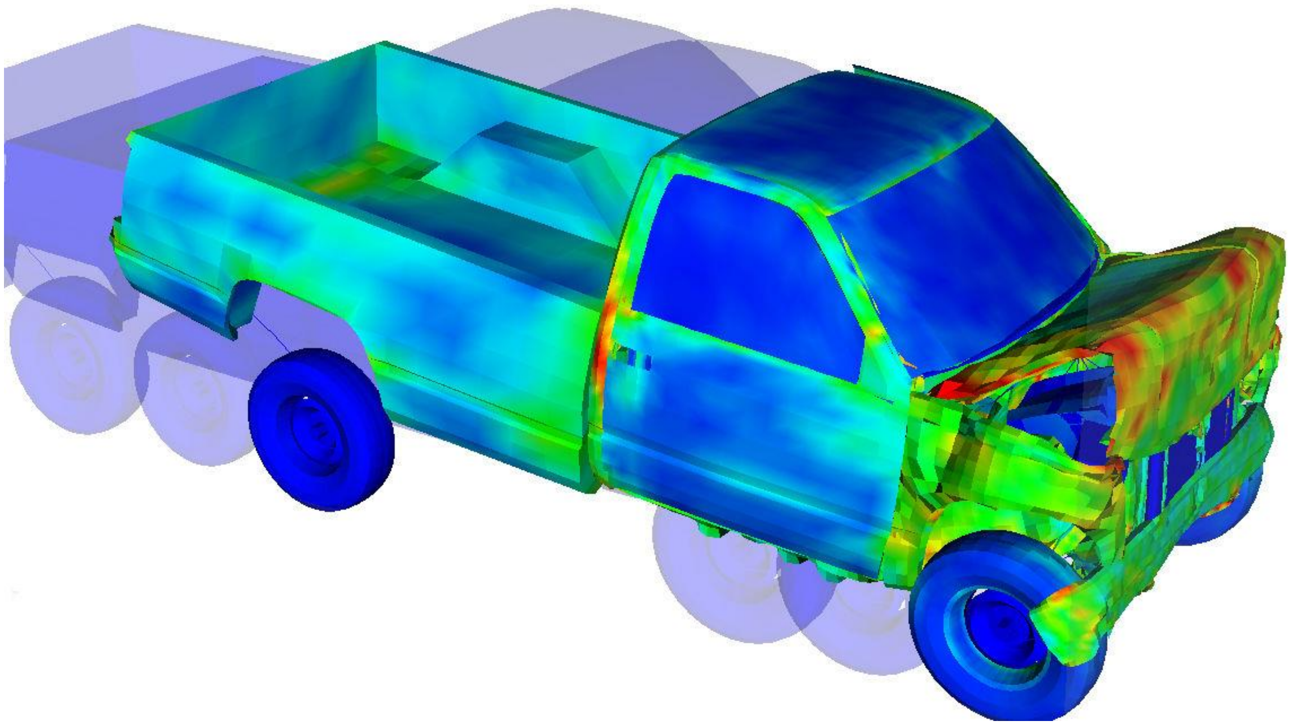
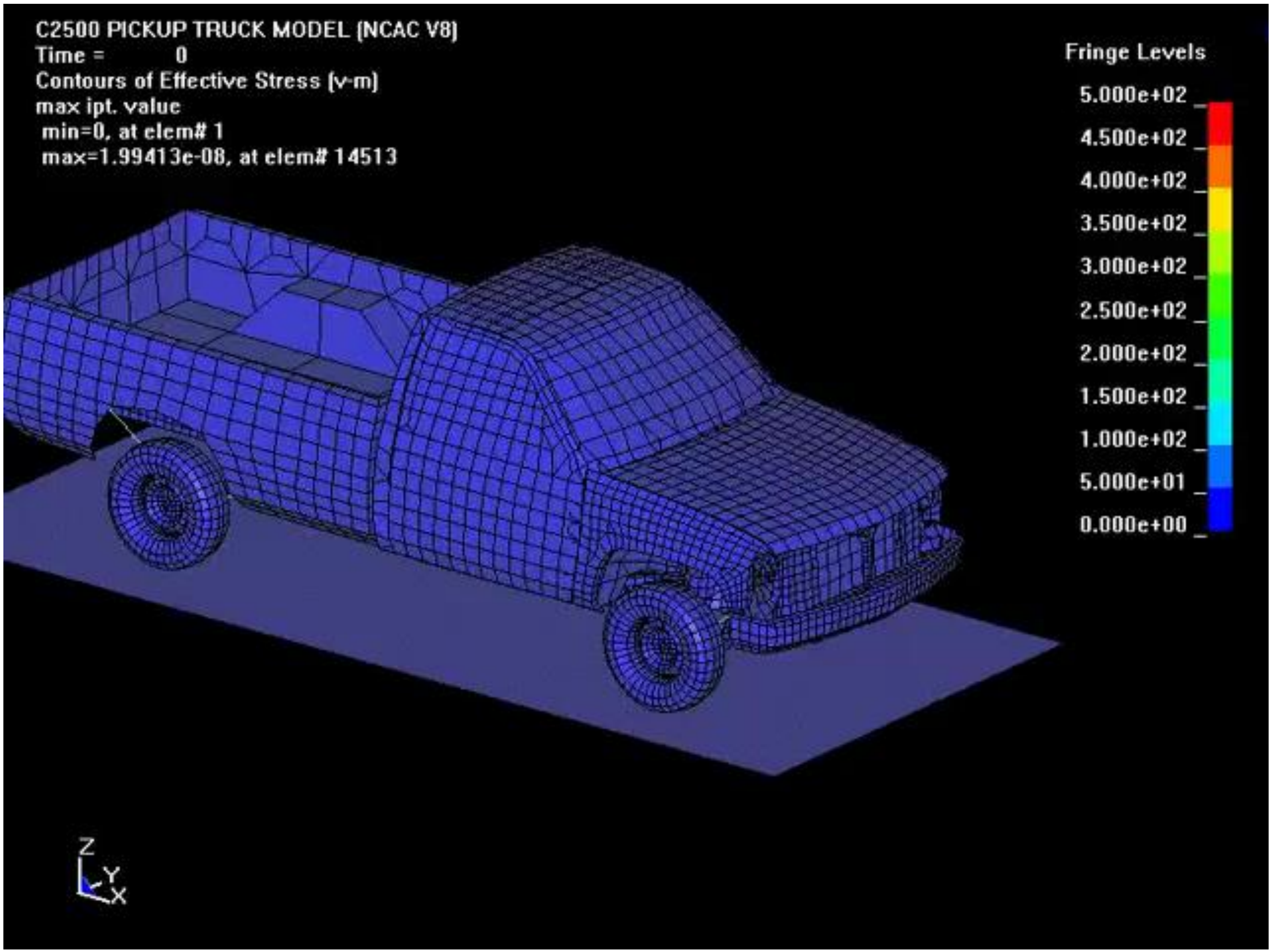
Continuum Mechanics and the Finite Element Method



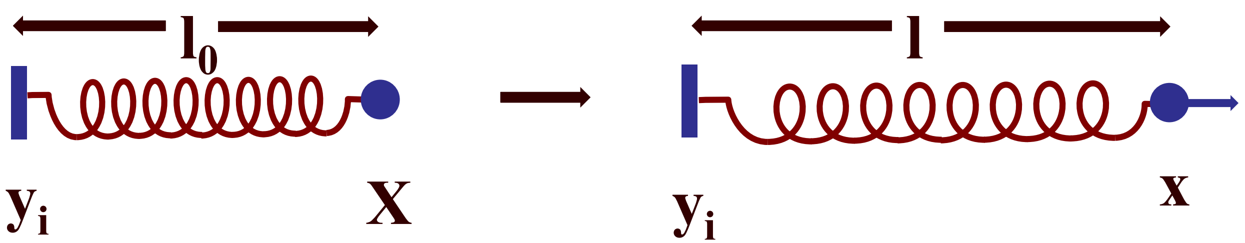
Assignment 2

- Due on March 2nd @ midnight

Suppose you want to simulate this…



The familiar mass-spring system



Spring length before/after

$\mathbf{l} = |\mathbf{x} - \mathbf{y_i}|$

$\mathbf{l_0} = |\mathbf{X} - \mathbf{y_i}|$

Deformation Measure

$$e = \left(\frac{l}{l_0} - 1\right)$$

Elastic Energy

$$W = \frac{1}{2} k e^2$$

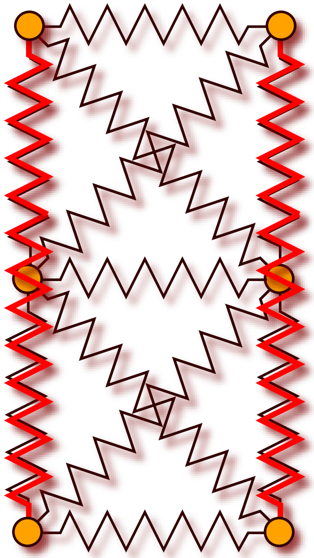
Forces

$$\mathrm{f_{int}} = -\frac{\partial W}{\partial x}$$

$$\mathbf{f}_{\mathrm{int}}(\mathbf{x}) = -k\left(\left(\frac{l}{l_0} - 1\right)\frac{\mathbf{x} - \mathbf{y}_i}{|\mathbf{x} - \mathbf{y}_i|}\right)$$

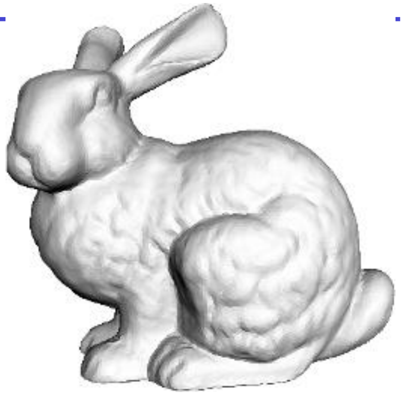
Mass Spring Systems

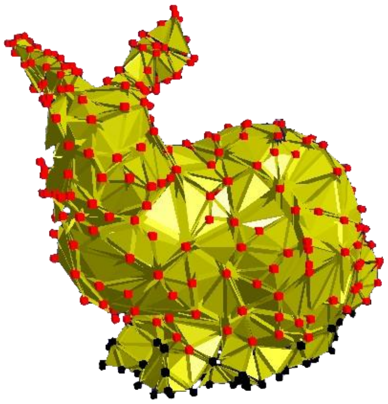
- Can be used to model arbitrary elastic/plastic objects, but…
  - Behavior depends on tessellation
    - Find good spring layout
    - Find good spring constants
    - Different types of springs interfere
    - No direct map to measurable material properties



Alternative…

- Start from continuum mechanics
- Discretize with Finite Elements
  - Decompose model into simple elements
  - Setup & solve system of algebraic equations
- Advantages
  - Accurate and controllable material behavior
  - Largely independent of mesh structure





Mass Spring vs Continuum Mechanics

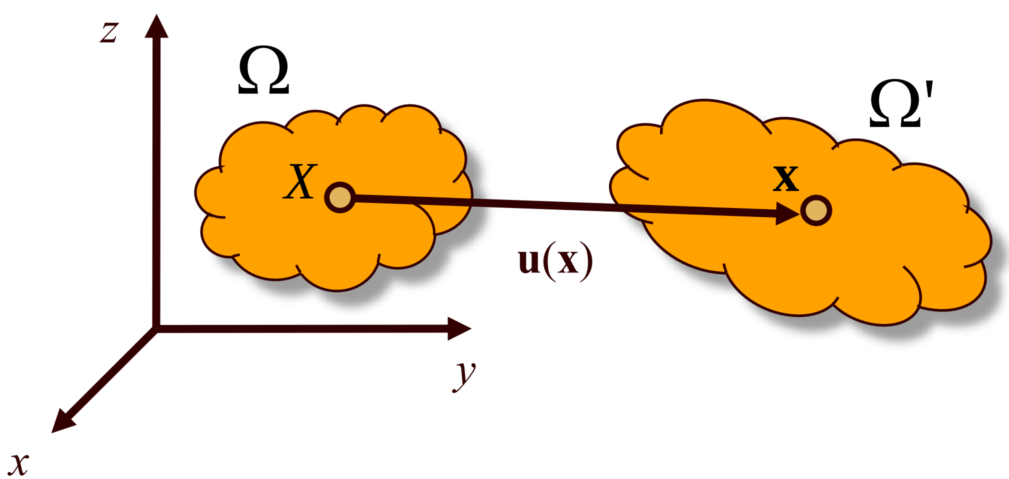
- Mass spring systems require:
  1. Measure of Deformation $\left(\frac{l}{l_0} - 1\right)$
  2. Material Model $\cdot k$
  3. Deformation Energy $W = \frac{1}{2} k e^2$
  4. Internal Forces $\mathrm{f}_{\mathrm{int}} = -\frac{\partial W}{\partial x}$

- We need to derive the same types of concepts using continuum mechanics principles

Continuum Mechanics: 3D Deformations

- For a deformable body, identify:
  - undeformed state $\Omega \subset \mathbf{R}^3$ described by positions X
  - deformed state $\Omega' \subset \mathbf{R}^3$ described by positions $\mathbf{x}$
- Displacement field $\mathbf{u}$ describes Ω' in terms of Ω

$$\mathbf{u} : \Omega \rightarrow \Omega' \qquad \mathbf{x} = X + \mathbf{u}(X)$$



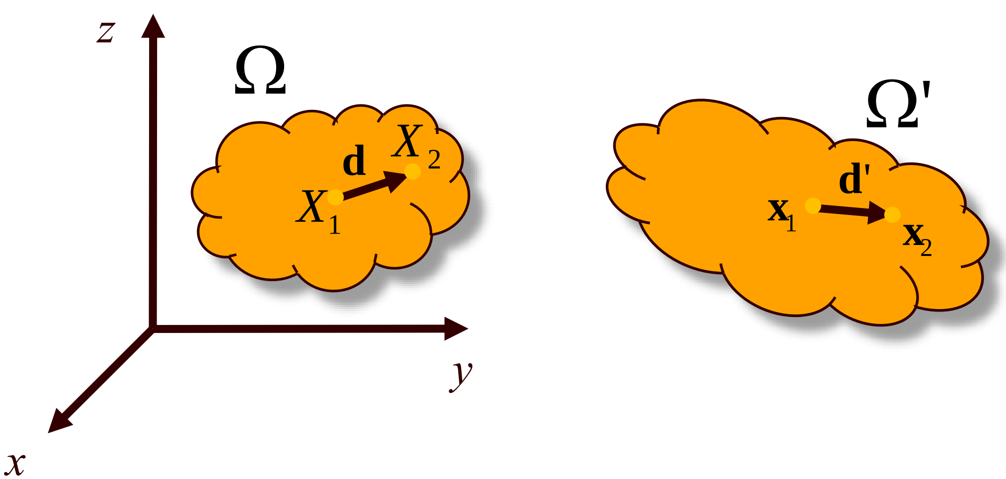
$$\mathbf{u}(X) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$$

Continuum Mechanics: 3D Deformations

- Consider material points X_1 and X_2 such that $|\mathbf{d}|$ is infinitesimal, where $\mathbf{d} = X_2 - X_1$
- Now consider deformed vector $\mathbf{d}'$

Deformation gradient $\mathbf{F} = \dfrac{\partial \mathrm{x}}{\partial X}$

$$\mathbf{d}' = \mathbf{x}_2 - \mathbf{x}_1 \approx \overbrace{(\mathbf{I} + \nabla \mathbf{u})}\, d$$



$$\nabla \mathbf{u} = \begin{pmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{pmatrix}$$

So…

- Displacement field transforms points

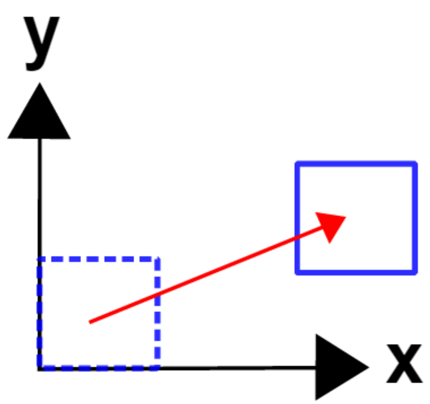
$$\mathbf{x} = X + \mathbf{u}(X)$$

- Jacobian of displacement field (deformation gradient) transforms differentials (infinitesimal vectors) from undeformed to deformed

$$\mathbf{d}' = (\mathbf{I} + \nabla\mathbf{u})\,\mathbf{d} = \mathbf{F}\mathbf{d} \qquad \mathbf{F} = \frac{\partial \mathrm{x}}{\partial X}$$

Displacement Field and Deformation Gradient

- In general, displacement field is not explicitly described. Nevertheless, toy examples:



$$x = X + 5$$
$$y = Y + 2$$

$$\mathbf{F} = \mathbf{I}$$

Displacement Field and Deformation Gradient

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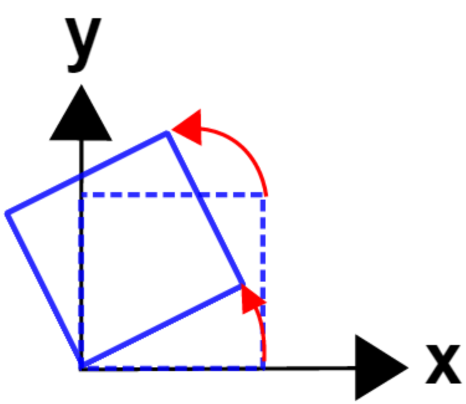

$$\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}$$

$$\mathbf{F} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Displacement Field and Deformation Gradient

