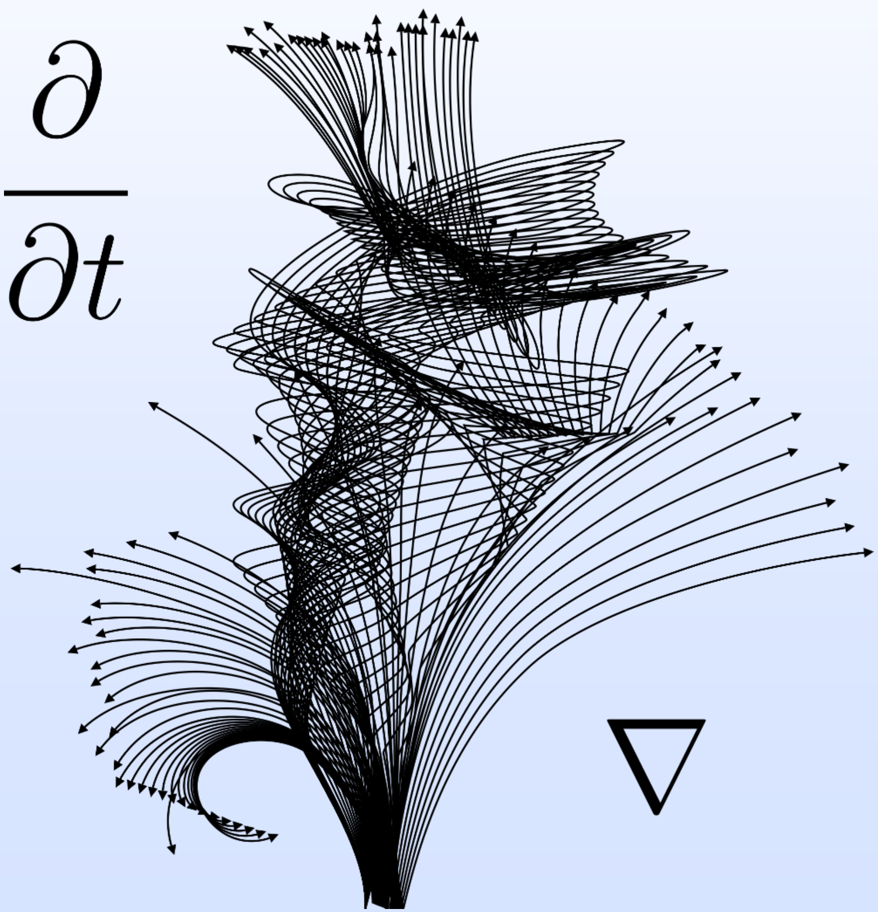
Partial Differential Equations

$$\frac{\partial}{\partial t}$$



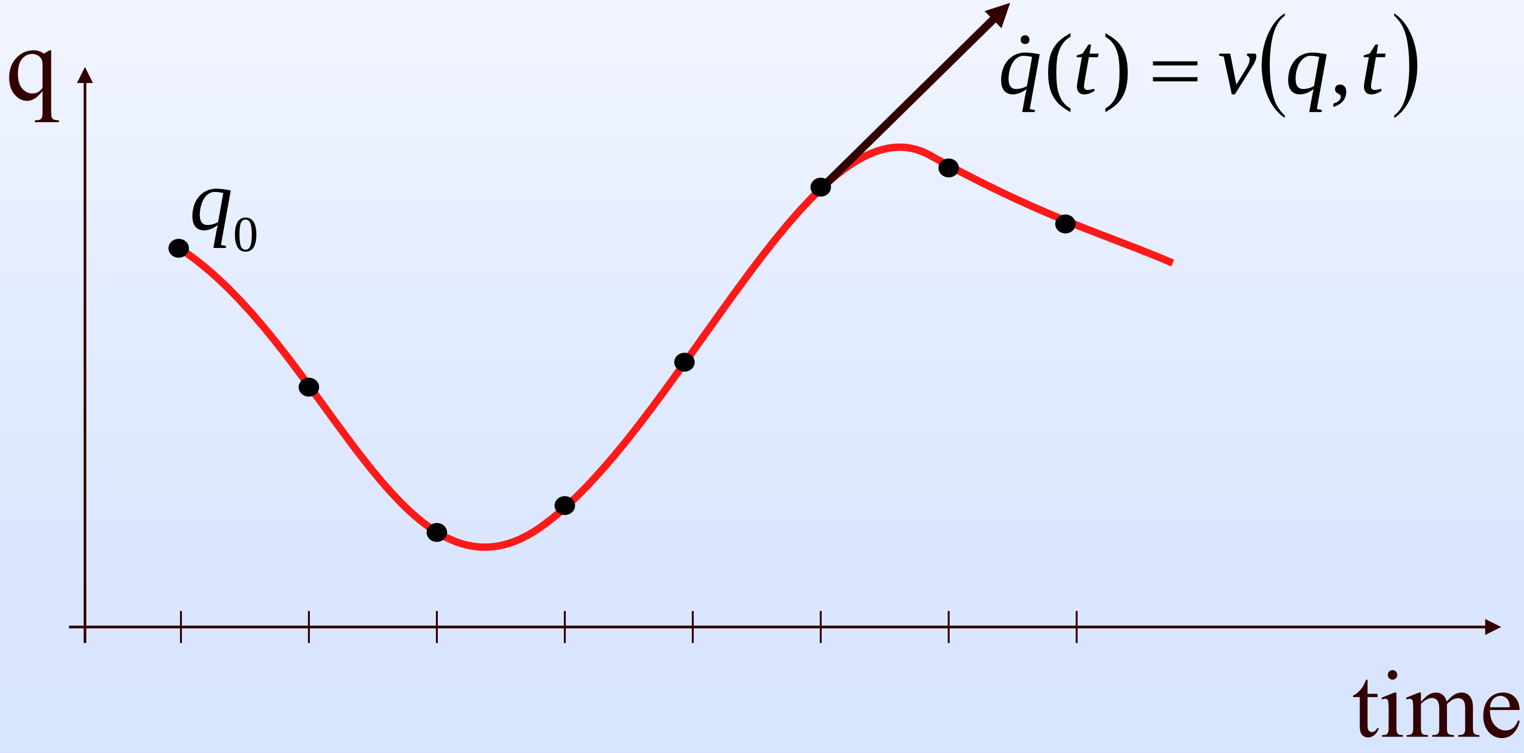
$$\nabla$$

Next week

- Tuesday: guest lecture (Prof. Keenan Crane)
- Thursday: in-class test

Recap

- ODEs: implicitly define a function through its time derivative



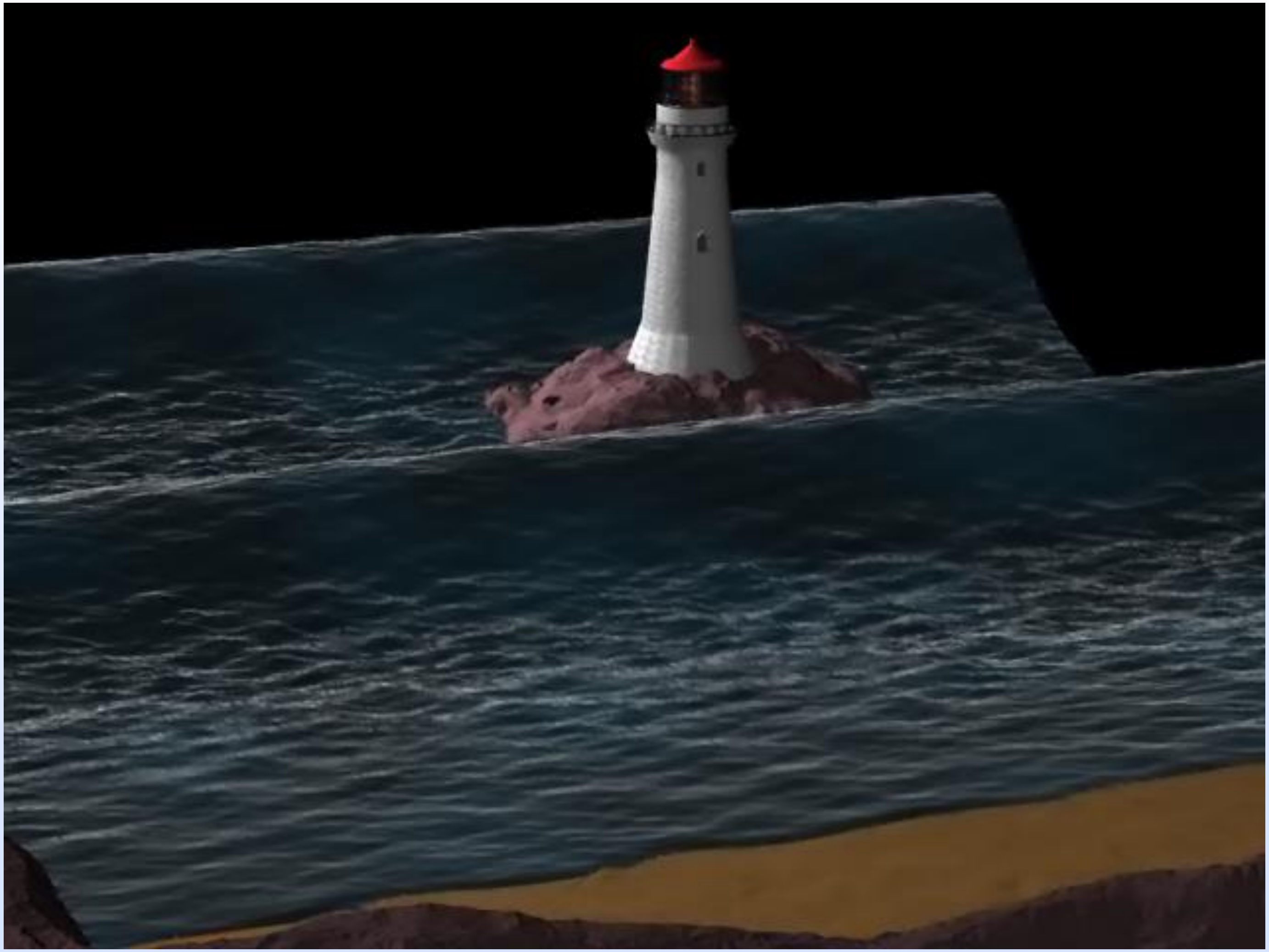
Recap

- What does an ODE for exponential decay look like?
  - It could be a very crude model for how the temperature of a particle changes through time…
  - But where does the heat go?
    - It's a PDE!

Partial Differential Equations

- PDEs: implicitly define a function through its derivatives with respect to time *and* space
- Most physical phenomena and processes can be described by partial differential equations

Partial Differential Equations



Partial Differential Equations

- ODEs: want to solve for function of time

$$\text{Example ODE:} \quad m\frac{d^2x(t)}{dt^2} = F(x(t))$$

- PDEs: want to solve for function of time and space

$$\text{Example PDE:} \quad \frac{\partial u(t,x)}{\partial t} = c\frac{\partial^2 u(t,x)}{\partial x^2}$$

Partial Differential Equations

- Definition of PDE

$$u(t,x)$$

- Function implicitly given in terms of derivatives

$$\dot{u}, \ddot{u}, \frac{d}{dt^3}u, \ldots$$

Any combination of time derivatives

$$\frac{\partial u}{\partial x_1}, \frac{\partial^2 u}{\partial x_2^2}, \frac{\partial}{\partial x_1}\frac{\partial u}{\partial x_2}, \ldots$$

Plus any combination of space derivatives

Partial Differential Equations

Abbreviation $$u_{tt} = \frac{\partial^2}{\partial t^2} u(t,..), \quad u_{xy} = \frac{\partial^2}{\partial x \, \partial y} u(x, y,..)$$

