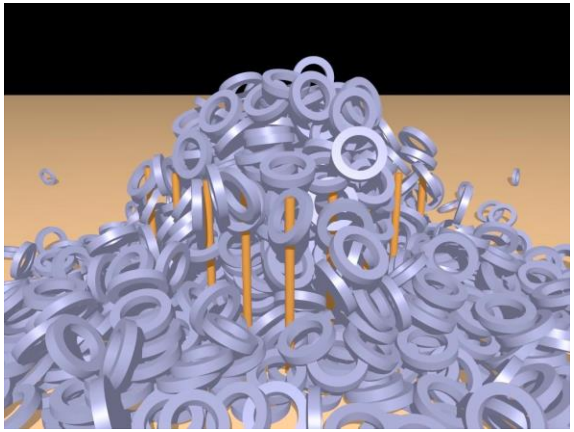
# Rigid Body Dynamics



# Project Proposal Presentation

- Plan for 20 min presentation if solo, 30 min if you are working in a team
  - Problem and motivation
  - Background
    - Choose 1-2 related research papers, and describe in detail – talk to us if unsure which papers to look at
  - Proposed technical method
  - Expected results (low and high bar)

# Project Proposal Presentation Schedule

March 29: Ziqi Guo + Jun Huo, Ye Yuan + Shuang Su

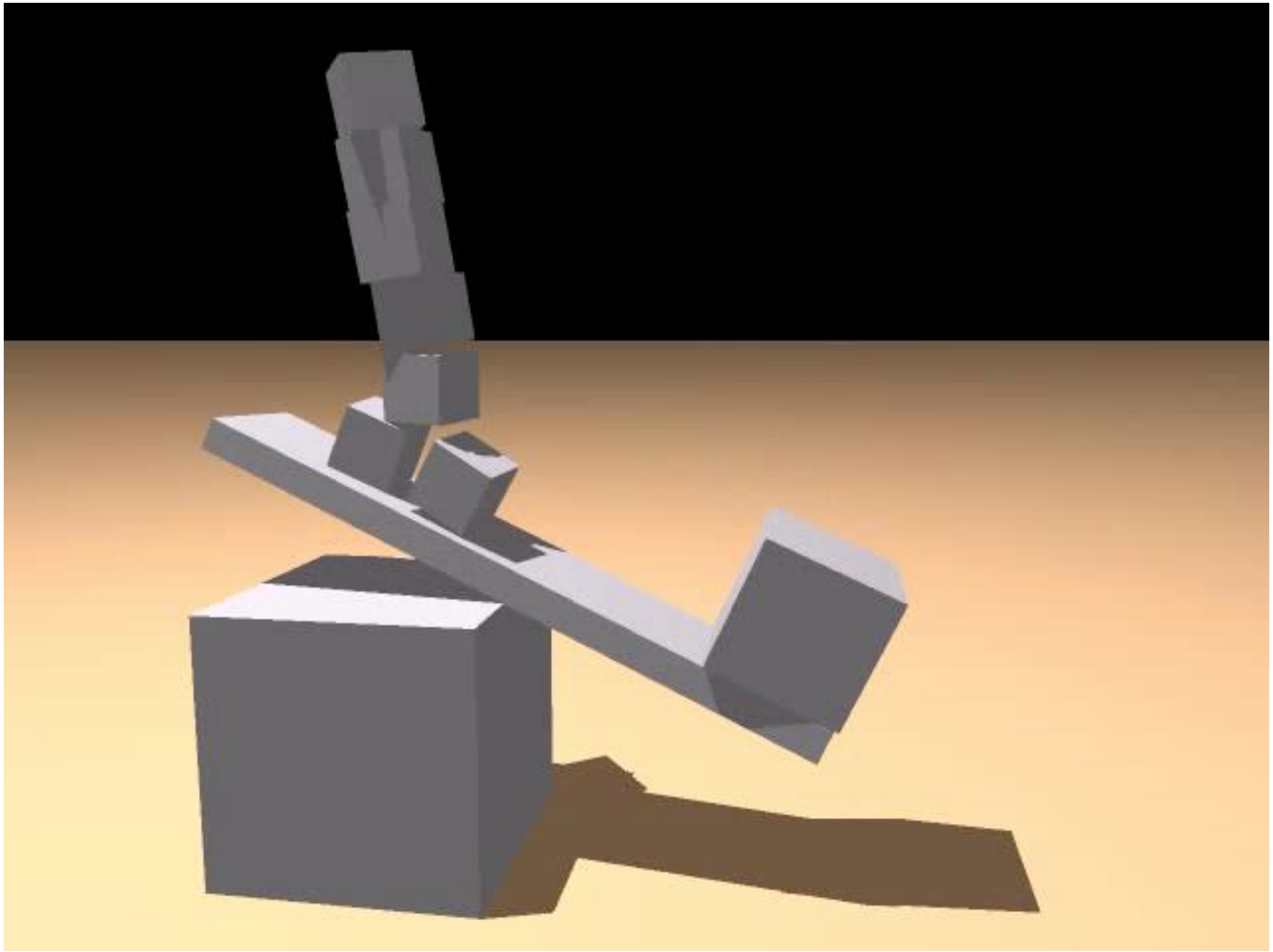
March 31: Che-Yuan Liang, Frances Tso, Blair Emanuel