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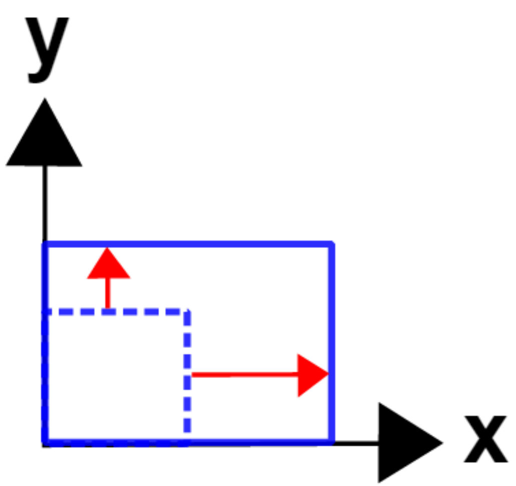
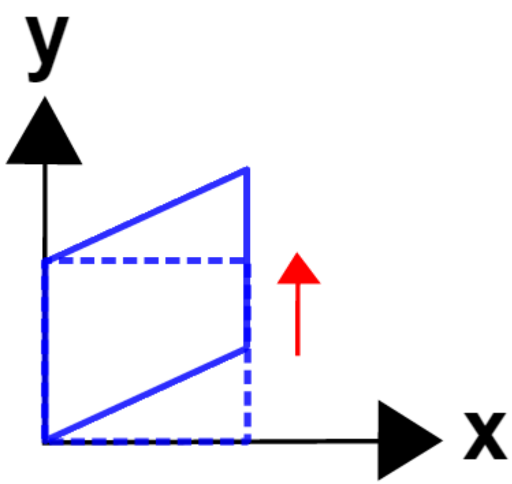


$$x = 2.0X + 0.0Y$$
$$y = 0.0X + 1.5Y$$

$$\mathbf{F} = \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.5 \end{bmatrix}$$

Displacement Field and Deformation Gradient

- In general, displacement field is not explicitly described. Nevertheless, toy examples:

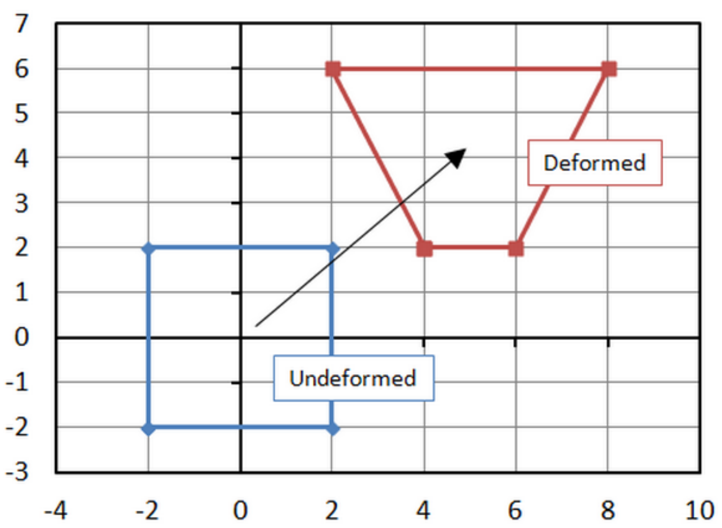


$$x = 1.0X + 0.0Y$$
$$y = 0.5X + 1.0Y$$

$$\mathbf{F} = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 1.0 \end{bmatrix}$$

Displacement Field and Deformation Gradient

- In general, displacement field is not explicitly described. Nevertheless, toy examples:



$$x = X + \frac{1}{4}XY + 5$$

$$y = Y + 4$$

$$\mathbf{F} = \begin{bmatrix} 1 + \frac{1}{4}Y & \frac{1}{4}X \\ 0 & 1 \end{bmatrix}$$

Measure of deformations

- Displacement field transforms points

$$\mathbf{x} = X + \mathbf{u}(X)$$

- Jacobian of displacement field (deformation gradient) transforms vectors

$$\mathbf{d}' = (\mathbf{I} + \nabla\mathbf{u})\,\mathbf{d} = \mathbf{F}\mathbf{d} \qquad \mathbf{F} = \frac{\partial \mathrm{x}}{\partial X}$$

How can we describe deformations?

Back to spring deformation

- Deformation measure (strain): $\left(\frac{l}{l_0} - 1\right)$

Similar to F

- Undeformed spring: $\frac{l}{l_0} = 1$

- Undeformed (infinitesimal) continuum volume:

$$\mathbf{F} = \mathrm{I}\ ?$$

Strain (description of deformation in terms of *relative* displacement**)**

- Deformation measure (strain): $\left(\frac{l}{l_0} - 1\right)$
- Desirable property: if spring is undeformed, strain is 0 (no change in shape)
- Can we find a similar measure that would work for infinitesimal volumes?

3D Nonlinear Strain

Idea: to quantify change in shape, measure change in squared length for any arbitrary vector

$$|\mathbf{d}'|^2 - |\mathbf{d}|^2 = \mathbf{d}'^T\mathbf{d}' - \mathbf{d}^T\mathbf{d} = \mathbf{d}^T(\mathbf{F}^T\mathbf{F} - \mathbf{I})\mathbf{d}$$

Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$

Strain (description of deformation in terms of *relative* displacement**)**

- Deformation measure (strain): $\left(\frac{l}{l_0} - 1\right)$
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Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$

3D Linear Strain

- Green strain is quadratic in displacements

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u})$$

- Neglecting quadratic term (small deformation assumption) leads to the linear

Cauchy strain (small strain)

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t) = \frac{1}{2}(\mathbf{F} + \mathbf{F}^t) - \mathbf{I}$$

- Written out:

$$\varepsilon = \frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_y u + \partial_x v & \partial_z u + \partial_x w \\ \partial_x v + \partial_y u & 2\partial_y v & \partial_z v + \partial_y w \\ \partial_x w + \partial_z u & \partial_y w + \partial_z v & 2\partial_z w \end{pmatrix}$$

Notation

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$$

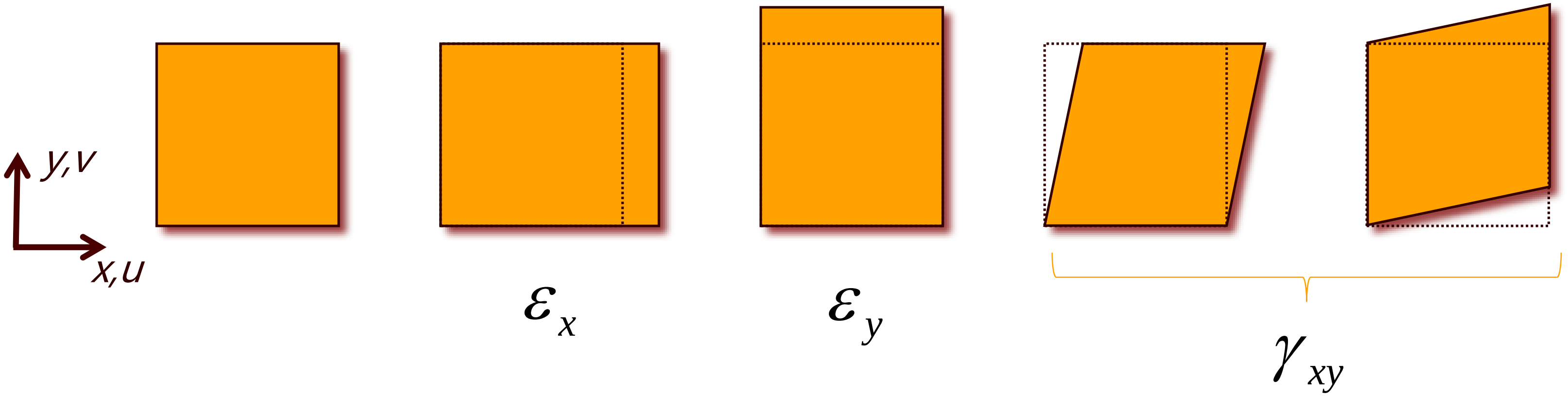
3D Linear Strain

- Linear Cauchy strain

$$\varepsilon = \frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_y u + \partial_x v & \partial_z u + \partial_x w \\ \partial_x v + \partial_y u & 2\partial_y v & \partial_z v + \partial_y w \\ \partial_x w + \partial_z u & \partial_y w + \partial_z v & 2\partial_z w \end{pmatrix} =: \begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{pmatrix}$$

ε_i : normal strains
γ_i : shear strains

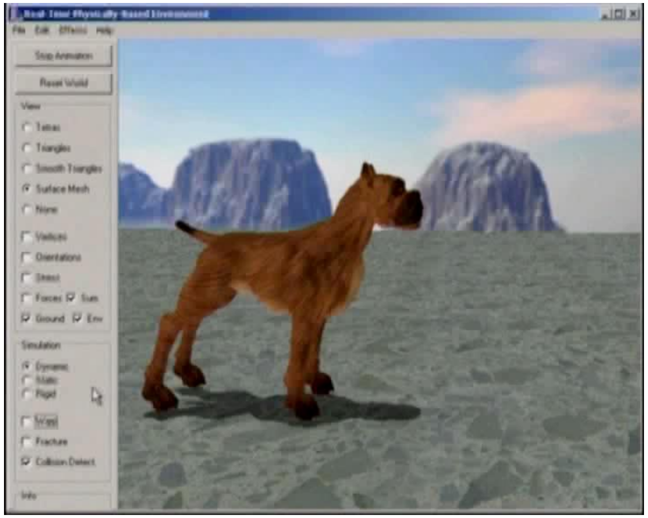
- Geometric interpretation



Cauchy vs. Green strain

- Nonlinear Green strain is rotation-invariant
  - Apply incremental rotation $\mathbf{R}$ to given deformation $\mathbf{F}$ to obtain $\mathbf{F}' = \mathbf{RF}$
  - Then $\mathbf{E}' = \frac{1}{2}(\mathbf{F}'^T \mathbf{F}' - \mathbf{I}) = \mathbf{E}$

- Linear Cauchy strain
  is not rotation-invariant
  $$\varepsilon' = \frac{1}{2}\left(\mathbf{F}' + \mathbf{F}'^t\right) \neq \varepsilon$$
  → artifacts for larger rotations



Mass Spring vs Continuum Mechanics

- Mass spring systems:
  1. Measure of Deformation $\left(\frac{l}{l_0} - 1\right)$
  2. Material Model k
  3. Deformation Energy $W = \frac{1}{2} k e^2$
  4. Internal Forces $f_{int} = -\frac{\partial W}{\partial x}$
- Continuum Mechanics:
  1. Measure of Deformation: Green or Cauchy strain
  2. Material Model

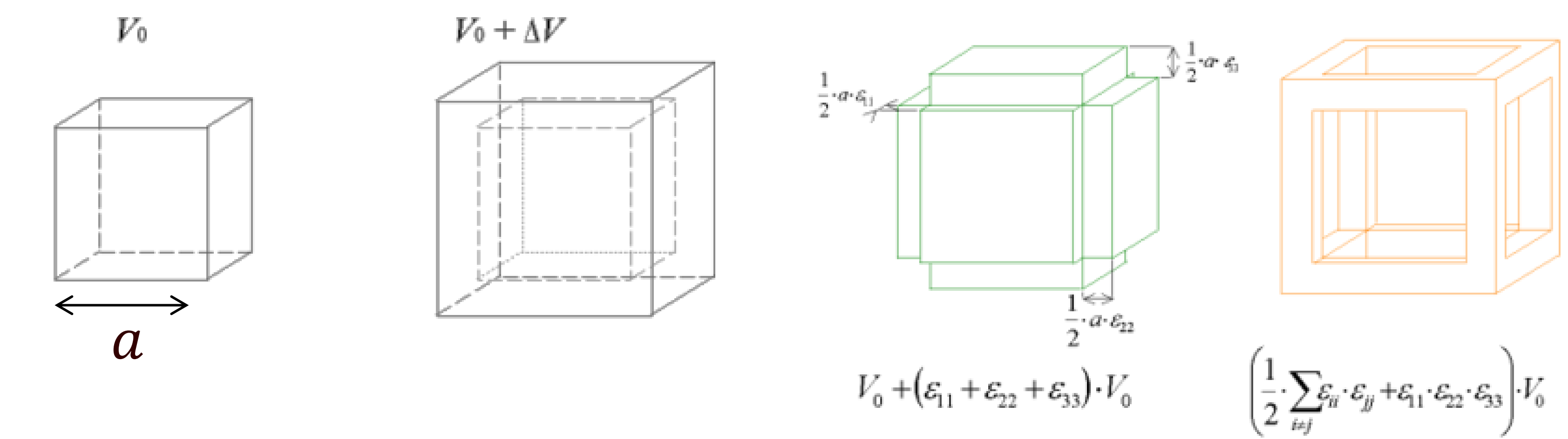
Material Model: linear isotropic material

- Material model links strain to energy (and stress)
- Linear isotropic material (*generalized Hooke's law*)
  - Energy density $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$
  - Lame parameters λ and μ are material constants related to Poisson Ratio and Young's modulus
- Interpretation
  - $\mathrm{tr}(\boldsymbol{\varepsilon}^2) = \|\varepsilon\|_F^2$ penalizes all strain components equally
  - $\mathrm{tr}(\boldsymbol{\varepsilon})^2$ penalizes dilatations, i.e., volume changes

Volumetric Strain (*dilatation, hydrostatic strain*)

- Consider a cube with side length a
- For a given deformation $\boldsymbol{\varepsilon}$, the volumetric strain is

$$\begin{aligned}\Delta V/V0 \quad &= (a(1+\varepsilon_{11}) \cdot a(1+\varepsilon_{22}) \cdot a(1+\varepsilon_{33}) - a^3)/a^3 \\ &= (\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + O(\boldsymbol{\varepsilon}^2) \approx \mathrm{tr}(\boldsymbol{\varepsilon})\end{aligned}$$

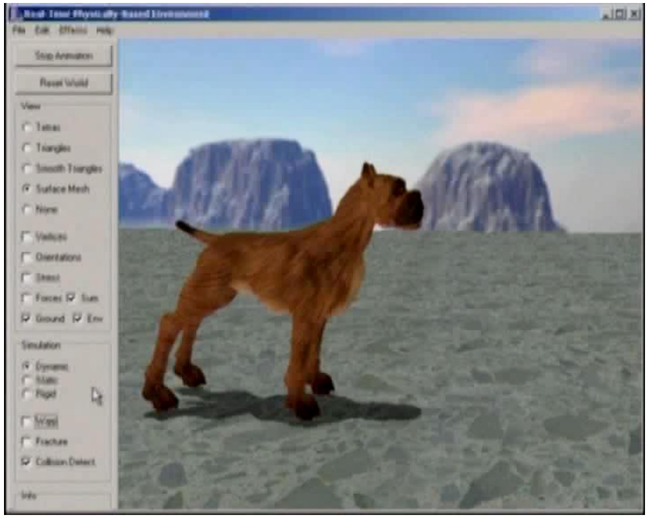


http://en.wikipedia.org/wiki/Infinitesimal_strain_theory

Linear isotropic material

Energy density: $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$

- Problem: Cauchy strain is not invariant under rotations
  → artifacts for rotations
- Solutions:
  - Corotational elasticity
  - Nonlinear elasticity



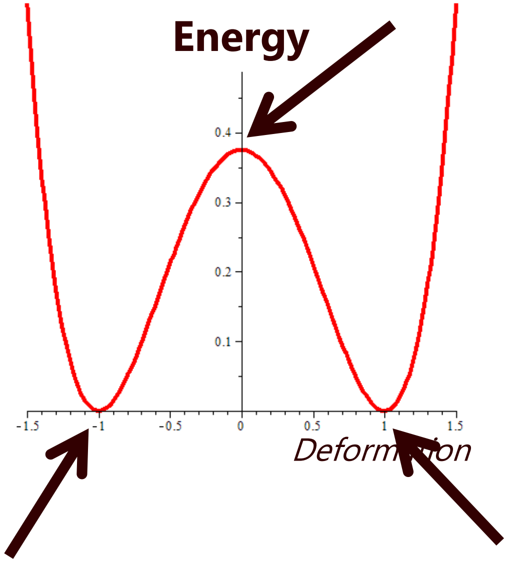
Material Model: non-linear isotropic model

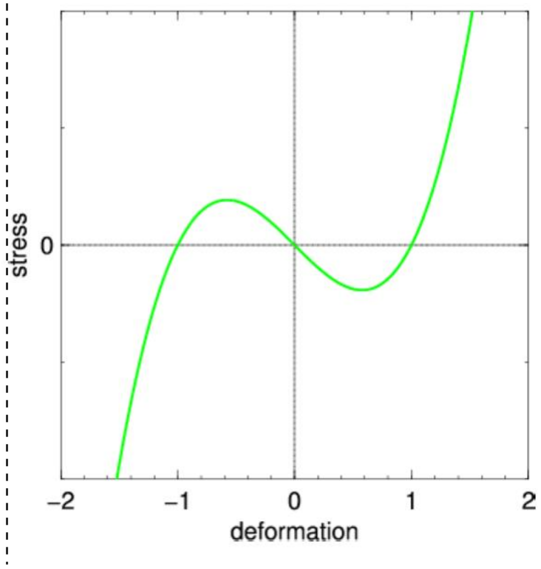
- Replace Cauchy strain with Green strain → *St. Venant-Kirchhoff material* (StVK)
- Energy density: $\Psi_{StVK} = \frac{1}{2}\lambda \mathrm{tr}(\mathbf{E})^2 + \mu \mathrm{tr}(\mathbf{E}^2)$
- Rotation invariant!

