinear, and very difficult to solve

Can be solved for:

- articulated chains

Has real problems with:

- branching hierarchies
- multiple constraints
- convincing motion

Idea #2

Let's differentiate!

$$\frac{\partial}{\partial t}\left(p_m(t) = C(q(t))\right)$$

$$\dot{p}_m = \frac{\partial C}{\partial q}\,\dot{q}$$

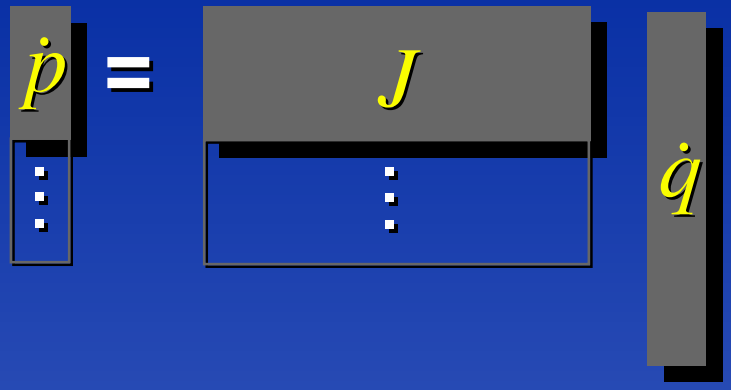
desired – actual

Jacobian J

state space velocity

A Diffy Q by any other name...

$$\dot{p} = J\dot{q}$$



Underdetermined linear system
Can solve for $\dot{q}$

Iterate during interaction using:

$$q(t+\Delta t) = q(t) + \Delta t\, \dot{q}(t)$$

Are we there yet?

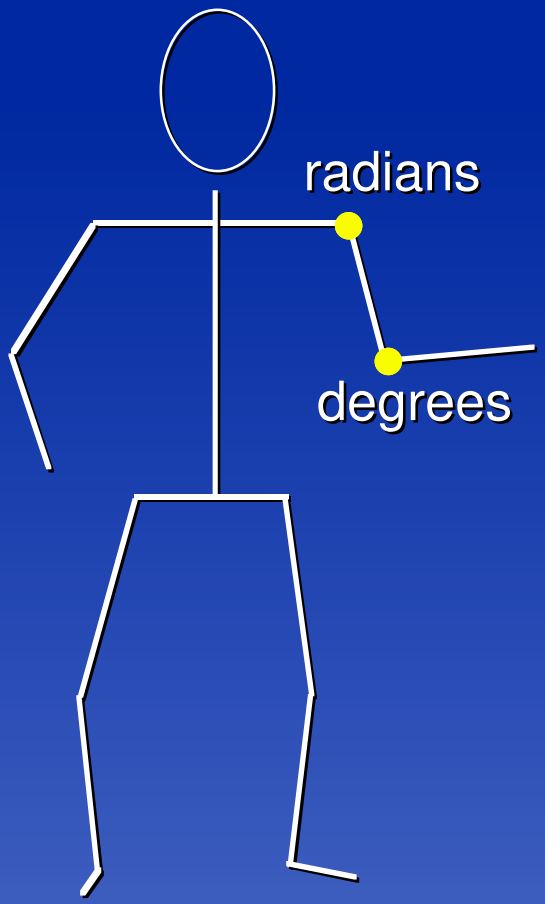
Alas, no.

We got *a* solution for $\dot{q}$

but it may not be a *reasonable* one.

Why not?

Scaling



When there's a choice,
least squares solver always
moves shoulder before elbow

Measuring both angles in radians
still leaves figure stiff-armed

What can we do about this?

Let's get physical!

Aristotlean physics:

$$f = Mv$$

Why not $f = Ma$ *?*

$$f_G + f_C = M\dot{q}$$

Generalized Force

Constraint Force

Mass Matrix

Blend on High for 2 minutes...

Recall

Constraints

$$p = C(q)$$

Solve

$$\dot{p} = J\dot{q}$$

Where

$$J = \frac{\partial C}{\partial q}$$

Define: $W = M^{-1}$

$$f_G + f_C = M\dot{q}$$

Lagrange Multipliers

$$f_G + J^T\lambda = M\dot{q}$$

substitute

get

$$\dot{q} = W\left(f_G + J^T\lambda\right)$$

$$\dot{p} = JWf_G + JWJ^T\lambda$$

Recalling Lagrange Multipliers

Then...

Assumption of *passive* constraints stipulated that f_C point along constraint gradient $\frac{\partial C}{\partial x} = N$

Now...