Linear vs non-linear PDEs

Linear example

$$u_{tt} = -c u_{xx}$$

wave equation

Nonlinear example

$$u_t + u \cdot u_x = u_{xx}$$

Burgers' equation

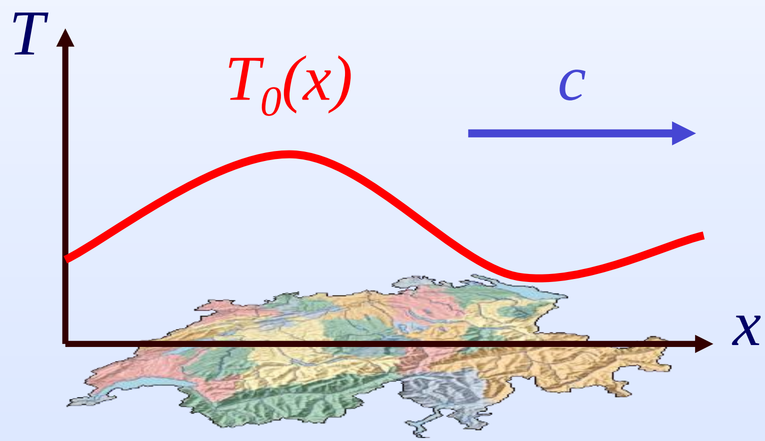
Order of a PDE: how many derivatives in space and time?

- wave equation: 2nd order in time, 2nd order in space
- Burger's equation: 1st order in time, 2nd order in space

PDEs: a first example

- A linear, first order PDE: 1D advection

Weather forecast: simulate temperature evolution.



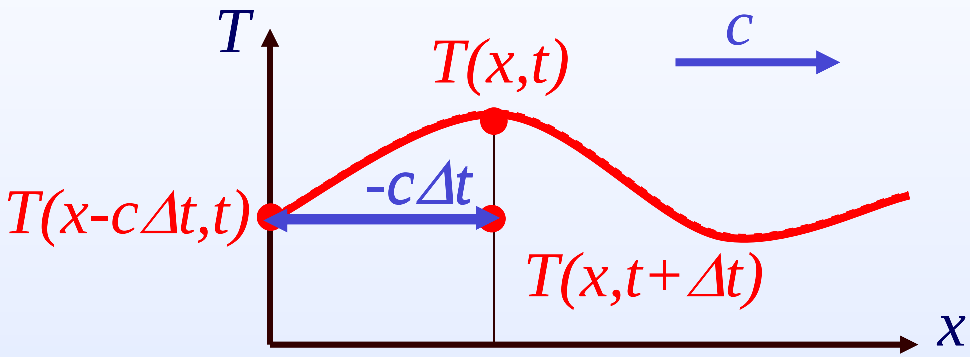
One space dimension + time

Given: initial temperature distribution $T_{0(x)} = T(x, 0)$, wind speed c.

Find: temperature distribution $T(x, t)$ for any t, x.

PDEs: a first example

- How does the temperature change over a time interval Δt?

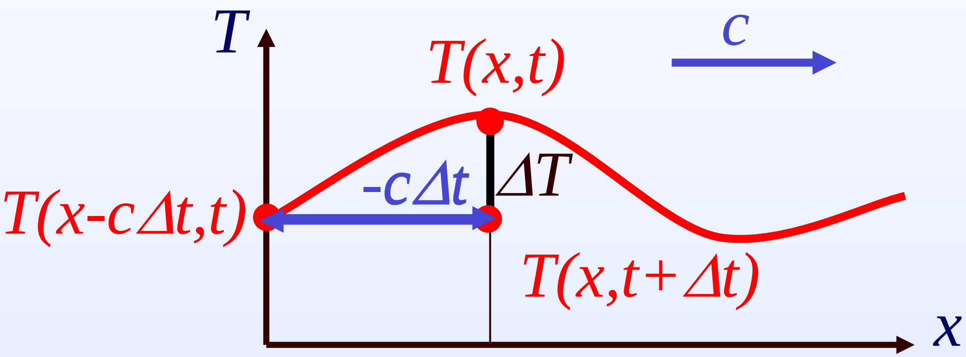


At a given location in space, some time later, what will the temperature be?