Problems with StVK

- StVK softens under compression

$$\Psi_{StVK} = \frac{1}{2}\lambda \mathrm{tr}(\mathbf{E})^2 + \mu \mathrm{tr}(\mathbf{E}^2)$$



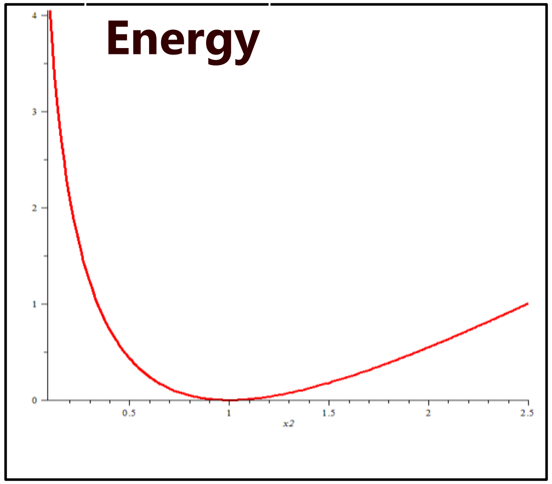


Advanced nonlinear materials

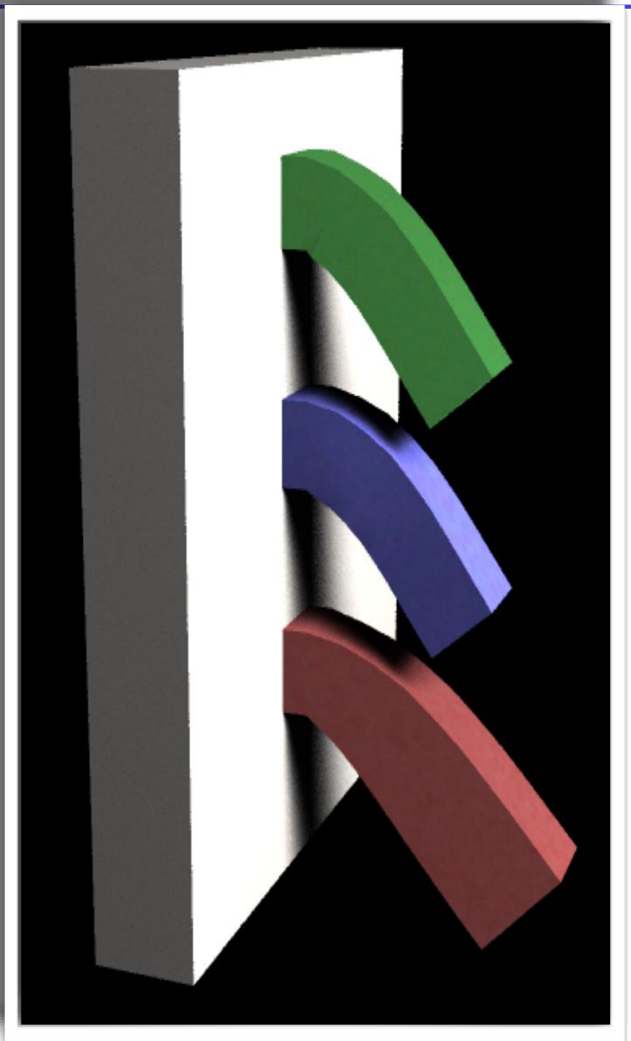
- Green Strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^t\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{C} - \mathbf{I})$
- Split into *deviatoric (i.e. shape changing/distortion)* and *volumetric (dilation, volume changing)* deformations

  Volumetric: $J = \det(\mathbf{F})$ Deviatoric: $\hat{\mathbf{C}} = \det(\mathbf{F})^{-2/3}\,\mathbf{C}$
- Neo-Hookean material:

$$\Psi_{NH} = \frac{\mu}{2}\mathrm{tr}\left(\hat{\mathbf{C}} - \mathbf{I}\right) - \mu\ln(J) + \frac{\lambda}{2}\ln(J)^2$$



Different Models



St. Venant-Kirchoff

Neo-Hookean

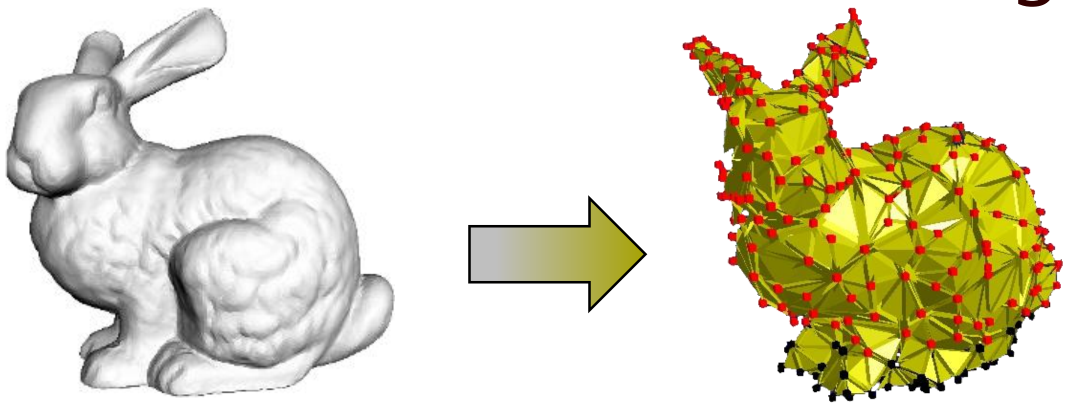
Linear

Mass Spring vs Continuum Mechanics

- Mass spring systems:
  1. Measure of Deformation $\left(\frac{l}{l_0} - 1\right)$
  2. Material Model k
  3. Deformation Energy $W = \frac{1}{2} k e^2$
  4. Internal Forces $\mathrm{f}_{\mathrm{int}} = -\frac{\partial W}{\partial x}$
- Continuum Mechanics:
  1. Measure of Deformation: Green or Cauchy strain
  2. Material Model: linear, StVK, Neo-Hookean, etc
  3. From Energy Density to Deformation Energy: Finite Element Discretization

Finite Element Discretization

◆ Divide domain into discrete elements, e.g., *tetrahedra*



- Explicitly store displacement values at nodes ($\mathbf{x}_i$).
- Displacement field everywhere else obtained through interpolation: $\mathbf{x}(X) = \sum N_i(X)\mathbf{x}_i$
- Deformation Gradient: $\mathbf{F} = \frac{\partial \mathbf{x}(X)}{\partial X} = \sum_i \mathbf{x}_i \left(\frac{\partial N_i}{\partial X}\right)^t$

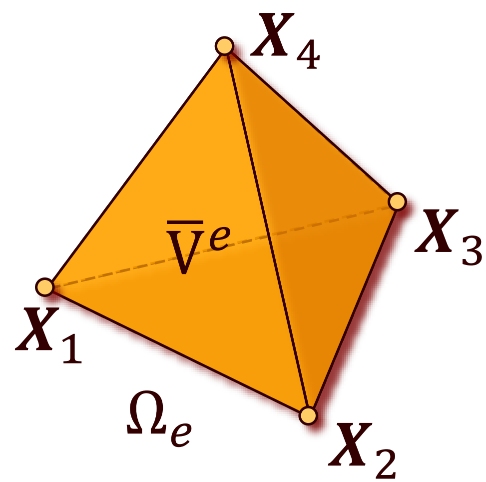
Basis Functions

- Basis functions (aka shape functions) $N_i(X_j)$: $\mathbf{R}^3 \rightarrow \mathbf{R}$
- Satisfy delta-property: $N_i(X_j) = \delta_{ij}$
- Simplest choice: linear basis functions

$$N_i(\bar{x}, \bar{y}, \bar{z}) = a_i \bar{x} + b_i \bar{y} + c_i \bar{z} + d_i$$

- Compute N_i (and $\frac{\partial N_i}{\partial X}$) through

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \end{pmatrix}$$



Constant Strain Elements (Linear Basis Functions)

- Displacement field is continuous in space
- Deformation Gradient, strain, stress are not
  - Constant Strain per element
- Deformation Gradient can be computed as
  - $\mathbf{F} = \boldsymbol{e}\boldsymbol{E}^{-1}$ where e and E are matrices whose columns are edge vectors in undeformed and deformed configurations

Constant Strain Elements: From energy density to deformation energy

- Integrate energy density over the entire element:

$$W^e = \int_{\Omega_e} \Psi(\mathbf{F})$$

- If basis functions are linear:
  - $\mathbf{F}$ is linear in $\mathbf{x}_i$
  - $\mathbf{F}$ is constant throughout element:

$$W^e = \int_{\Omega_e} \Psi(\mathbf{F}) = \Psi(\mathbf{F}) \cdot \bar{V}^e$$

Mass Spring vs Finite Element Method

- Mass spring systems:
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  3. Deformation Energy $W = \frac{1}{2} k e^2$
  4. Internal Forces $f_{int} = -\frac{\partial W}{\partial x}$
- Continuum Mechanics:
  1. Measure of Deformation: Green or Cauchy strain
  2. Material Model: linear, StVK, Neo-Hookean, etc
  3. Deformation Energy: integrate over elements
  4. Internal Forces: $f_{int} = -\frac{\partial W}{\partial x}$

FEM recipe

- Discretize into elements (triangles/tetraderons, etc)
- For each element
  - Compute deformation gradient $\mathbf{F} = \boldsymbol{e}\boldsymbol{E}^{-1}$
  - Use material model to define energy density $\Psi(\mathbf{F})$
  - Integrate over elements to compute energy: *W*
  - Compute nodal forces as: $\mathrm{f}_{\mathrm{int}} = -\frac{\partial W}{\partial x}$

FEM recipe

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu \|\mathbf{E}\|_{\mathrm{F}} + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_{\mathrm{F}}^2, \quad J = \det \mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu \log(J) + \frac{\lambda}{2}\log^2(J)$$

Area/volume of element

$$\boldsymbol{f} = -\frac{\partial W}{\partial \boldsymbol{x}} = -V\frac{\partial \Psi}{\partial \boldsymbol{F}}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$$

First Piola-Kirchhoff stress tensor **P**

FEM recipe

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$$

$$\Psi = \mu\|\mathbf{E}\|_F + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P} = \mathbf{F}\left[2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}\right]$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det \mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu\log(J) + \frac{\lambda}{2}\log^2(J)$$

$$\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda\log(J)\mathbf{F}^{-T}$$

Area/volume of element

$$\boldsymbol{f} = -\frac{\partial W}{\partial \boldsymbol{x}} = -V\frac{\partial \Psi}{\partial \boldsymbol{F}}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$$

First Piola-Kirchhoff stress tensor **P**

FEM recipe

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu\|\mathbf{E}\|_F + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

$$\mathbf{P} = \mathbf{F}\,[2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}]$$

Neohookean elasticity

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det\mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu\log(J) + \frac{\lambda}{2}\log^2(J)$$

$$\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda\log(J)\mathbf{F}^{-T}$$

For a tetrahedron, this works out to:

$$[\boldsymbol{f}_1 \;\; \boldsymbol{f}_2 \;\; \boldsymbol{f}_3] = -V\boldsymbol{P}\boldsymbol{E}^{-T}; \boldsymbol{f}_4 = -\boldsymbol{f}_1 - \boldsymbol{f}_2 - \boldsymbol{f}_3$$

Additional reading: http://www.femdefo.org/

Material Parameters

St. Venant-Kirchhoff material

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\Psi = \mu\|\mathbf{E}\|_F + \frac{\lambda}{2}\mathrm{tr}^2(\mathbf{E})$$

Neohookean elasticity

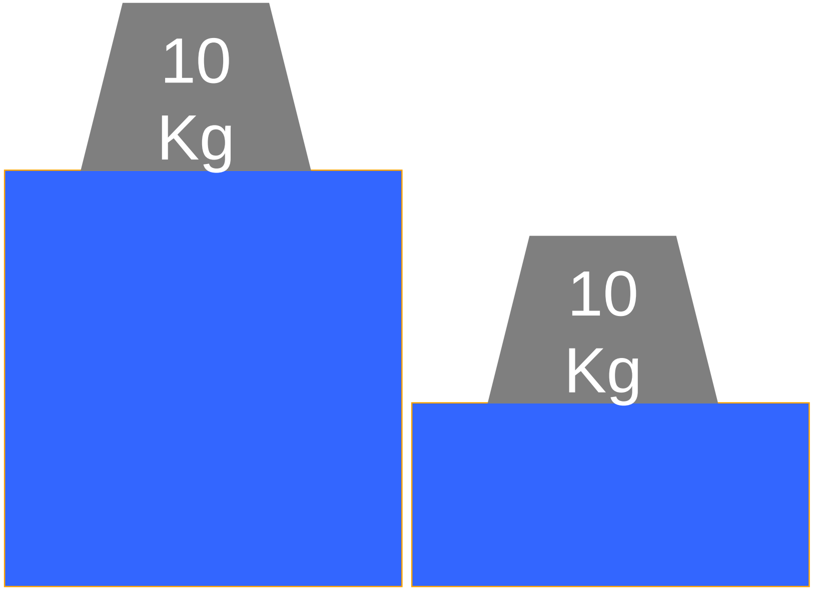
$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det \mathbf{F}$$

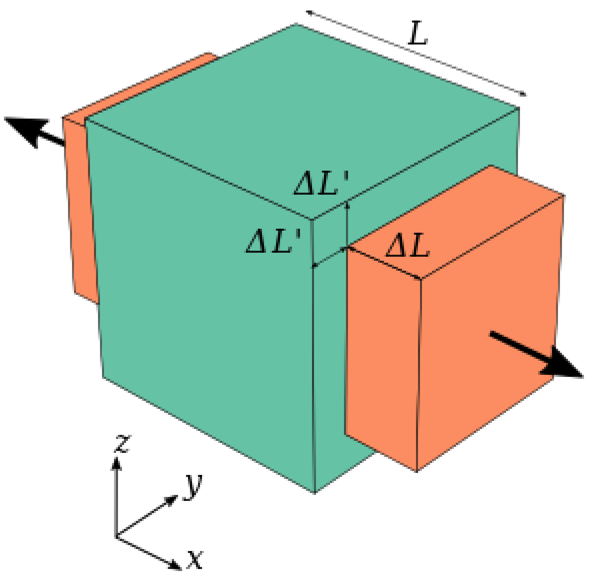
$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu \log(J) + \frac{\lambda}{2}\log^2(J)$$

Lame parameters λ and μ are material constants related to the fundamental physical parameters: Poisson's Ratio and Young's modulus (http://en.wikipedia.org/wiki/Lamé_parameters)

Young's Modulus and Poisson Ratio

Lame parameters λ and μ are material constants related to the fundamental physical parameters: Poisson's Ratio and Young's modulus (http://en.wikipedia.org/wiki/Lamé_parameters)





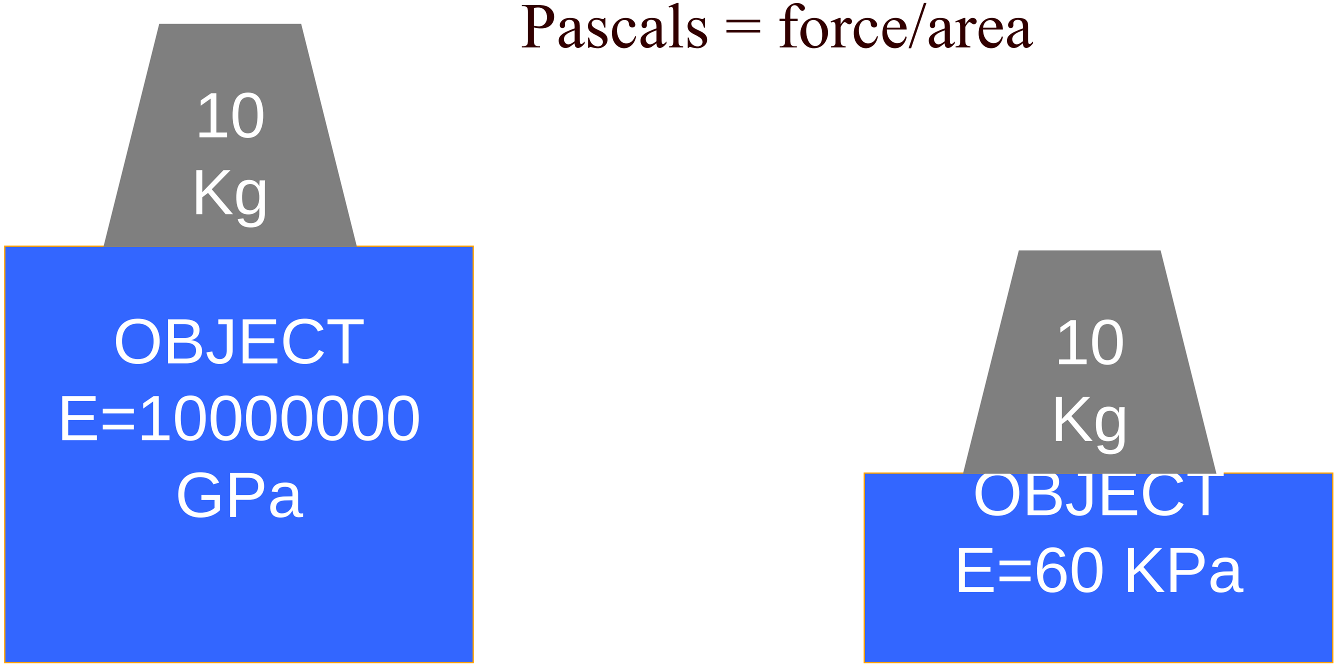
Young's modulus (E), measure of stiffness