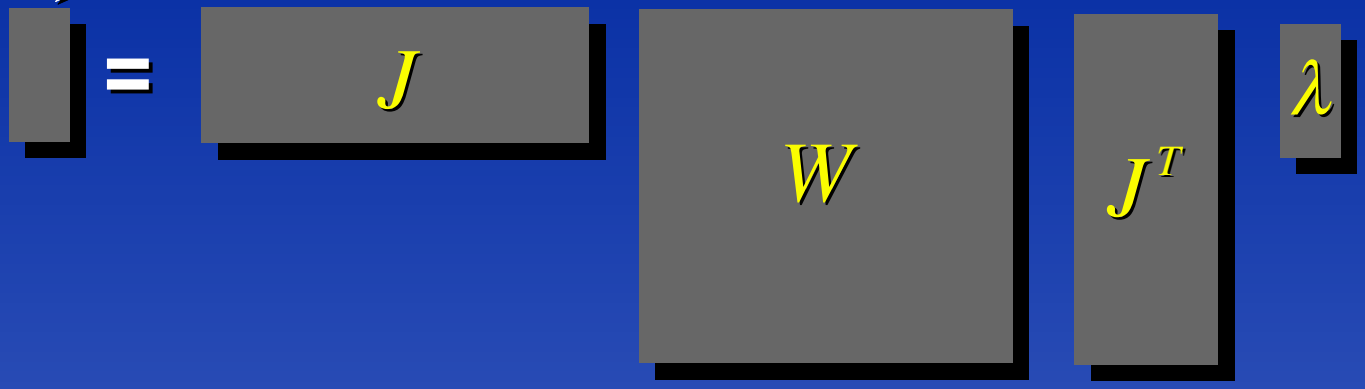
Constraint gradients: $\frac{\partial C}{\partial q} = J$

f_C is a *linear combination* of constraint gradients:

$$f_C = J^T \lambda$$

and wind up with:

$$\dot{p} - JWf_G = JWJ^T\lambda$$



Can solve for λ, plug into formula for $\dot{q}$

Have we solved the scaling problem?

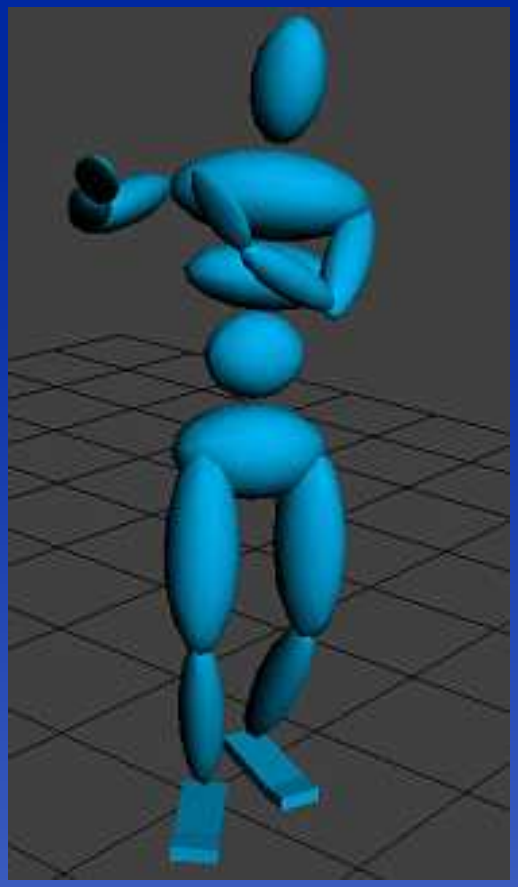
We get a velocity $\dot{q}$ that obeys

the "physical law" $f = Mv$ ...

M determines how the system responds to applied forces.

So it all depends on what M is.

How well can we do?

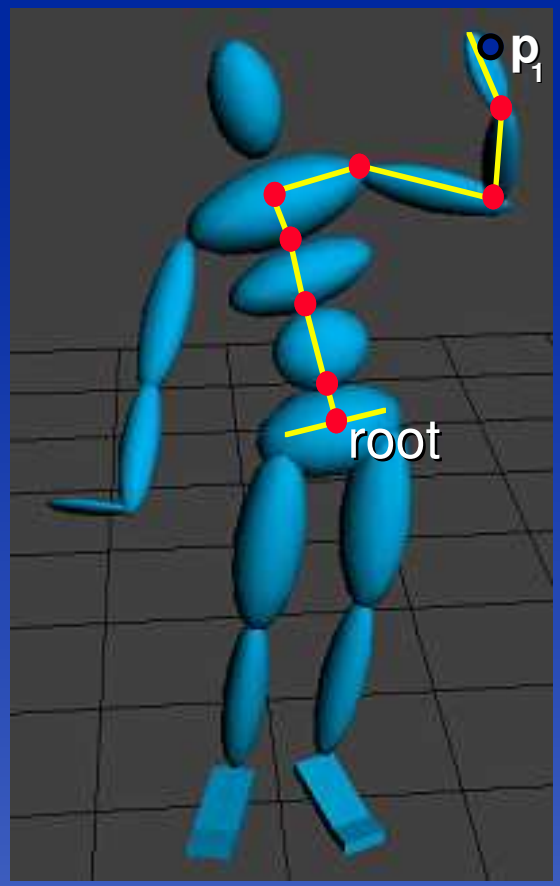


If approximate figure by rigid, uniform density, simple shapes...

Can derive and compute an $M(q)$ from minimizing kinetic energy of articulated figure

Minimum energy consumption is good!

A word about J



p_1 only depends on ancestors in hierarchy

Therefore, J is sparse. Exact pattern depends on ordering of q.

Can compute all derivatives efficiently in recursive tree traversal

What's in the online notes

- More detailed derivations
- Formula for computing Mass Matrix
- Pseudocode for tree traversal
- Bibliography

That's Just the Beginning...

- One sided constraints $\rightarrow$ joint limits
- Hierarchical constraints
- Spring "muscles"
- Build complex constraint functions for higher level behavior