

What is the Right Solution Concept for No-Limit Poker?

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Abstract

We analyze one of the simplest no-limit poker games, which has been previously studied. We show that the game has infinitely many Nash equilibria, all of which are extensive-form perfect, extensive-form proper, and normal-form perfect, but only one of which is normal-form proper; however, we argue that one of the equilibria is more intuitively compelling than the others, which differs from the normal-form proper equilibrium. This suggests that a new refinement concept is needed to more appropriately model no-limit poker.

In the *no-limit clairvoyance game* [Ankenman and Chen, 2006], player 2 is given no private cards, and player 1 is given a single card drawn from a distribution that is half winning hands and half losing hands. Both players have stacks of size n , and they both ante \$0.50. P1 is allowed to bet any amount $x \in [0, n]$. Then P2 is allowed to call or fold (but not raise). The game is small enough that its solution can be computed analytically [Ankenman and Chen, 2006]:

- P1 bets n with prob. 1 with a winning hand.
- P1 bets n with prob. $\frac{n}{1+n}$ with a losing hand (and checks otherwise).
- For all $x \in [0, n]$, P2 calls a bet of size x with prob. $\frac{1}{1+x}$.

It was shown by Ankenman and Chen [Ankenman and Chen, 2006] that this strategy profile constitutes a Nash equilibrium. (They also show that these frequencies are optimal in many other poker variants.) Here is a sketch of that argument.

Proposition 1. *The strategy profile presented above is a Nash equilibrium of the clairvoyance game.*

Proof. Player 2 must call a bet of size x with probability $\frac{1}{1+x}$ in order to make player 1 indifferent between betting x and checking with a losing hand. For a given x , player 1 must bluff $\frac{x}{1+x}$ as often as he value bets for player 2 to be indifferent between calling and folding. Given these values, the expected payoff to player 1 of betting size x is $v(x) = \frac{x}{2(1+x)}$. This function is monotonically increasing, and therefore player 1 will maximize his payoff with $x = n$. \square

Despite the simplicity of this game, the solution has been used in order to interpret bet sizes for the opponent that fall outside an abstracted game model by many of the strongest agents for full no-limit Texas hold 'em [Ganzfried and Sandholm, 2013; Ganzfried, 2015; Jackson, 2013].

It turns out that player 2 does not need to call a bet of size $x \neq n$ with exact probability $\frac{1}{1+x}$: he need only not call with such an extreme probability that player 1 has an incentive to change his bet size from n to x (with either a winning or losing hand). In particular, it can be shown that player 2 need only call a bet of size x with any probability (which can be different for different values of x) in the interval $\left[\frac{1}{1+x}, \min\left\{\frac{n}{x(1+n)}, 1\right\}\right]$ in order to remain in equilibrium. Only the initial equilibrium is reasonable, however, in the sense that we would expect a rational player 2 to maintain the calling frequency $\frac{1}{1+x}$ for all x so that he plays a properly-balanced strategy in case player 1 happens to bet x .

To provide further intuition, if the opponent bets x as opposed to the “optimal” size n that he should bet in equilibrium, then a reasonable deduction is that he isn't even aware that n would have been the optimal size, and believes that x is optimal. Therefore, it would make sense to play a strategy that is an equilibrium in the game where the opponent is restricted to only betting x (or to betting 0, i.e., checking). Doing so would correspond to the particular equilibrium that I have prescribed. The other equilibria pay more heed to the concern that the opponent could exploit us by deviating to bet x instead of n ; but in fact, I argue that we need not be as concerned about this possibility, since a rational opponent who knew to bet n would not be betting x .

Interestingly, the equilibrium I have singled out does not coincide to any traditional Nash equilibrium refinements. One popular refinement is the normal-form proper equilibrium (NFPE). Based on personal communication with Troels Sørensen and Jiří Čermák, we have computed the unique NFPE when player 1 is allowed to bet 0, 1, or 2. It differs from the equilibrium I propose; in the NFPE P2 calls vs. a bet of 1 with probability $\frac{5}{9}$, while in the one I prescribe above he calls with probability $\frac{1}{2}$. All of the (infinitely many) equilibria for this game satisfy each of several other popular refinement concepts besides NFPE (extensive-form trembling-hand perfect equilibrium, extensive-form proper equilibrium, and normal-form trembling-hand perfect equilibrium).

References

- [Ankenman and Chen, 2006] Jerrod Ankenman and Bill Chen. *The Mathematics of Poker*. ConJelCo LLC, 2006.
- [Ganzfried and Sandholm, 2013] Sam Ganzfried and Tuomas Sandholm. Action translation in extensive-form games with large action spaces: Axioms, paradoxes, and the pseudo-harmonic mapping. In *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI)*, 2013.
- [Ganzfried, 2015] Sam Ganzfried. Reflections on the first man vs. machine no-limit Texas hold 'em competition. *SIGecom Exchanges*, 4.2, 2015.
- [Jackson, 2013] Eric Jackson. Slumbot NL: Solving large games with counterfactual regret minimization using sampling and distributed processing. In *AAAI Workshop on Computer Poker and Incomplete Information*, 2013.