Poisson's ratio (ν), relative transverse to axial deformation

What Do These Parameters Mean

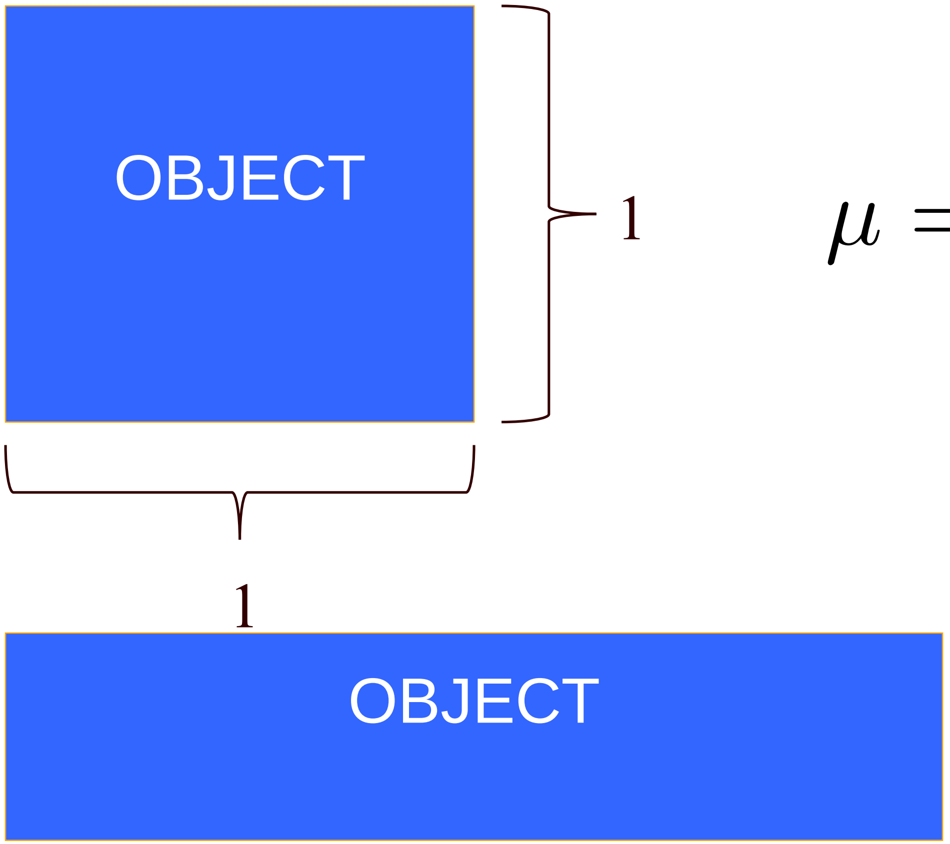
- Stiffness is pretty intuitive

Pascals = force/area



What Do These Parameters Mean

Poisson's Ratio controls volume preservation



$$\mu = -\frac{\Delta y}{\Delta x}$$

What Do These Parameters Mean

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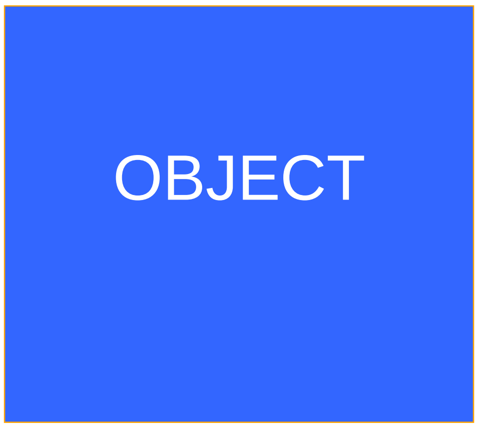

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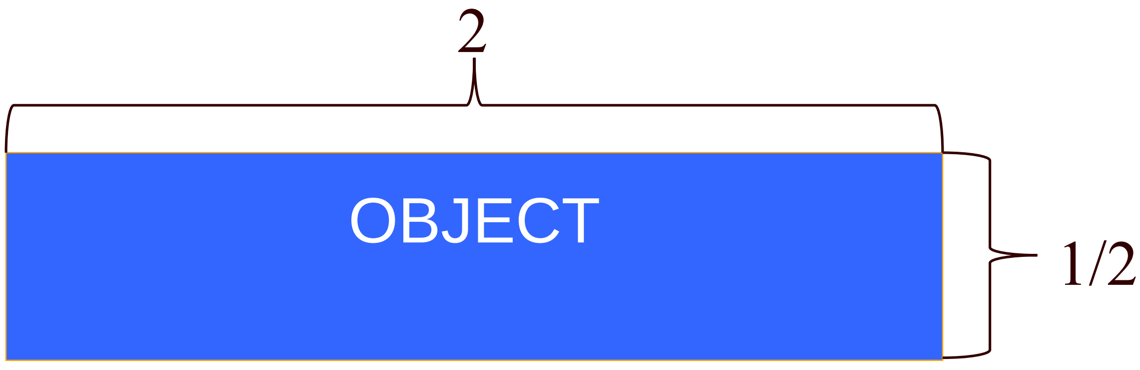

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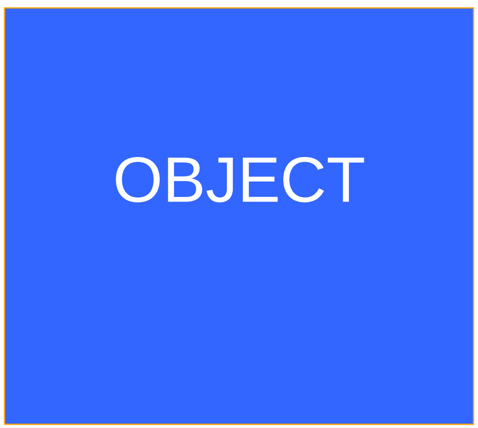
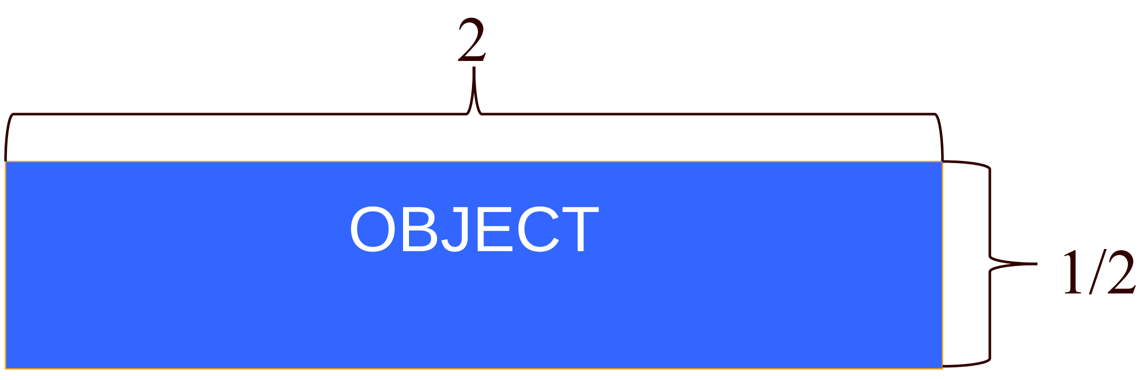


$$\mu = -\frac{\Delta y}{\Delta x}$$

Poisson's Ratio (ν) = ?



What Do These Parameters Mean

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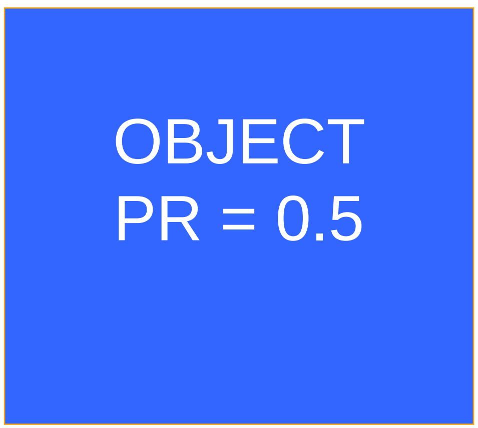
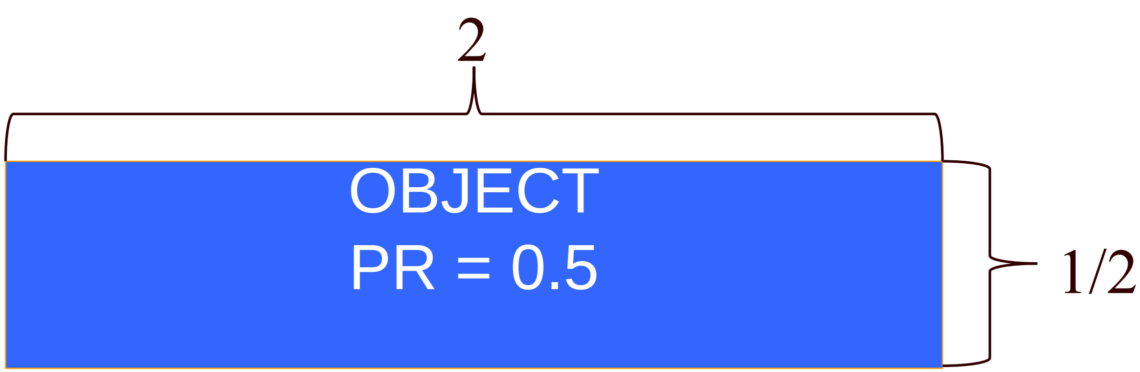
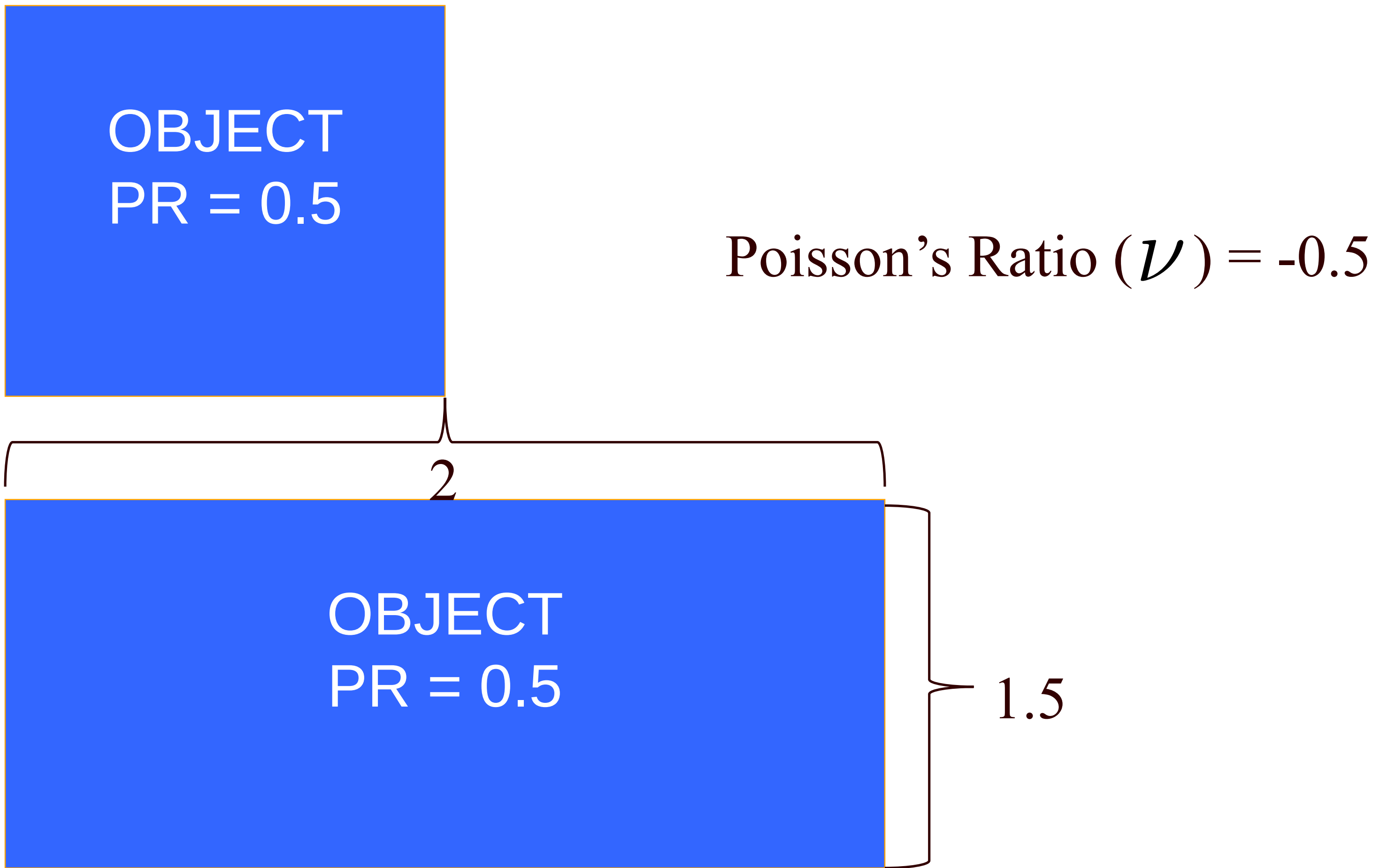


Poisson's Ratio (ν) = 0.5



What Do These Parameters Mean

Poisson's ratio is between -1 and 0.5

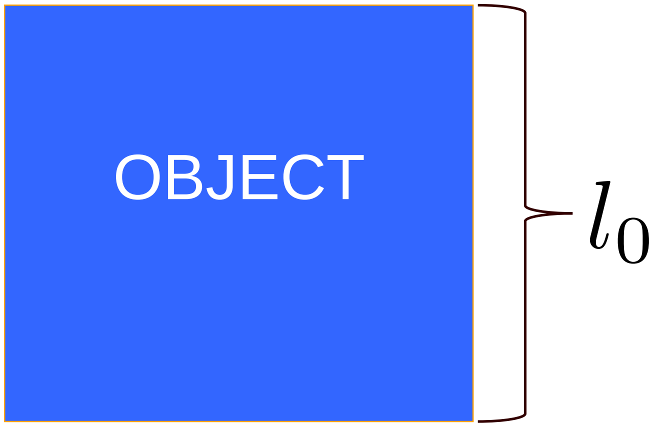


Negative Poisson’s Ratio

Measurement

- Where do material parameters come from?

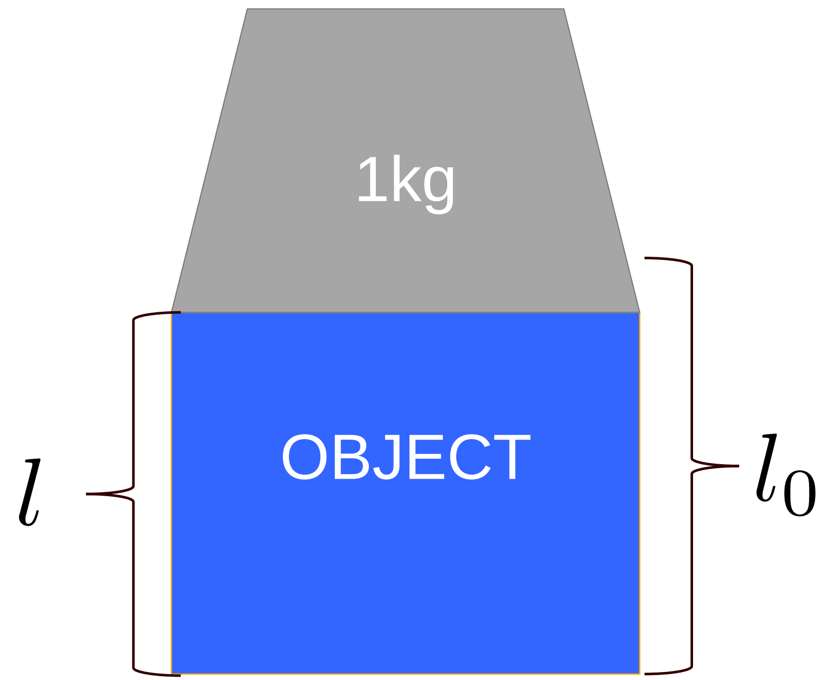
Simple Measurement: Stiffness



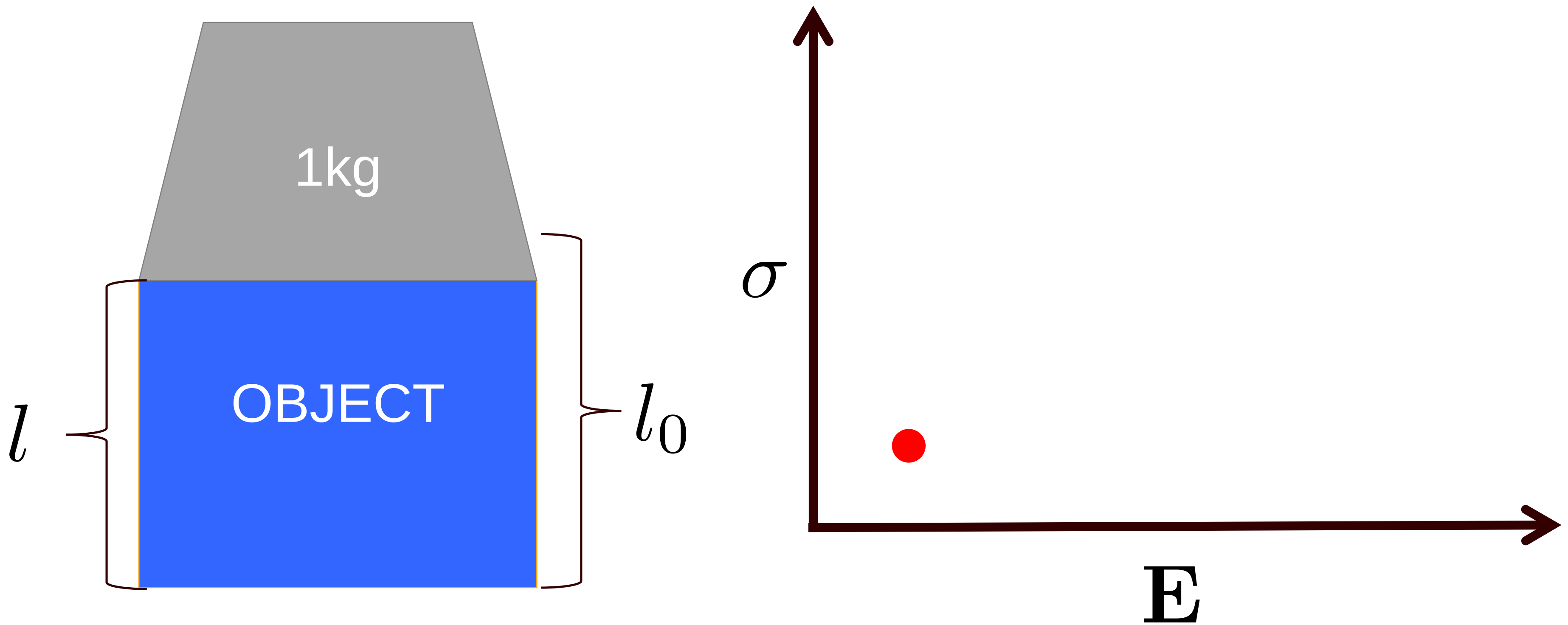
Simple Measurement

What's the Force (Stress)?