April 5: Nick Sharp, Jordan Tick, Michael Rivera

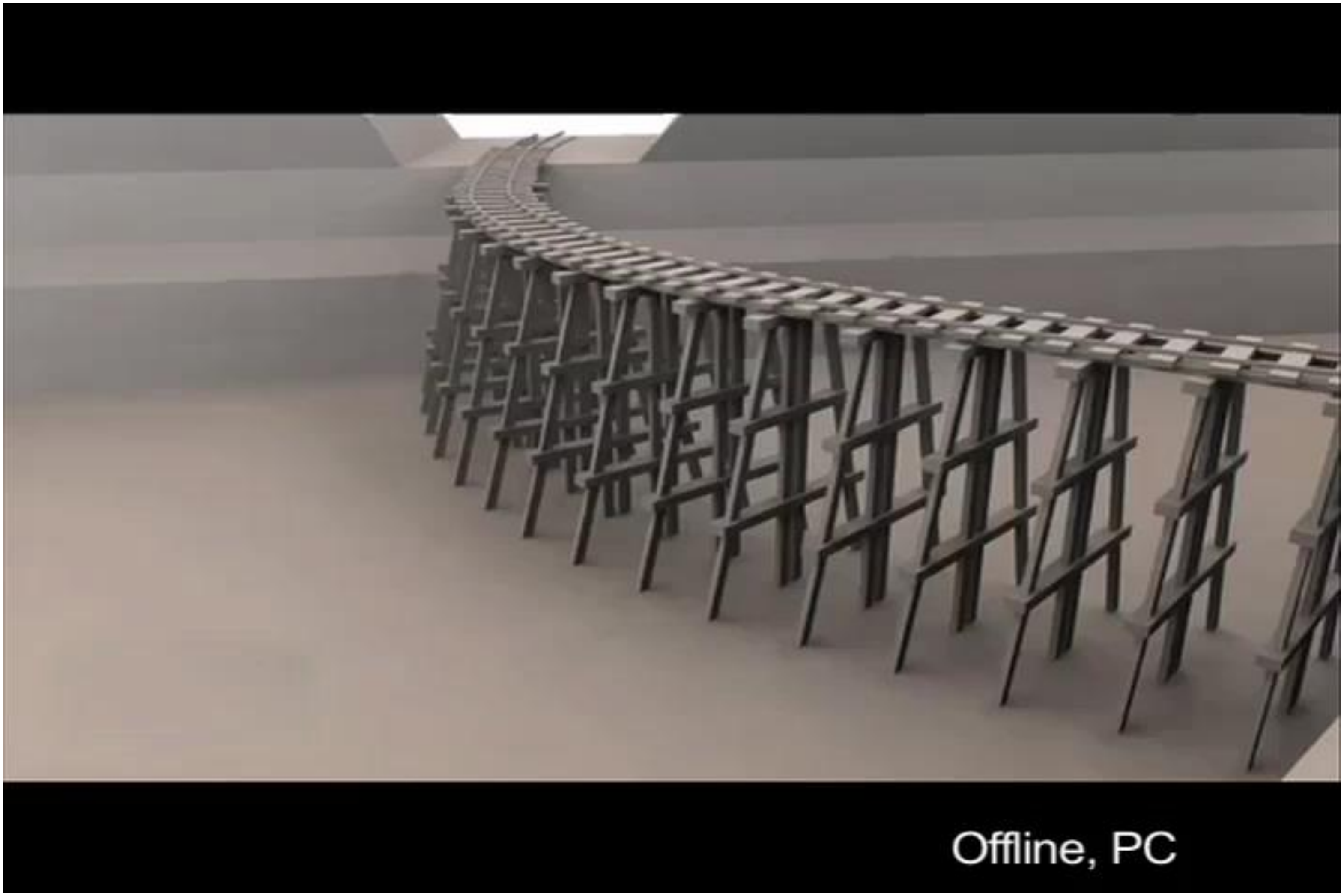
April 7: James Bern, Vidya Narayanan, Shih-Tsui Kuo