Where was this air front at time *t*?

PDEs: a first example

- How does the temperature change over a time interval Δt?



$$T(x,t+\Delta t) = T(x-c\Delta t,t)$$

$$\Delta T = T(x,t+\Delta t) - T(x,t)$$

$$T(x-c\Delta t,t) = T(x,t) - \frac{\partial T}{\partial x} c\Delta t + O(\Delta t^2) = T(x,t+\Delta t)$$

$$\frac{\Delta T}{\Delta t} \approx -c\frac{\partial T}{\partial x} \quad \xrightarrow{\Delta t \to 0} \quad \frac{\partial T}{\partial t} = -c\frac{\partial T}{\partial x}$$

1D advection equation

PDEs: a first example

- Analytical solution:
  - Want a function $T(x,t)$ that satisfies $$\frac{\partial T}{\partial t} = -c\frac{\partial T}{\partial x}$$

- Any $T(x,t)$ of the form $T(x,t) = f(x-ct)$ will work! How do we choose one of the many options?

- The solution also needs to satisfy the initial condition:

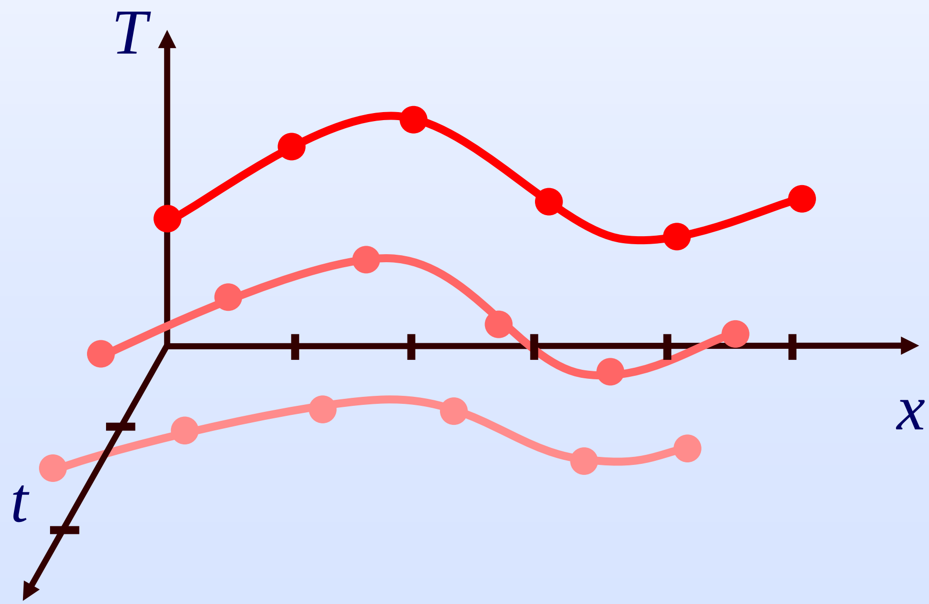
$$T(x,0) = T_0(x)$$

- The solution is thus $T(x,t) = T_0(x-ct)$

Note: *only simple PDEs can be solved analytically!*

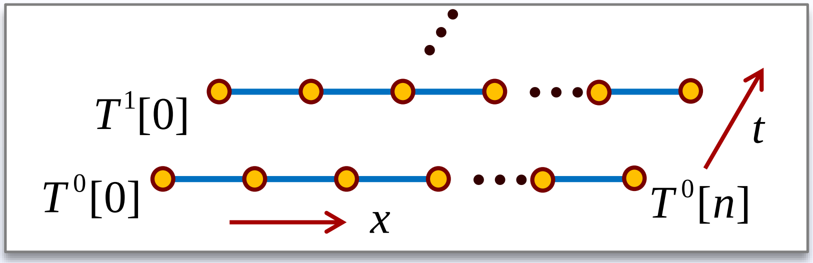
PDEs: a first example

- Numerical solution:
  - Want to approximate function *T*(*x*,*t*) by discretizing in space and time (estimate T(x,t) at different (x_i,t_i))