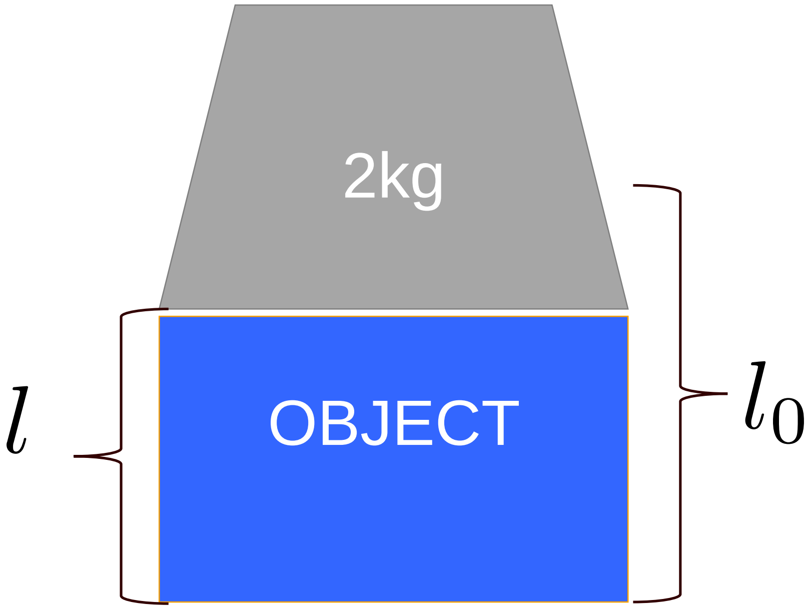
What's the Deformation (Strain)?

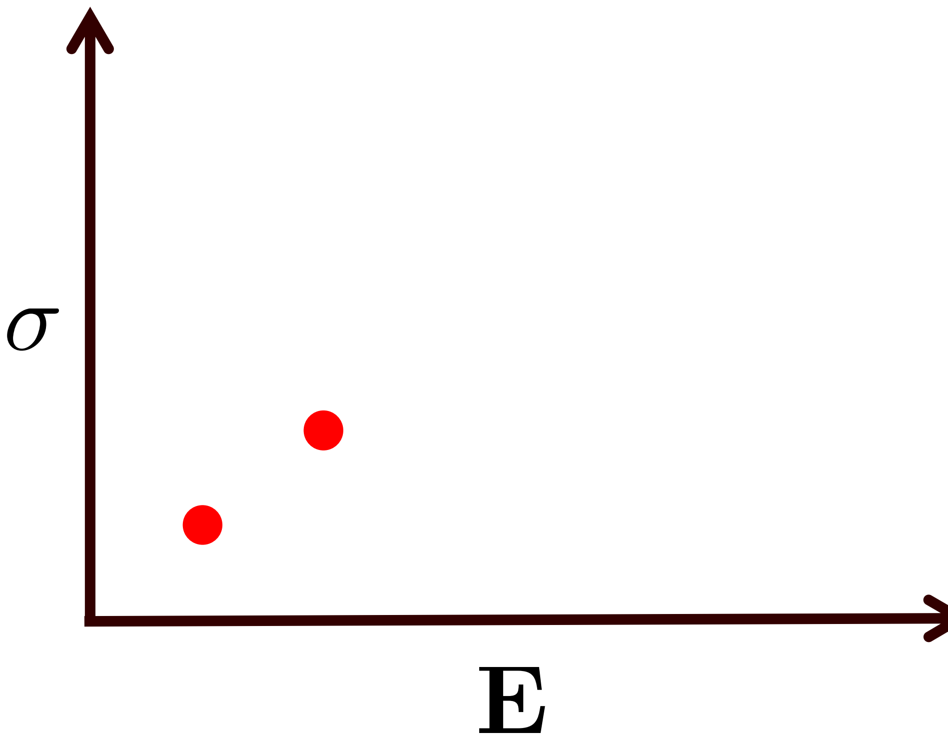


Simple Measurement



Simple Measurement





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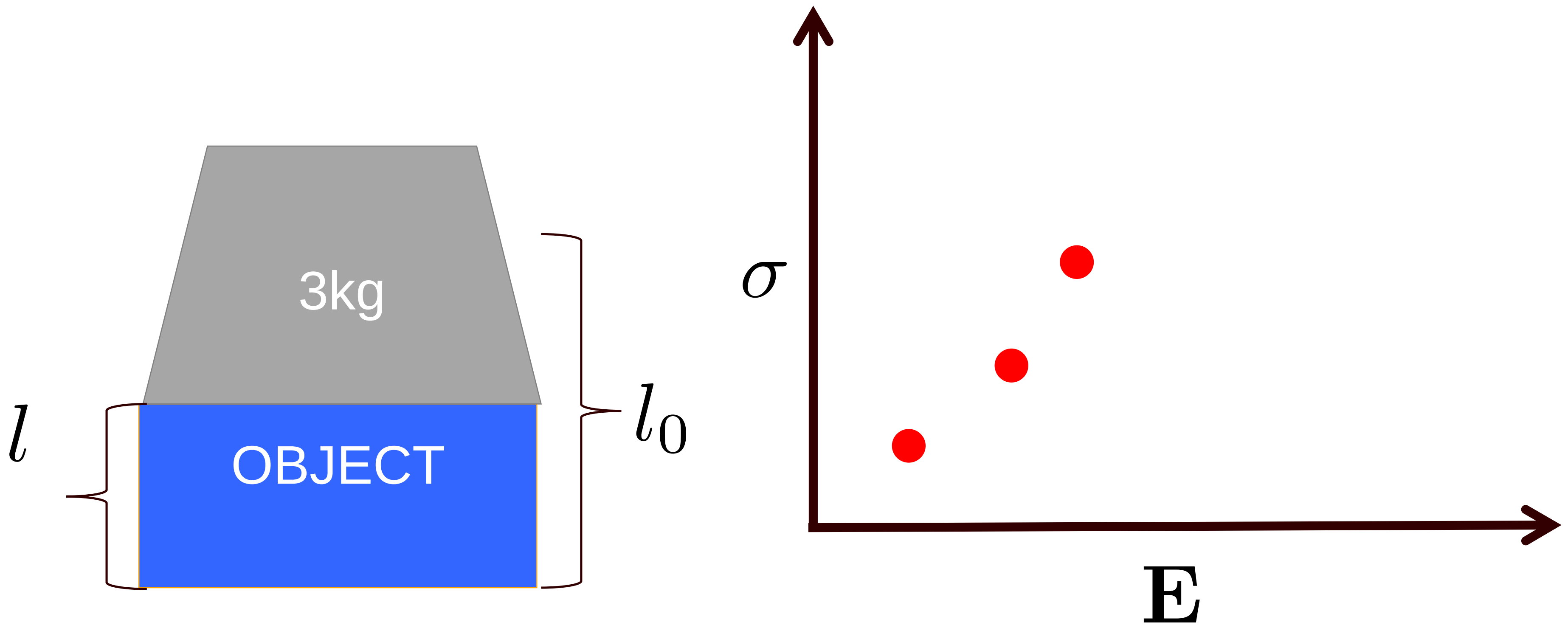

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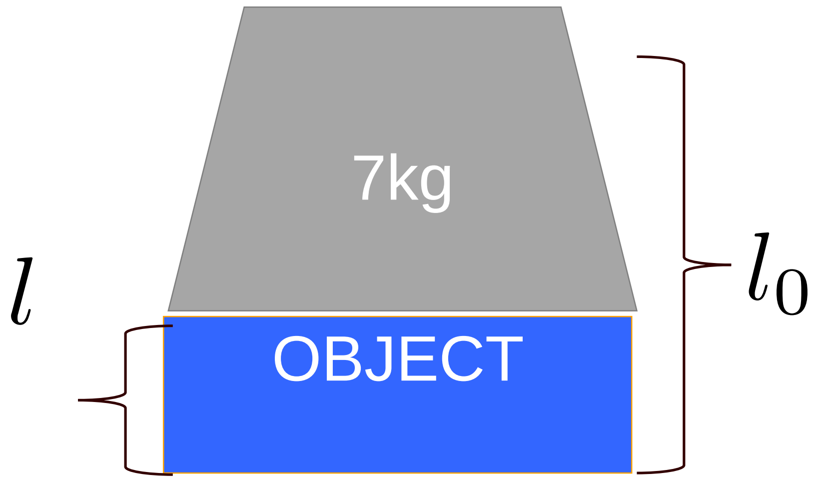
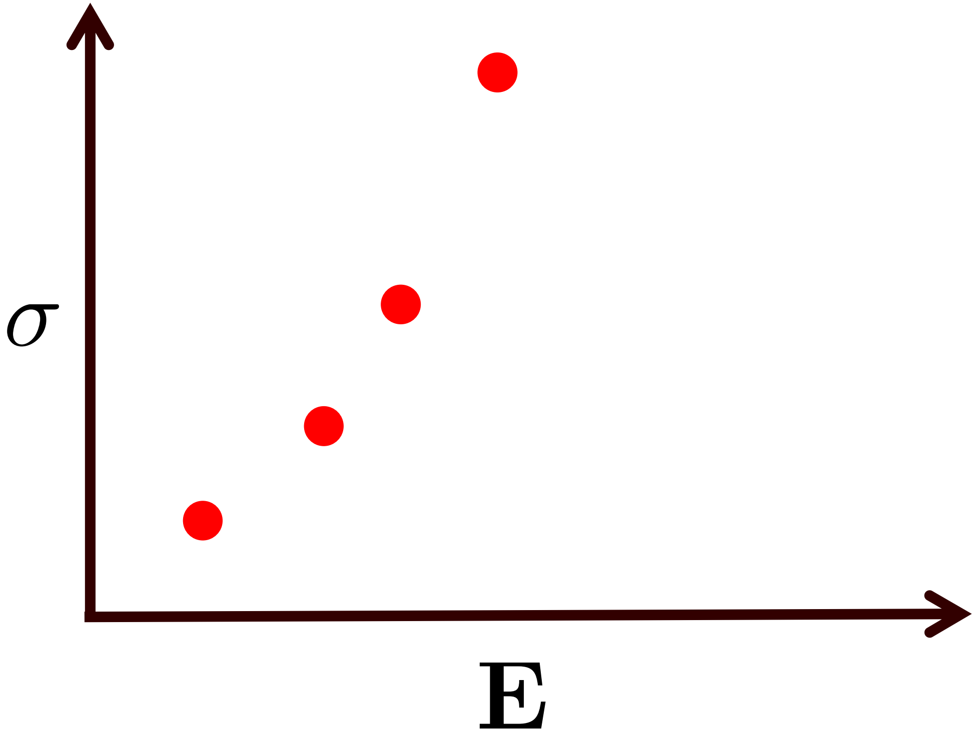
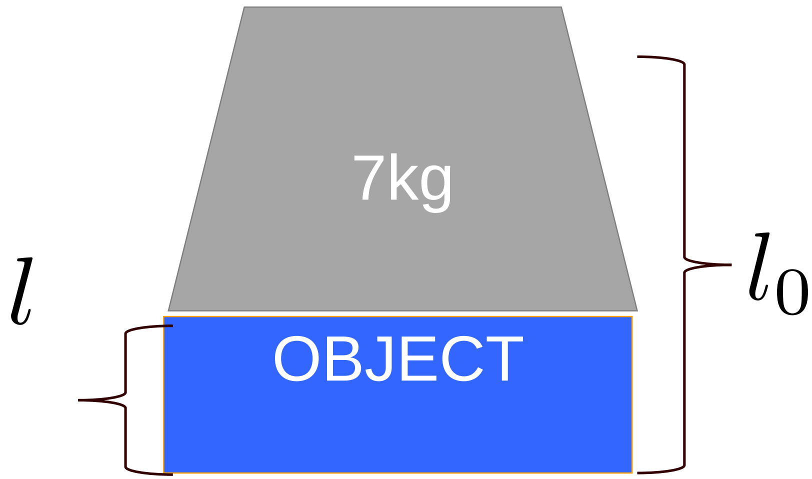
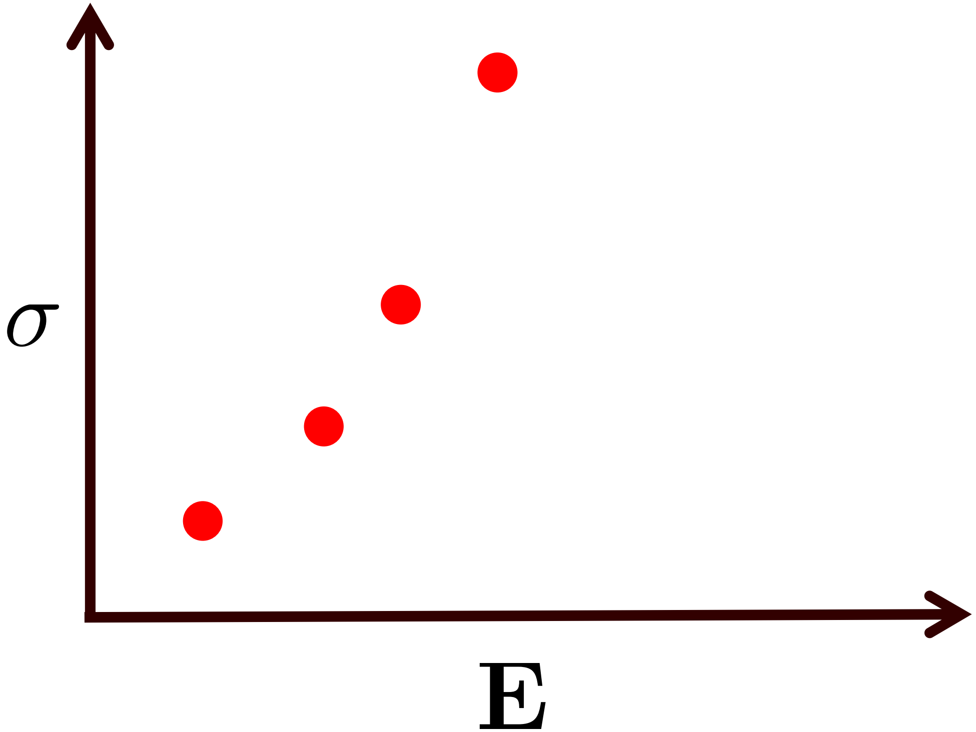




Simple Measurement

How do we get the stiffness ?





Simple Measurement

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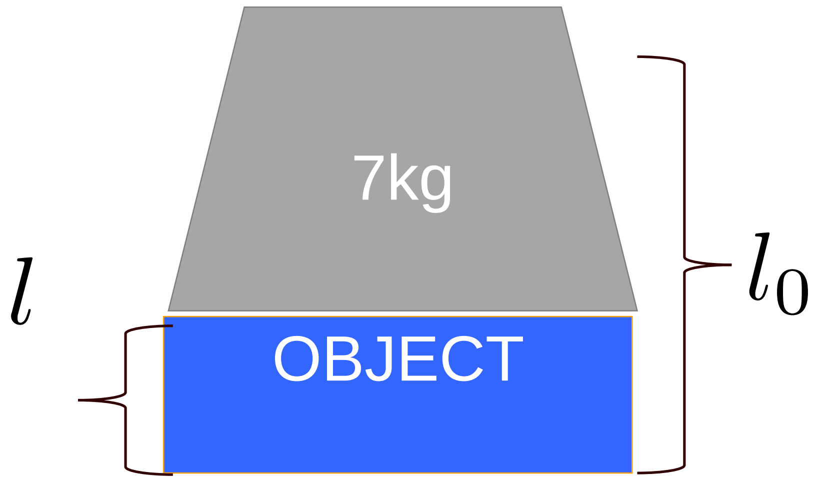