FANDANGO
MOVIECLIPS

# Rigid Bodies

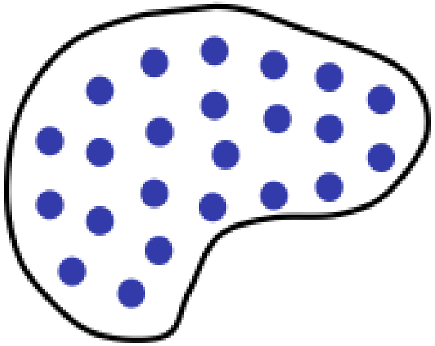


# Rigid Bodies



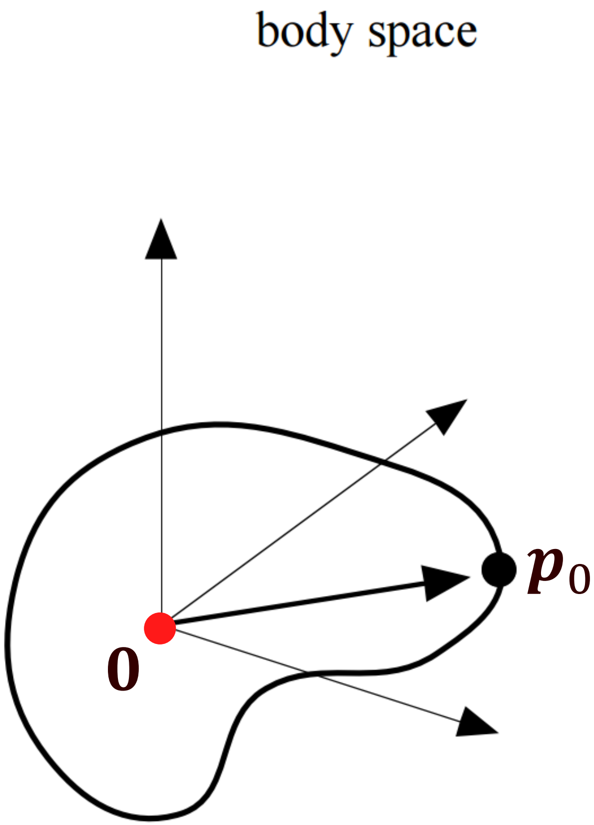

Eric G. Parker and James F. O'Brien. "**Real-Time Deformation and Fracture in a Game Environment**". Symposium on Computer Animation, 2009.

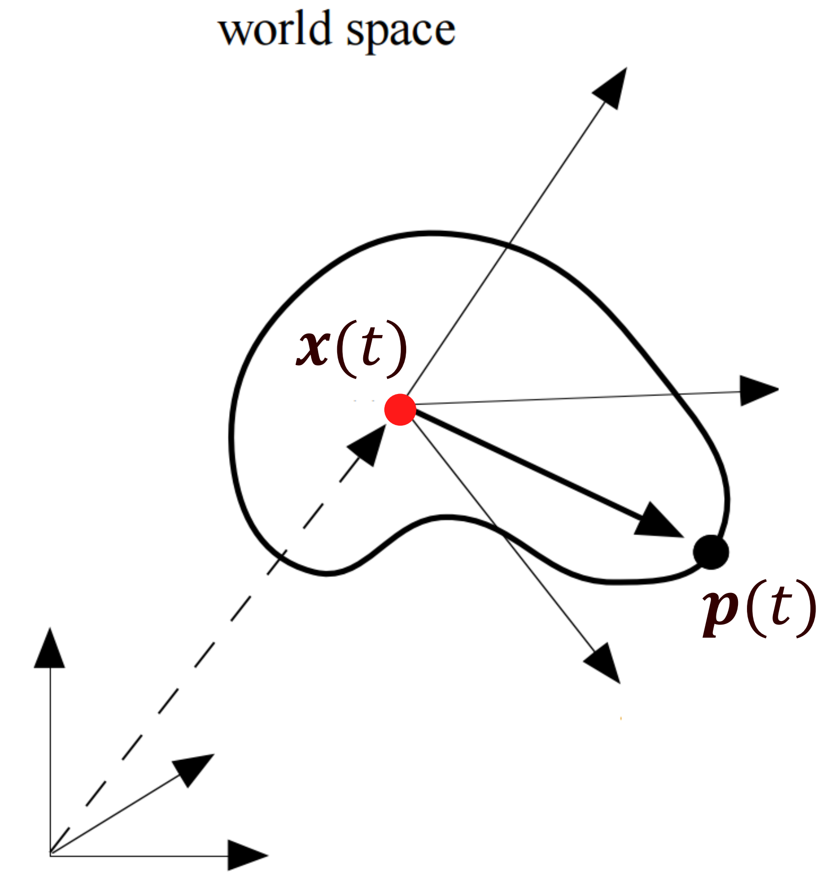
# A rigid body



- Collection of particles
- Distance between any two particles is always constant
- What types of motions preserve these constraints?
  - Translation, rotation

# Rigid Body Parameterization (reduced coordinates)



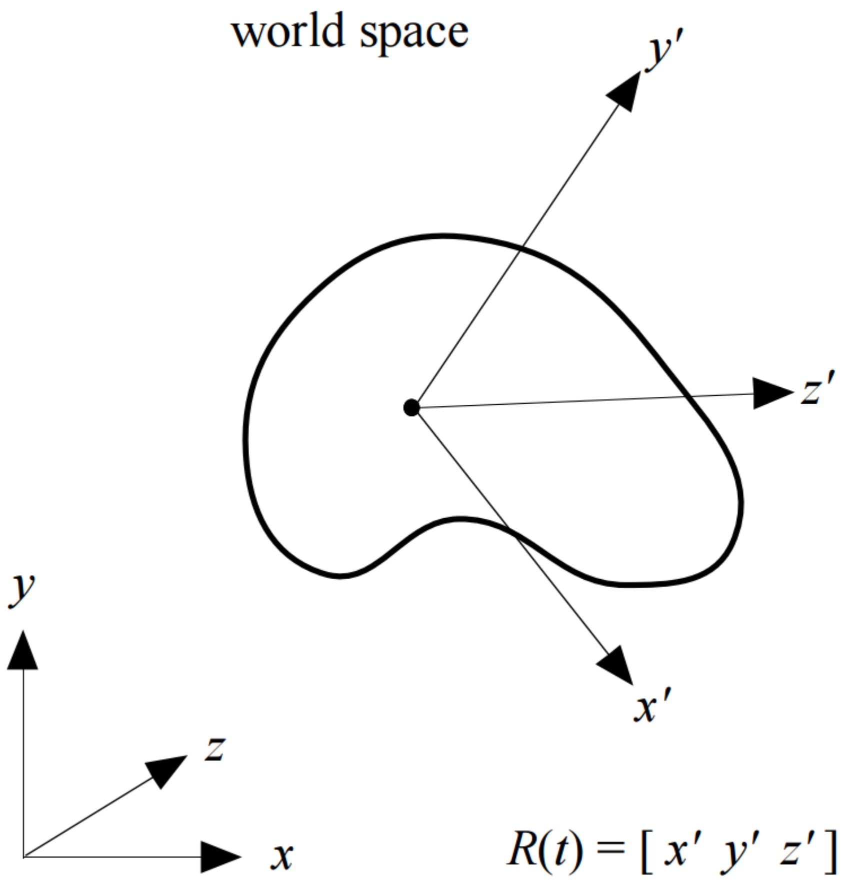

$$\mathbf{p}(t) = \mathbf{x}(t) + \mathbf{R}(t)\mathbf{p}_0$$

$x(t) \in R^3$ *position*

$R(t) \in R^{3x3}$ *orientation*

# Body Orientation

- Rotates vectors from body to world coordinates
  - Columns of $\boldsymbol{R}(t)$ encode world coordinates of body axes