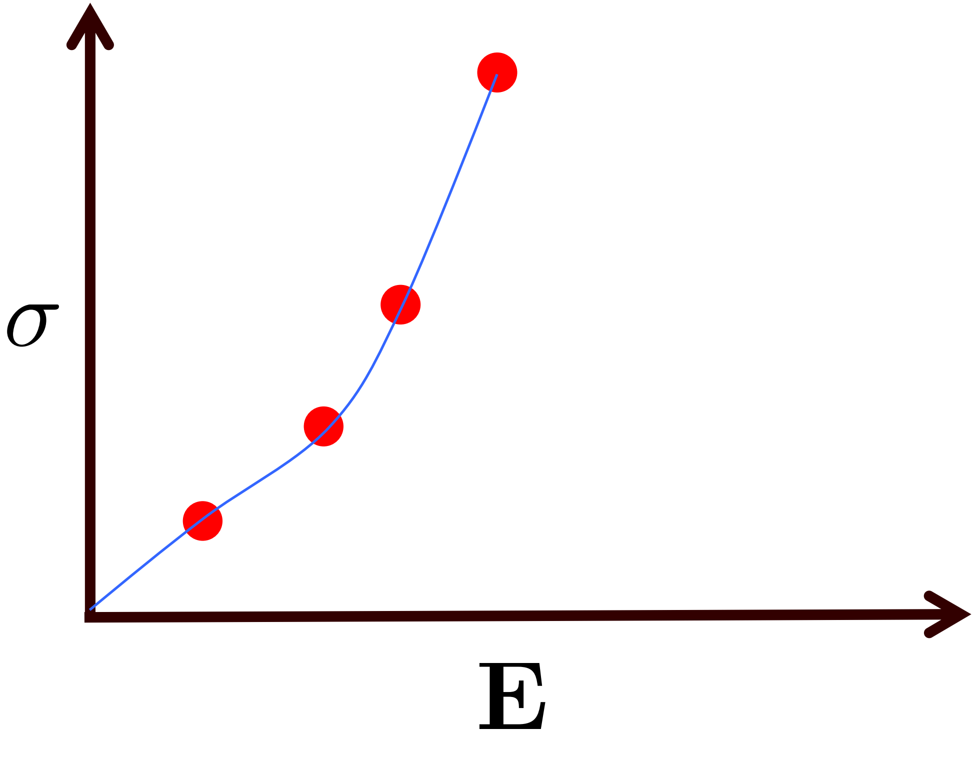

Simple Measurement

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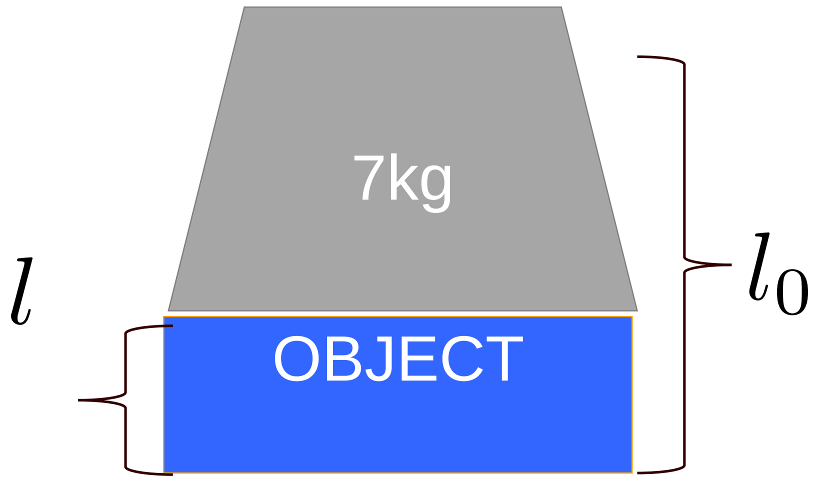
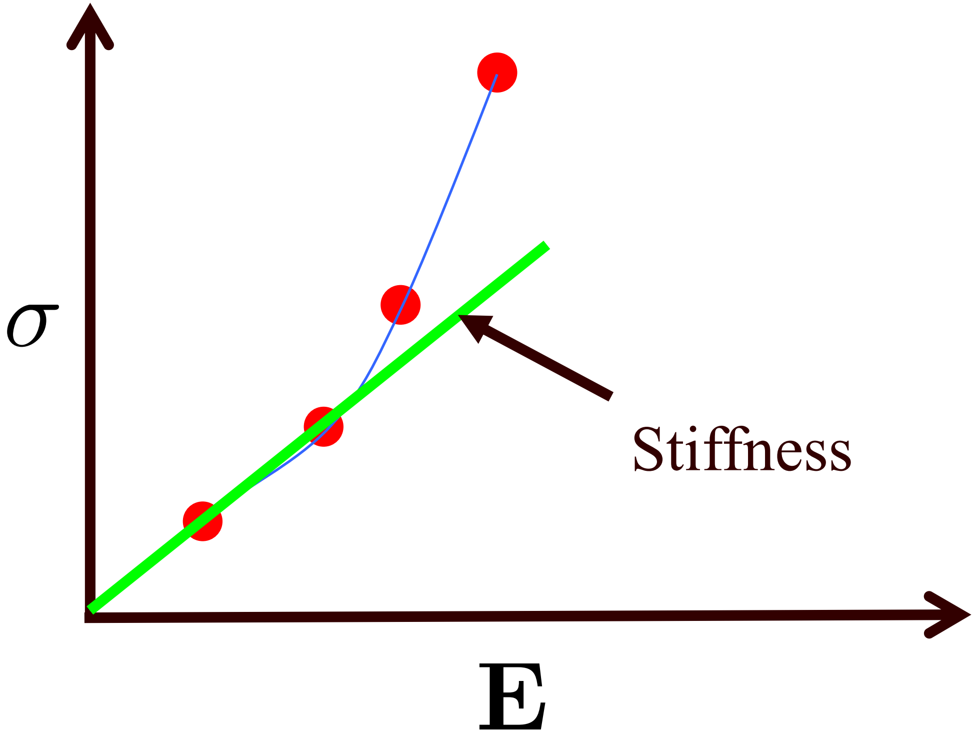
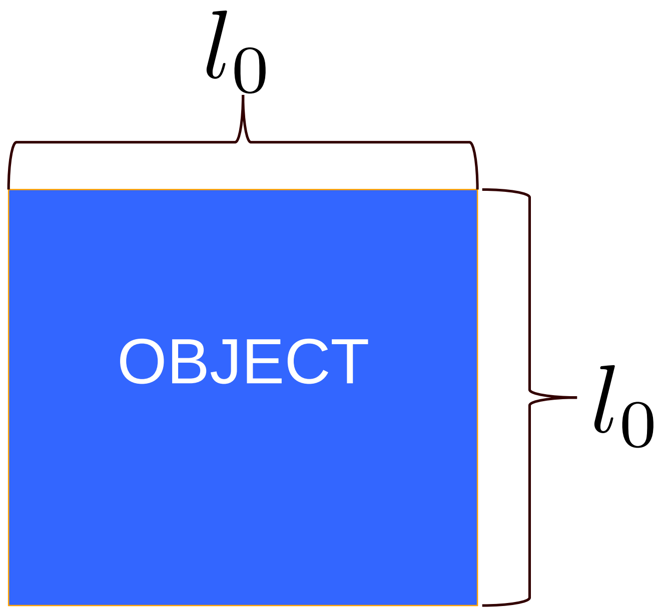
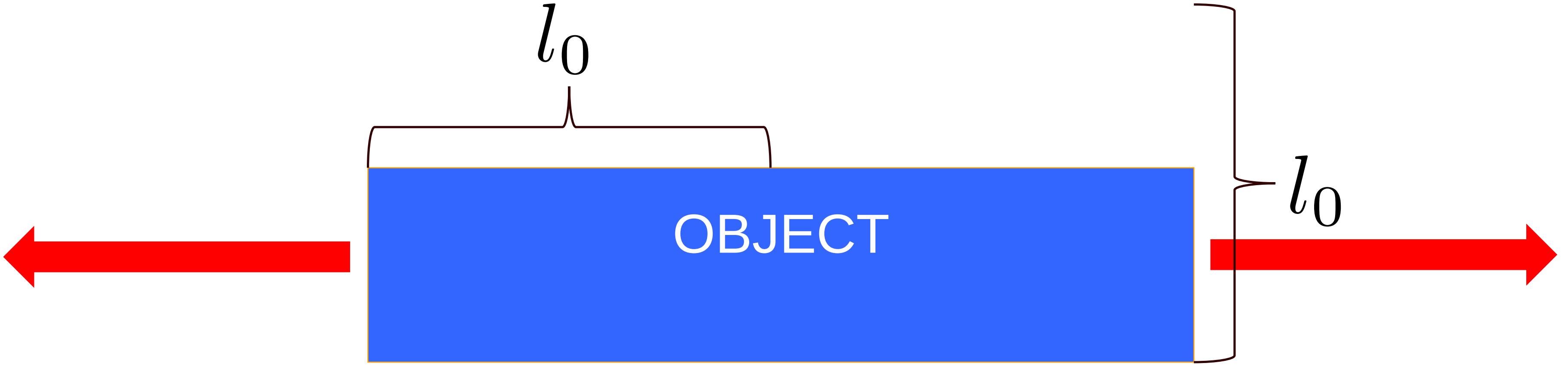




Simple Measurement: Poisson's Ratio

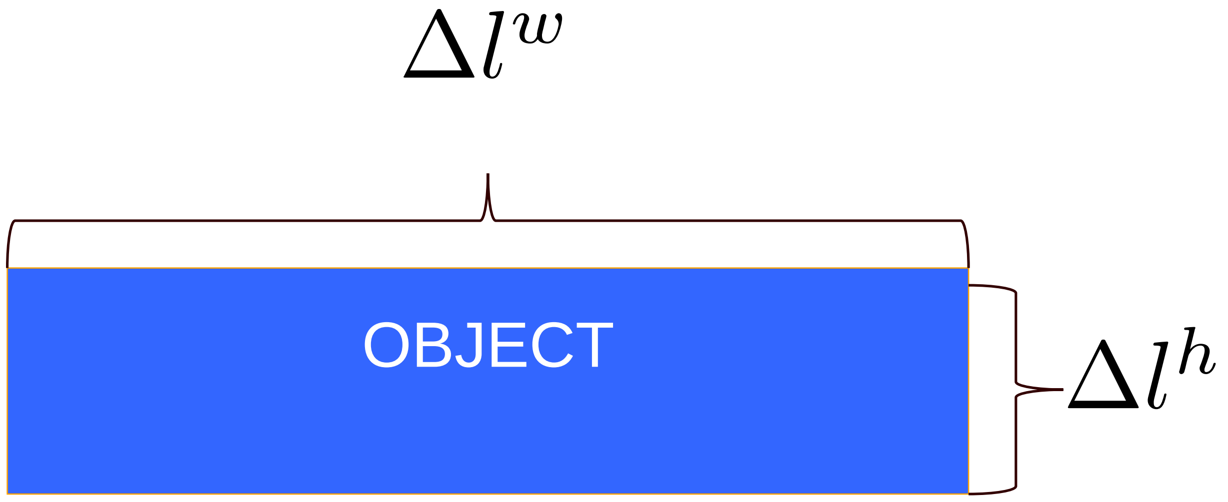


Simple Measurement: Poisson's Ratio



Simple Measurement: Poisson's Ratio

Compute changes in width and height

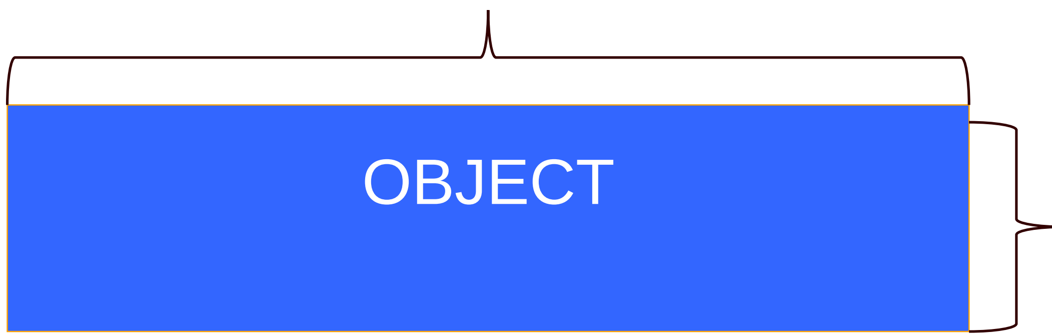


Simple Measurement: Poisson's Ratio

$$\nu = \frac{\Delta l_h}{\Delta l_w}$$

Poisson's Ratio

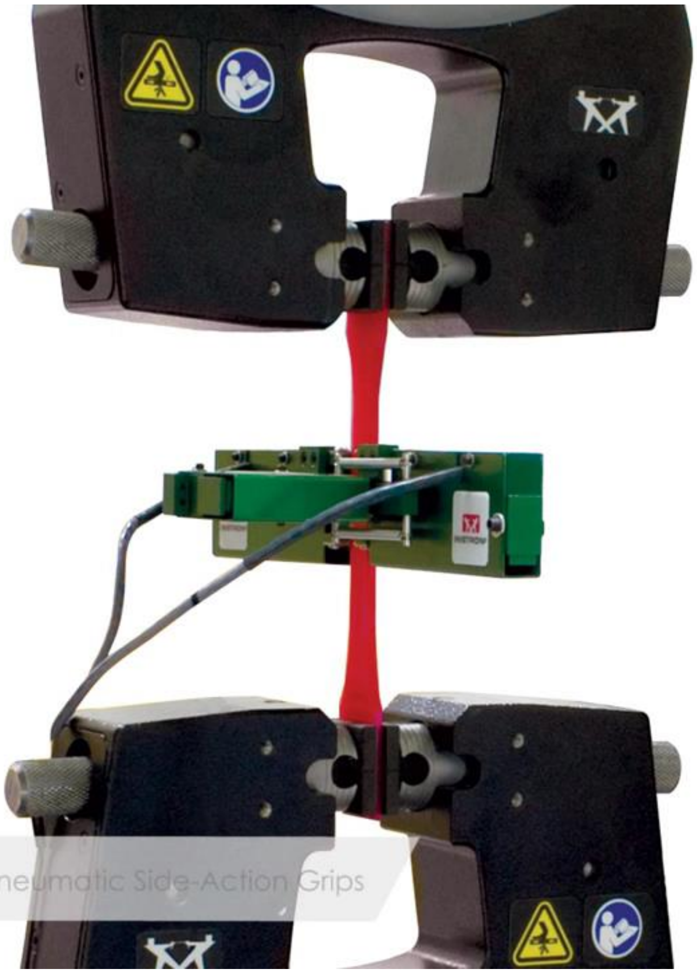
$$\frac{\Delta l^w}{l_0^w}$$



$$\frac{\Delta l^h}{l_0^h}$$

Measurement Devices

INSTRON

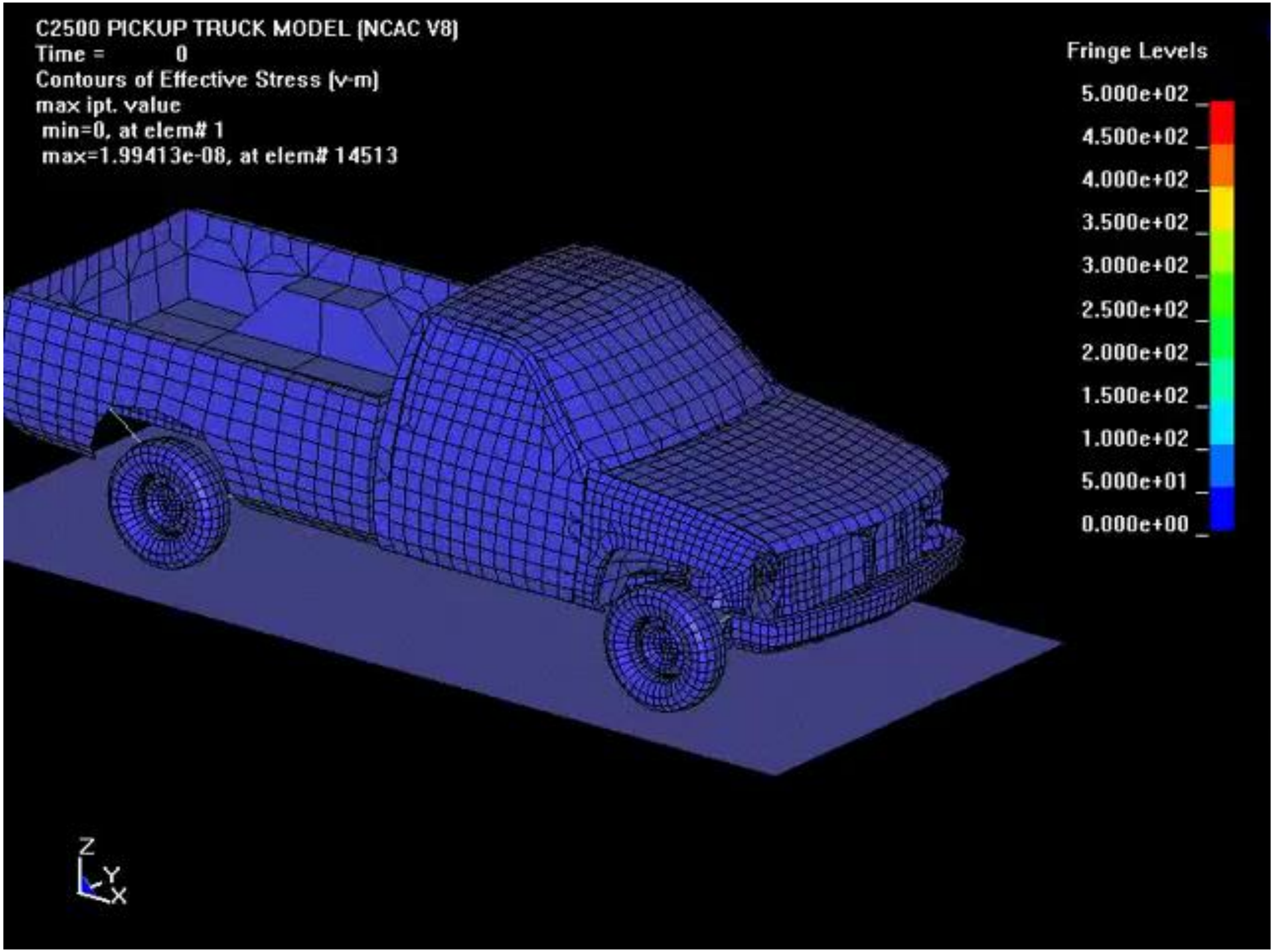


Bi-Axial Extensometer with 5 kN Pneumatic Side-Action Grips

Simulating Elastic Materials with CM+FEM

- You now have all the mathematical tools you need

Suppose you want to simulate this…



Plastic and Elastic Materials

- Elastic Materials
  - Objects return to their original shape in the absence of other forces
- Plastic Deformations:
  - Object does not always return to its original shape