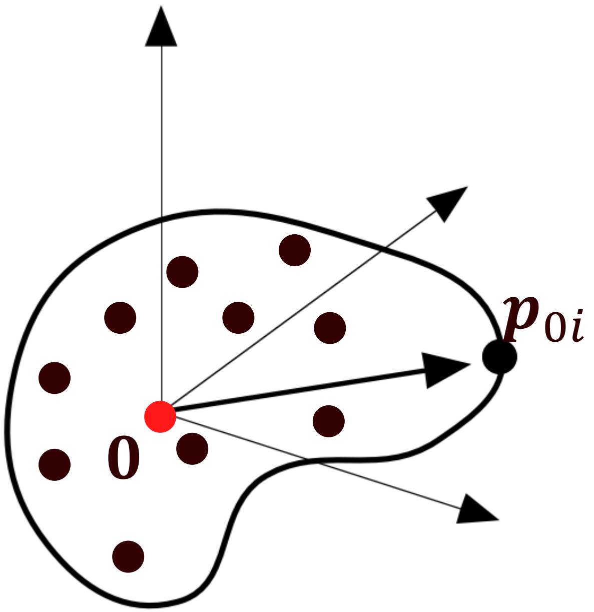
# Center of Mass (COM)

- Geometric center of the body
  - (0,0,0) in body coordinates
  - $\boldsymbol{x}(t)$ in world coordinates

Geometric center: $\sum m_i \boldsymbol{p}_{0i} \coloneqq (0,0,0)$

$$M = \sum m_i$$

$$\boldsymbol{COM}: \frac{\sum m_i \boldsymbol{p}_i(\boldsymbol{t})}{M} = \boldsymbol{x}(\boldsymbol{t})$$



# Body velocities

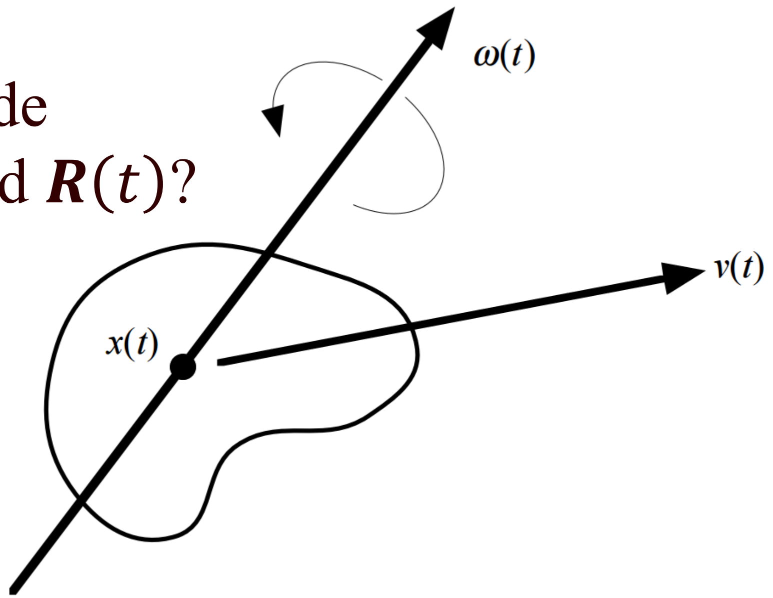
- How do the COM position and orientation change with time?

Linear velocity: $\boldsymbol{v}(t) = \frac{d\boldsymbol{x}(t)}{dt} = \dot{\boldsymbol{x}}(t)$

Angular velocity: $\boldsymbol{\omega}(t) = ?$

$\boldsymbol{\omega}(t)$ encodes spin direction and magnitude

What is the relationship between $\boldsymbol{\omega}(t)$ and $\boldsymbol{R}(t)$?