Example: Crushing a Coke Can

Old Reference State **New Reference State**

Example: Crushing a van

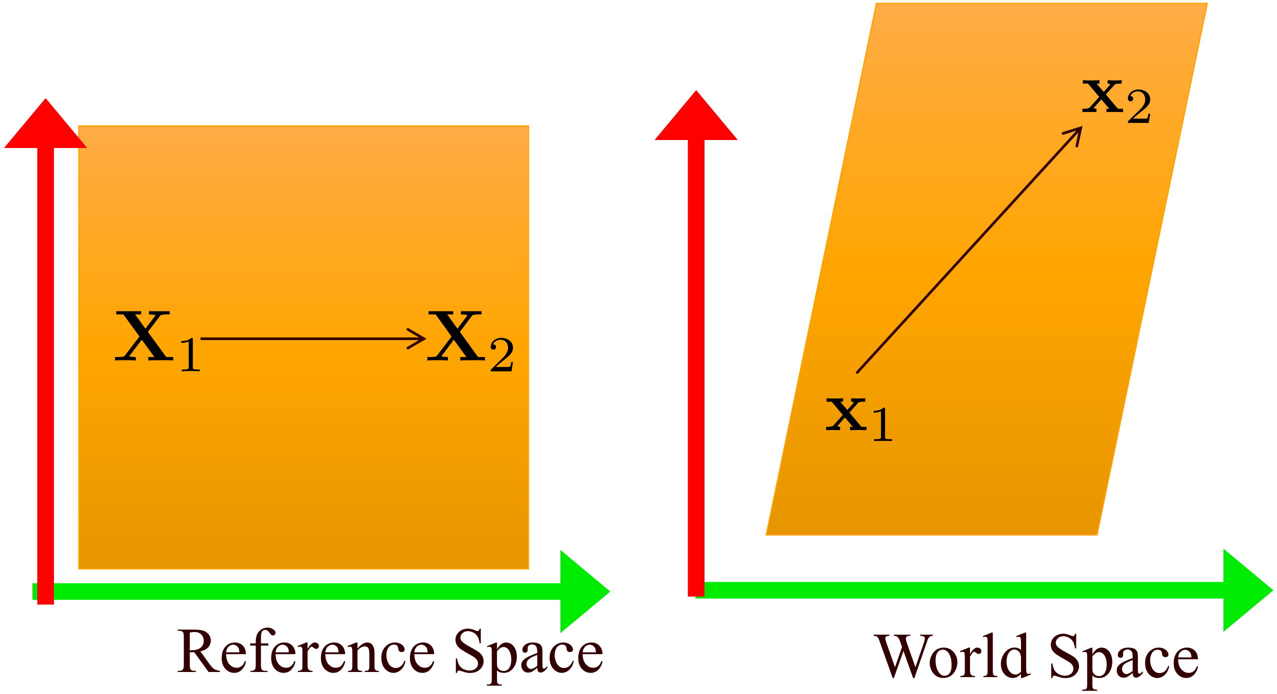
A Simple Model For Plasticity

- Recall our model for strain: $\frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$
- Let's consider how to encode a change of reference shape into this metric
  - Changing undeformed mesh is not easy!
- We want to exchange **F** with ${}^{w}_{p}\mathbf{F}$, a deformation gradient that takes into account the new shape of our object

New Reference State

Continuum Mechanics: Deformation

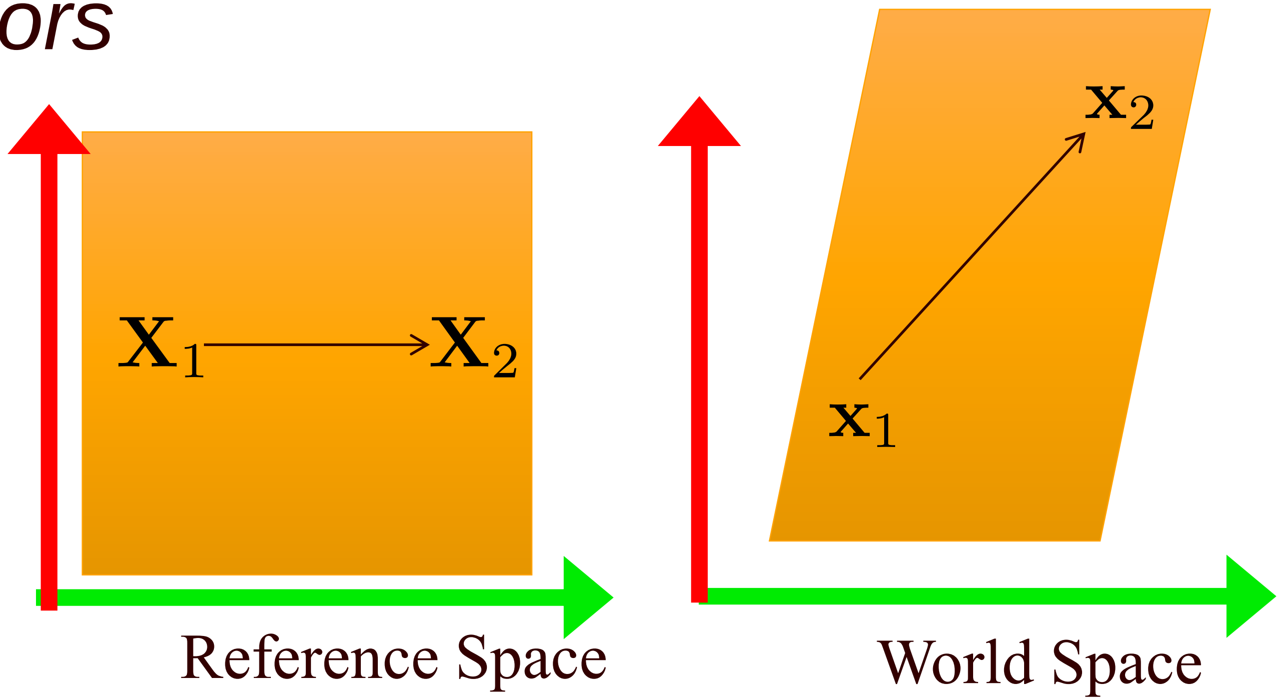
- ***deformation gradient*** *maps undeformed vectors (local) to deformed (world) vectors*



$$\mathbf{dx} \approx \mathbf{F}\mathbf{dX}$$

Continuum Mechanics: Deformation

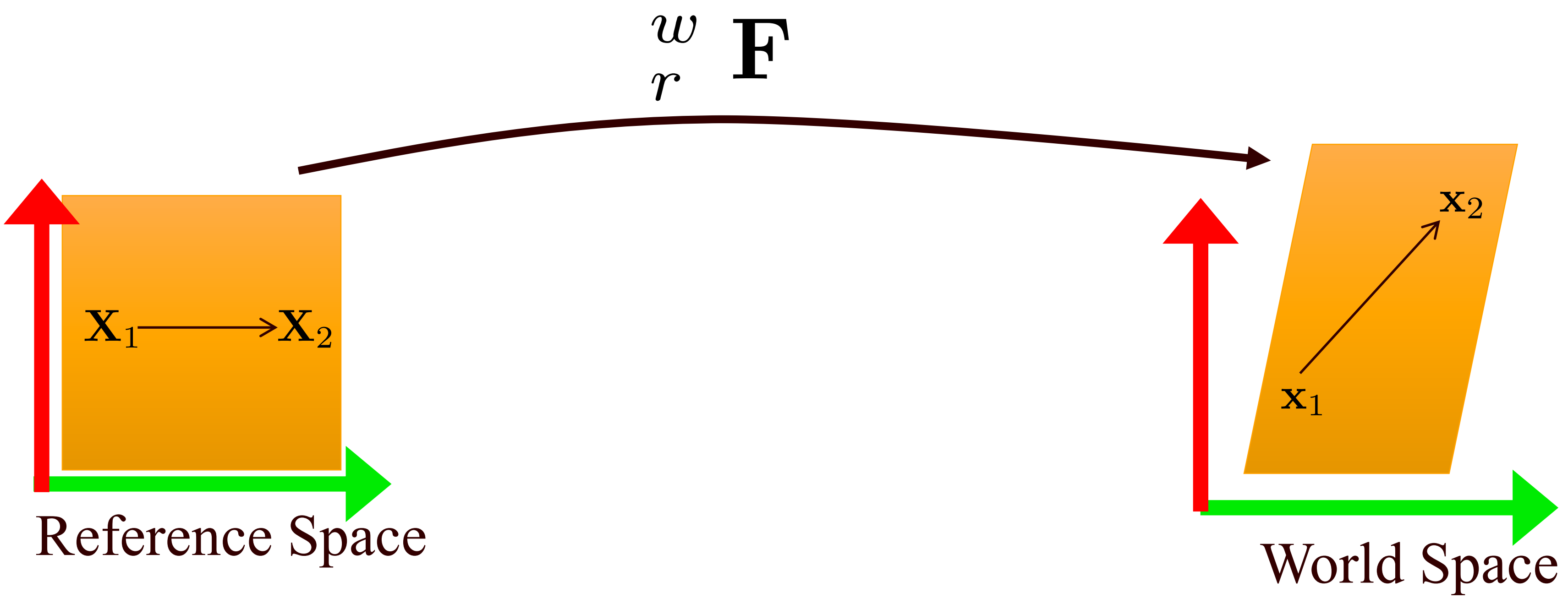
- ***deformation gradient*** *maps undeformed vectors (reference) to deformed (world) vectors*



$$\mathbf{dx} \approx_r^w \mathbf{F}\mathbf{dX}$$

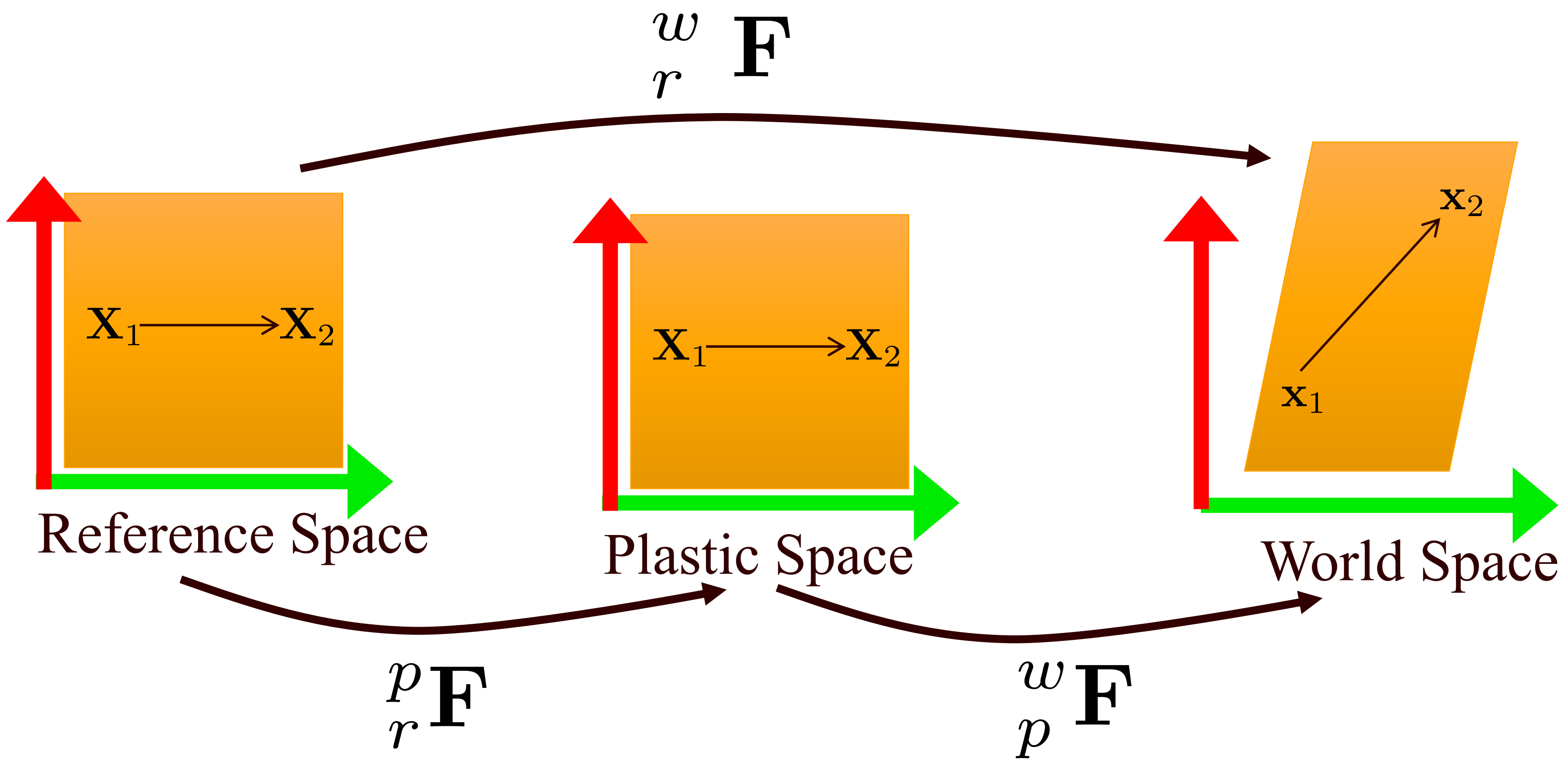
Continuum Mechanics: Deformation

- $\mathbf{F}$ transforms a vector from Reference space to World Space



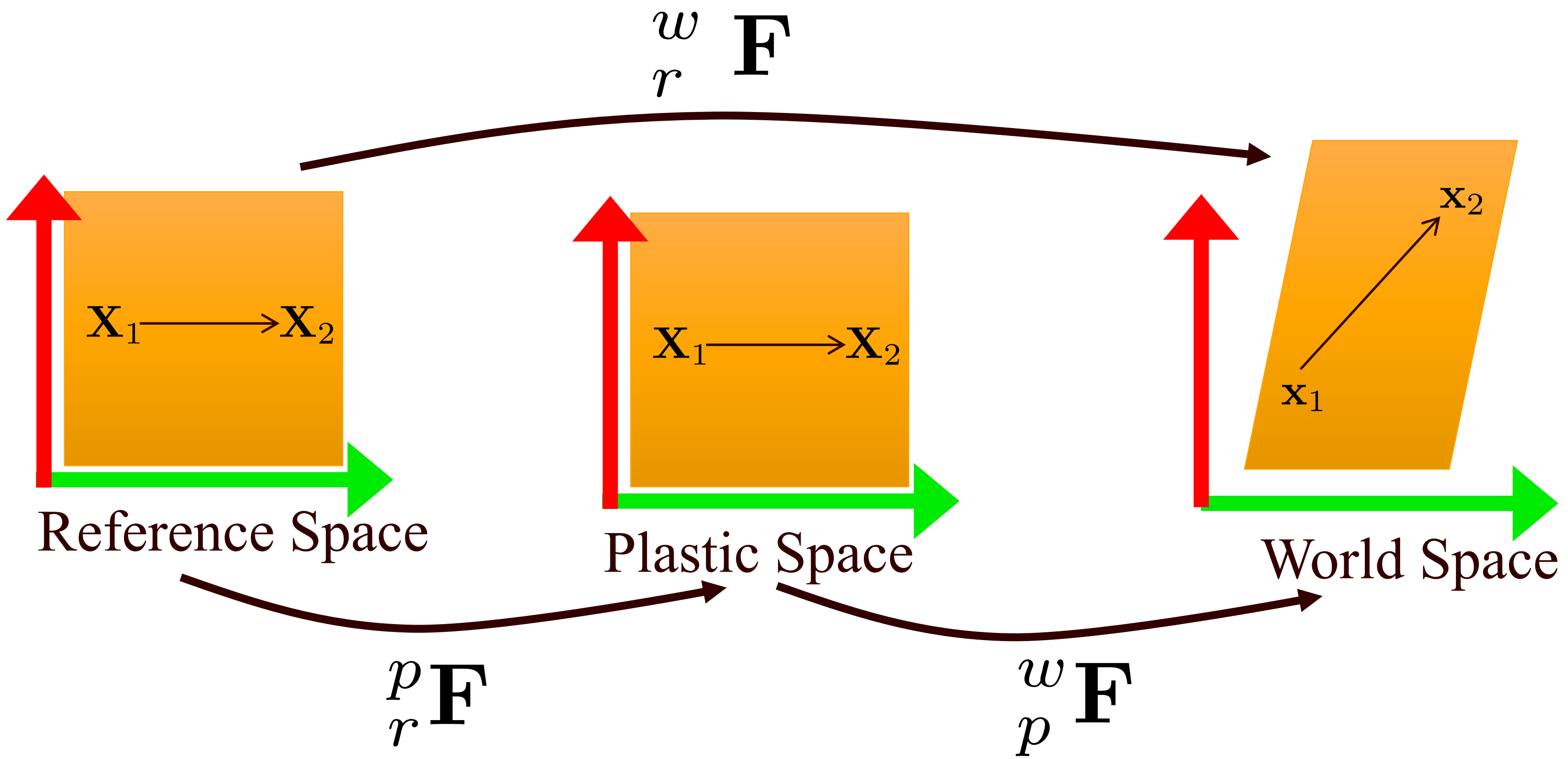
Continuum Mechanics: Deformation

- Introduce a new space



Continuum Mechanics: Deformation

- Our goal is to use ${}^{w}_{p}\mathbf{F}$ but we only have access to ${}^{w}_{r}\mathbf{F}$