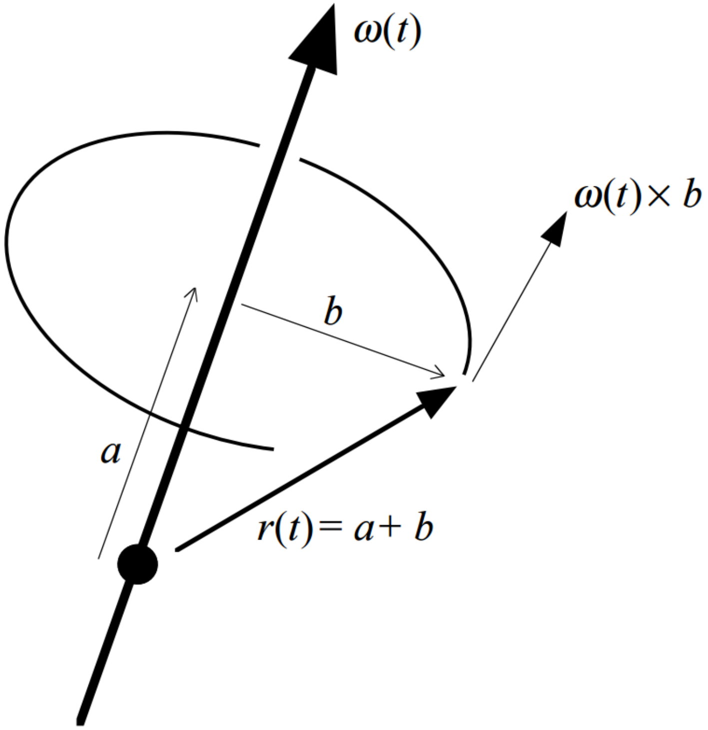


# Angular velocity

- Consider vector $\boldsymbol{r}(t)$. What is $\dot{\boldsymbol{r}}(t)$?

$$\dot{\boldsymbol{r}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{r}(t)$$

- Relation to $\dot{\boldsymbol{R}}(\boldsymbol{t}) = \frac{d\boldsymbol{R}(t)}{dt}$?



# Angular Velocity

- **R** rotates vectors from body to world coords
  - Columns of $\boldsymbol{R}(t)$: world coordinates of body axes
  - Columns of $\dot{\boldsymbol{R}}(t)$: change of body axes world coordinates wrt time

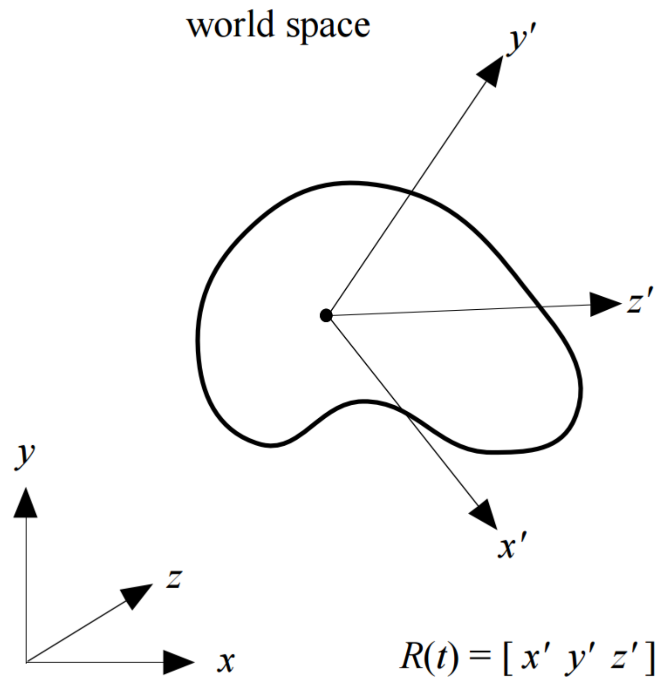
$$\dot{\boldsymbol{x}}'(t) = \boldsymbol{\omega}(t) \times \boldsymbol{x}'(t)$$

$$\dot{\boldsymbol{y}}'(t) = \boldsymbol{\omega}(t) \times \boldsymbol{y}'(t)$$

$$\dot{\boldsymbol{z}}'(t) = \boldsymbol{\omega}(t) \times \boldsymbol{z}'(t)$$

Putting these all together:

$$\dot{\boldsymbol{R}}(t)=\boldsymbol{\omega}(t)_{\times}\boldsymbol{R}(t)$$



# Recall

$$a \times b = \begin{pmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

$$\boldsymbol{a}_\times$$

# Summary

- Kinematics: *how does the body move?*

$$\boldsymbol{p}(t), \boldsymbol{R}(t)$$

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{x}(t)}{dt} = \dot{\boldsymbol{x}}(t)$$