Continuum Mechanics: Deformation

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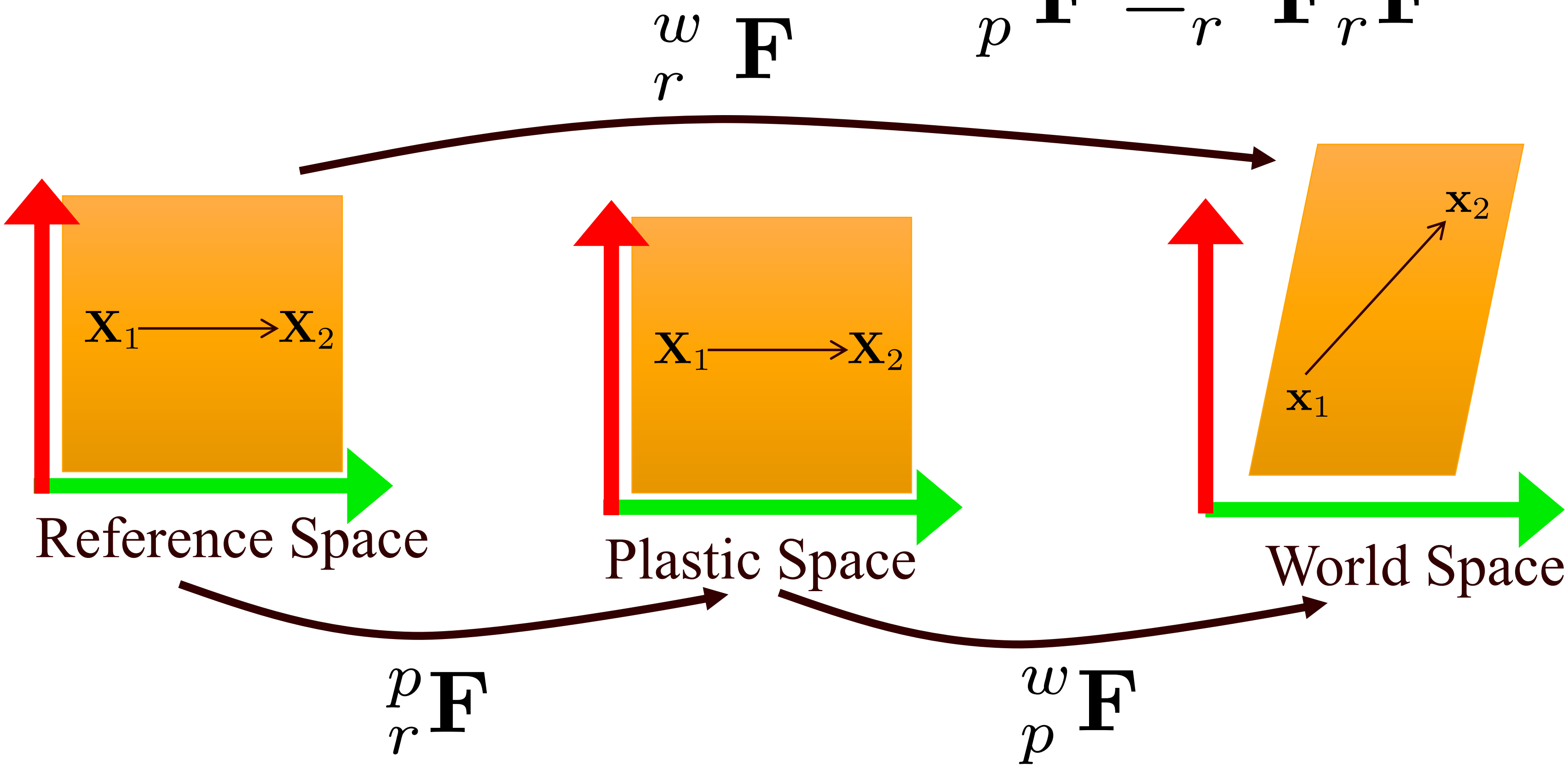
$${}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1}$$



Continuum Mechanics: Deformation

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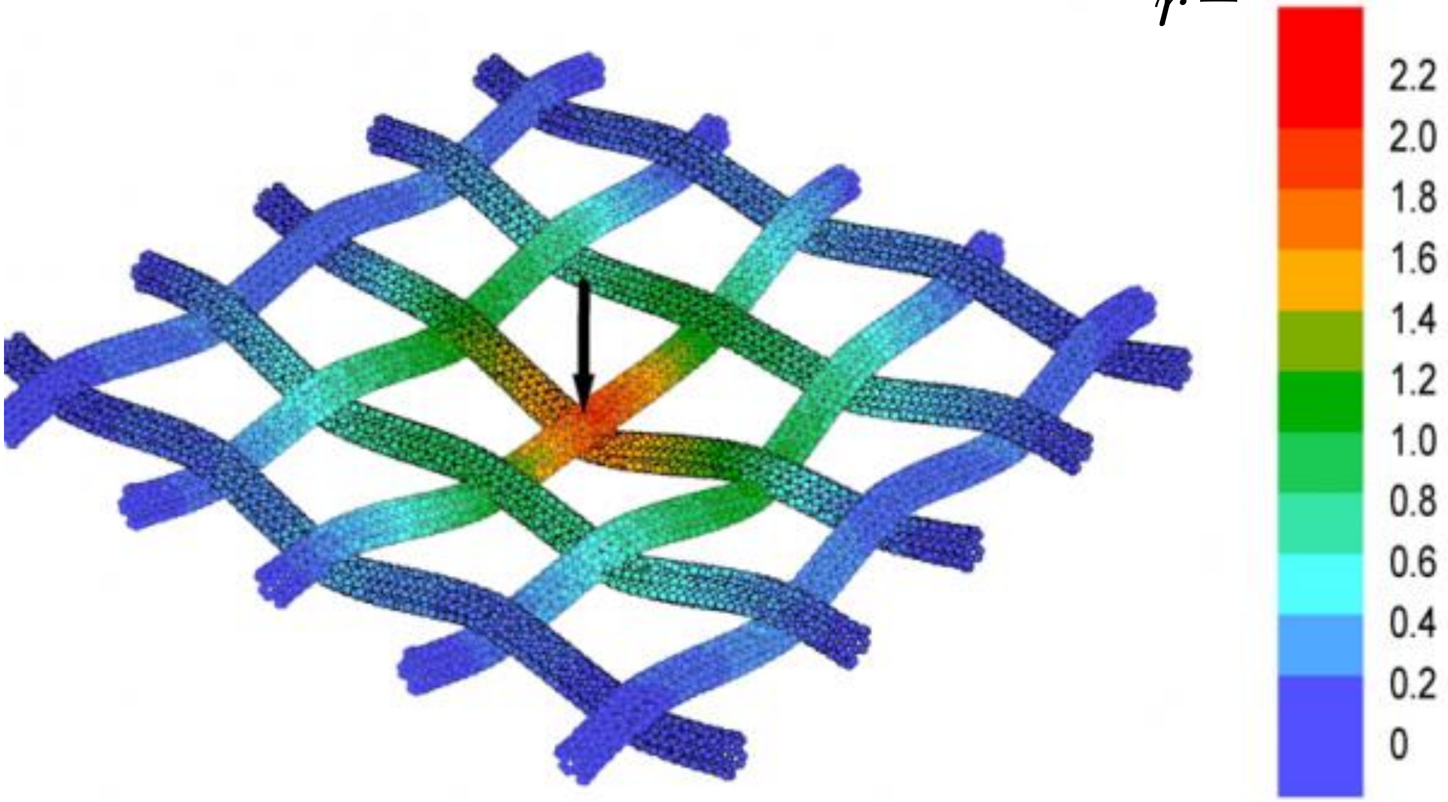
$$ {}^w_p\mathbf{F} = {}^w_r\mathbf{F}\,{}^p_r\mathbf{F}^{-1} $$

- Keep an estimate of ${}^p_r\mathbf{F}^{-1}$ per element, built incrementally

How to Compute the Plastic Deformation Gradient

- We compute the strain/stress for each element during simulation
- When it gets above a certain threshold store **F** as ${}^{p}_{r}\mathbf{F}$



How to Compute the Plastic Deformation Gradient

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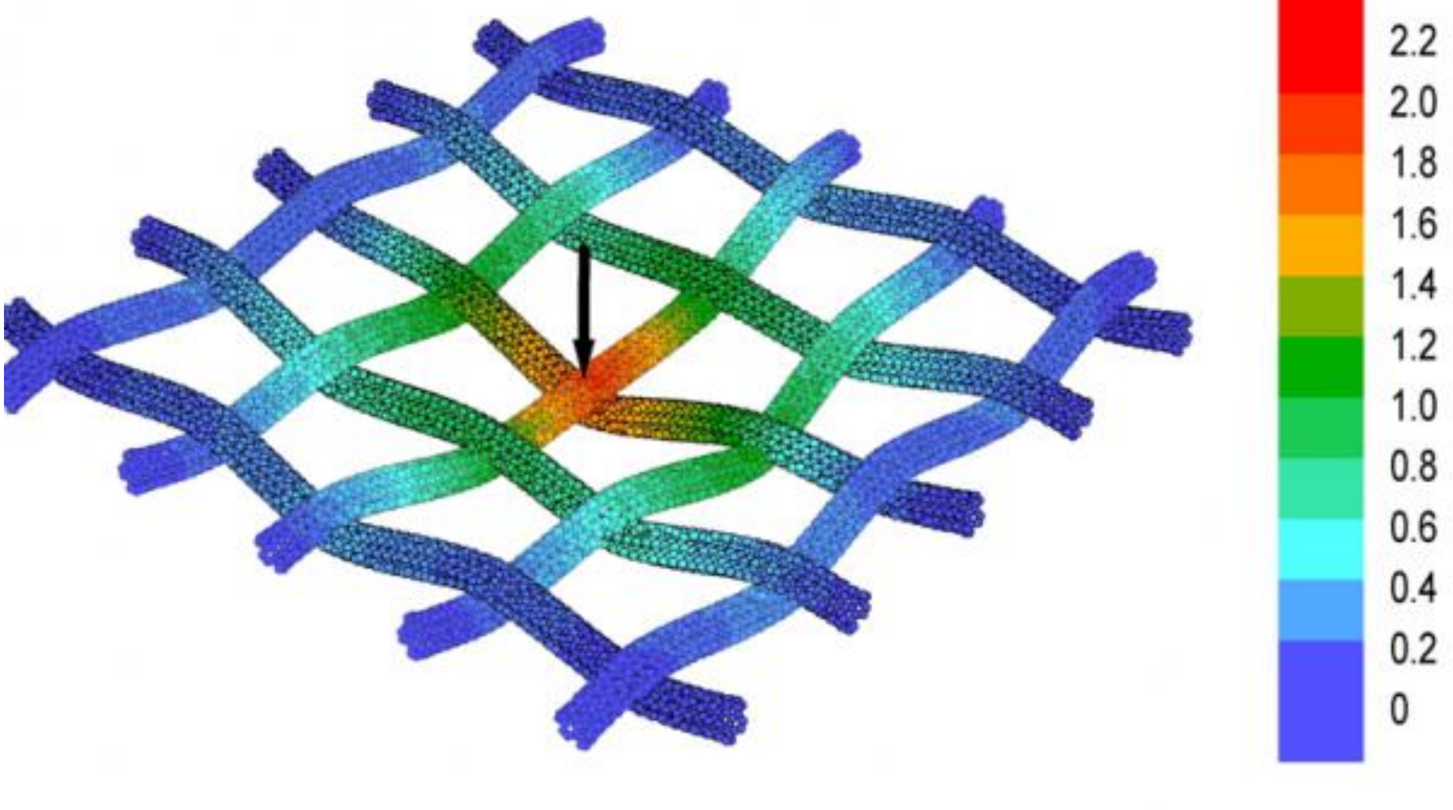
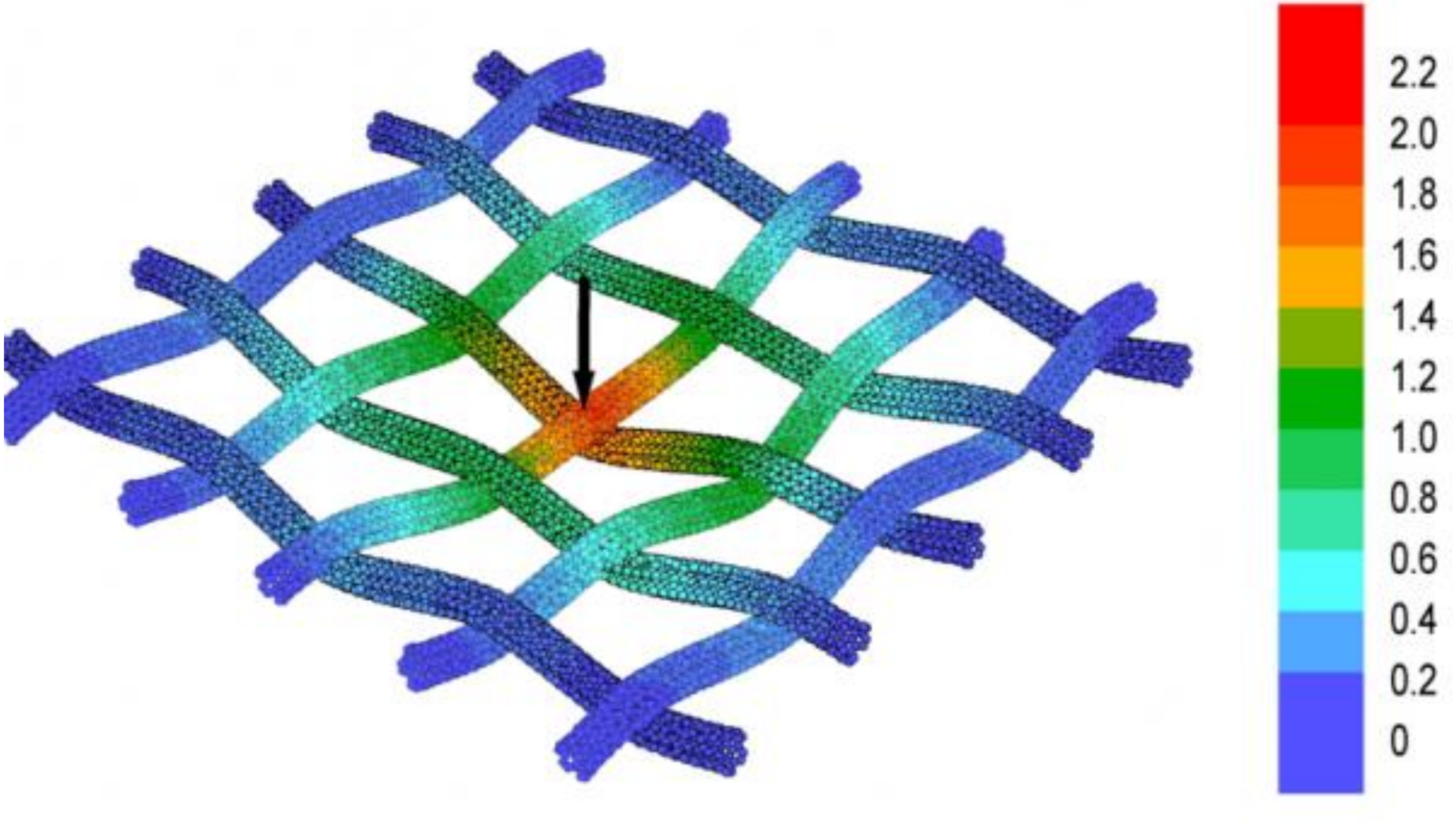


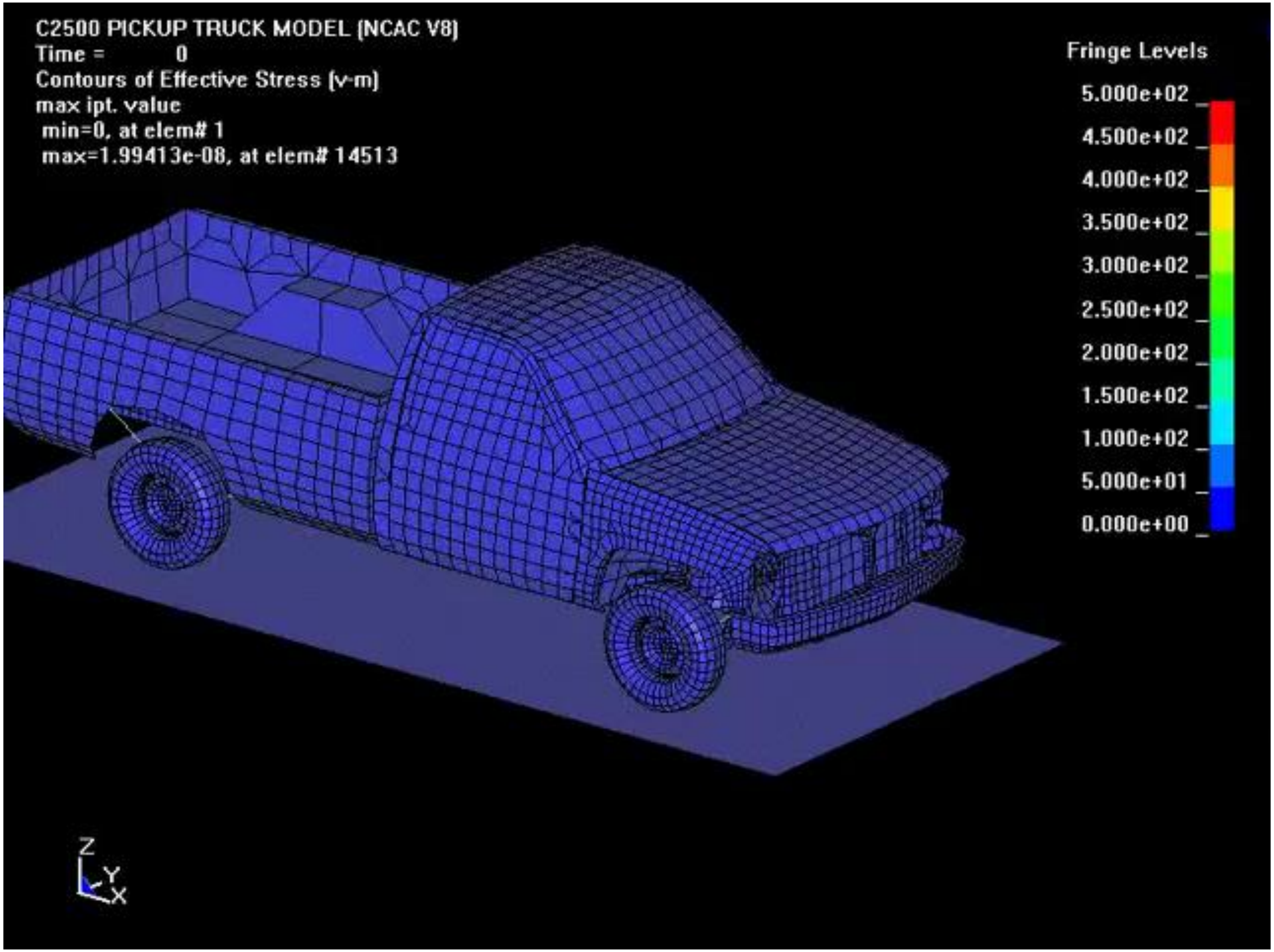
How to Compute the Plastic Deformation Gradient

- Each subsequent simulation step uses

$$ {}^{w}_{p}\mathbf{F} = {}^{w}_{r}\mathbf{F}\,{}^{p}_{r}\mathbf{F}^{-1} $$



So now you too can simulate this…



Questions?