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{\omega}(t)_{\times}\boldsymbol{R}(t)$$

$$\boldsymbol{p}(t) = \boldsymbol{x}(t) + \boldsymbol{R}(t)\boldsymbol{p}_0$$

$$\dot{\boldsymbol{p}}(t) = \dot{\boldsymbol{x}}(t) + \boldsymbol{\omega}(t)_{\times}\boldsymbol{R}(t)\boldsymbol{p}_0$$
$$= \dot{\boldsymbol{x}}(t) + \boldsymbol{\omega}(t)_{\times}(\boldsymbol{p}(t) - \boldsymbol{x}(t))$$

- Dynamics: *what causes this motion?*

# Forces and Torques

- External forces: acting on individual particles

$$\text{Net force on body: } \boldsymbol{F} = \sum \boldsymbol{F}_i$$

- Conservation of linear momentum (Newton's 2nd law):

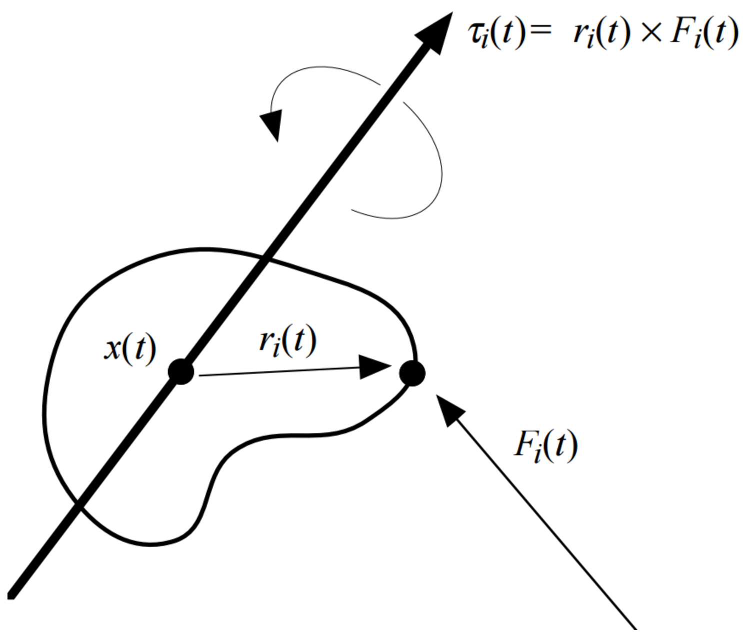
$$\boldsymbol{p} = M\boldsymbol{v}; \dot{\boldsymbol{p}} = \boldsymbol{F}$$

- Analogous concepts for angular motion

# Forces and Torques

- Forces on individual particles generate torques
  - (consequence of constant inter-particle distance)

Net torque on body: $\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \boldsymbol{r_i} \times \boldsymbol{f}_i$



# Forces and Torques

- Forces on individual particles generate torques
  - (consequence of constant inter-particle distance)

$$\text{Net torque on body: } \boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \boldsymbol{r_i} \times \boldsymbol{f}_i$$

- Conservation of angular momentum:

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}; \dot{\boldsymbol{L}} = \boldsymbol{\tau}$$

- What is $\boldsymbol{I}$?
  - Moment of Inertia tensor

# The Inertia Tensor

- Analogous to mass, but for rotational motions
  - quantifies distribution of mass as a 2nd order tensor

$$\boldsymbol{I} = \boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T$$

$$\boldsymbol{I}_b = \sum_{\boldsymbol{i}} m_i(\boldsymbol{p}_{0i}{}^T\boldsymbol{p}_{0i}\mathbf{1} - \boldsymbol{p}_{0i}\boldsymbol{p}_{0i}{}^T)$$

- Body-coords MOI is constant, can be precomputed
- World-cords MOI changes with time!

# Conservation of Linear and Angular Momenta

- Linear Momentum:

$$\boldsymbol{p} = M\boldsymbol{v}; \dot{\boldsymbol{p}} = \boldsymbol{F}; \ \dot{\boldsymbol{v}} = \frac{1}{\boldsymbol{M}}\boldsymbol{F}$$

- Angular Momentum:

$$\boldsymbol{L} = \boldsymbol{I\omega}; \dot{\boldsymbol{L}} = \boldsymbol{\tau}; \ \dot{\boldsymbol{\omega}} = \boldsymbol{I}^{-1}(\boldsymbol{\tau} - \boldsymbol{\omega} \times \boldsymbol{I\omega})$$

- Note: they are decoupled!

# Numerical Integration

- COM Acceleration → Velocity → Position
  - Easy: $\boldsymbol{v}_{t+1} = \boldsymbol{v}_t + \Delta t \dot{\boldsymbol{v}}$; $\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \Delta t \boldsymbol{v}_{t+1}$

- Angular Acceleration → Angular Velocity
  - Easy:$\boldsymbol{\omega}_{t+1} = \boldsymbol{\omega}_t + \Delta t \dot{\boldsymbol{\omega}}$

- Angular Velocity to Rotations?
  - A bit trickier: $\boldsymbol{R}_{t+1} = \boldsymbol{R}_t + \Delta t \dot{\boldsymbol{R}}_{t+1}$?

# Updating Rotations

- $$\begin{aligned}\boldsymbol{R}_{t+1} &= \boldsymbol{R}_t + \Delta t \dot{\boldsymbol{R}}_{t+1} \\ &= \boldsymbol{R}_t + \Delta t \boldsymbol{\omega}_{t+1_\times} \boldsymbol{R}_t = \left(I + \Delta t \boldsymbol{\omega}_{t+1_\times}\right) \boldsymbol{R}_t\end{aligned}$$

- No longer a rotation matrix!
  - Option 1: orthonormalize (Gram–Schmidt)
  - Option 2: explicitly compute rotation $\boldsymbol{R}_{\Delta t}$ due to spinning with angular speed $\boldsymbol{\omega}_{t+1}$ for $\Delta t$ seconds, apply incremental rotations: $\boldsymbol{R}_{t+1} = \boldsymbol{R}_{\Delta t} \boldsymbol{R}_t$
  - NOTE: same concept applies if other rotation parameterizations (i.e. quaternions) are employed

# Quaternions

Generalization of complex numbers: $\boldsymbol{q} = s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = [\boldsymbol{s}, \boldsymbol{v}]$

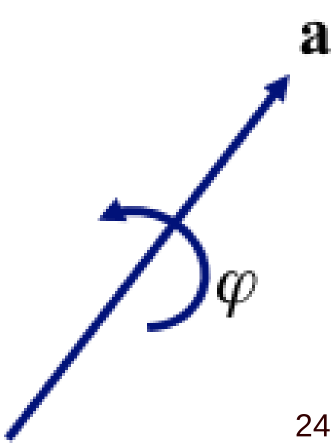
- $\boldsymbol{i} * \boldsymbol{i} = \boldsymbol{j} * \boldsymbol{j} = \boldsymbol{k} * \boldsymbol{k} = \boldsymbol{ijk} = -\boldsymbol{1}; \boldsymbol{i} * \boldsymbol{j} = -\boldsymbol{j} * \boldsymbol{i} = \boldsymbol{k}; \boldsymbol{j} * \boldsymbol{k} = -\boldsymbol{k} * \boldsymbol{j} = \boldsymbol{i}; \boldsymbol{k} * \boldsymbol{i} = -\boldsymbol{i} * \boldsymbol{k} = \boldsymbol{j}$
- $\boldsymbol{q}_1 * \boldsymbol{q}_2 = [s_1 s_2 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2, s_1 \boldsymbol{v}_2 + s_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2]$

Much more compact parameterization for rotations

- $\boldsymbol{q} = [\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2}\boldsymbol{a}]$
- $\mathrm{Rot}(\boldsymbol{q}, \boldsymbol{v}) = \boldsymbol{q} * \boldsymbol{v} * \boldsymbol{q}^{-1}$

Compositions of rotations is given by quaternion multiplication:

$$\boldsymbol{q}_{t+1} = [\cos\frac{\Delta t|\boldsymbol{\omega}_{t+1}|}{2}, \sin\frac{\Delta t|\boldsymbol{\omega}_{t+1}|}{2}\frac{\boldsymbol{\omega}_{t+1}}{|\boldsymbol{\omega}_{t+1}|}]\boldsymbol{q}_t$$




You might encounter:

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{\omega}\boldsymbol{q}; \boldsymbol{q}_{t+1} = \boldsymbol{q}_t + \Delta t\dot{\boldsymbol{q}}_{t+1}$$