PDEs: a first example

- Sample temperature $T(x,t)$ on 1D grids $T^t[i] = T(i \cdot h, t \cdot \Delta t)$ with $i \in (1,..,n), t \in (0,1,2..)$



- Discretize derivatives with **finite differences** (*space & time*)

$$\frac{\partial T}{\partial t} = -c \frac{\partial T}{\partial x} \Rightarrow \frac{T^{t+1}[i] - T^t[i]}{\Delta t} = -c \frac{T^t[i] - T^t[i-1]}{h}$$

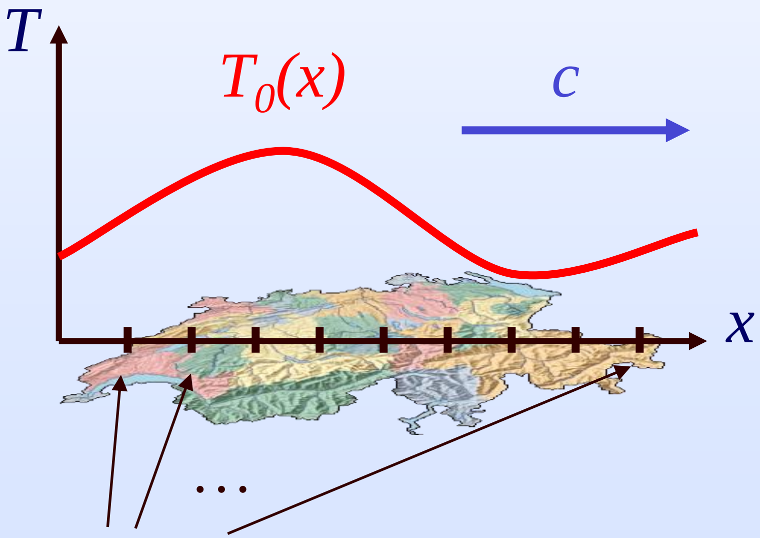
- Solving for $T^{t+1}[i]$ yields update rule

$$T^{t+1}[i] = T^t[i] - \Delta t \cdot c \frac{T^t[i] - T^t[i-1]}{h}$$

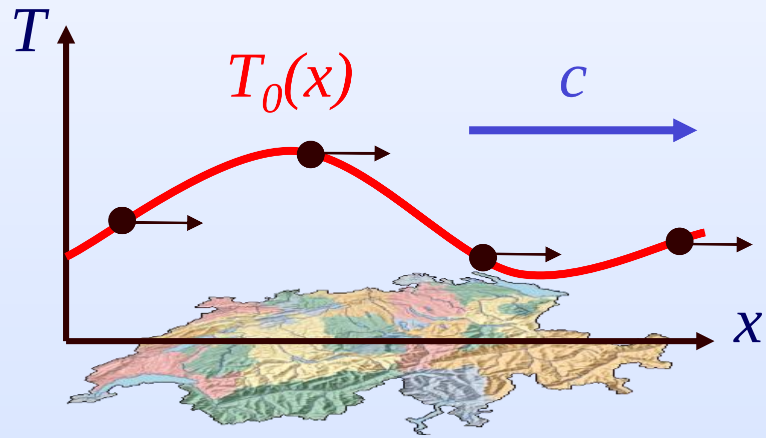
- Provide initial values $T^0[i]$
- Need some boundary conditions $T^t[0]$!

PDEs: a first example

- Eulerian vs Lagrangian viewpoint



Monitor temperature at fixed locations in space



Release weather balloons and see where they end up (the temperature they record does not change)!

Basic Recipe for solving PDEs numerically

- Pick a spatial discretization
- Pick a time discretization (forward Euler, backward Euler, etc)
- Run a time-stepping algorithm using resulting update rule
- We will see a few more examples…

Mathematical Background

- Many, many types of PDEs
- 2nd order linear PDEs are most important for us
- They involve the Laplace operator ("average" curvature)

  - Nabla operator $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right)^t \quad \nabla s = \left(\frac{\partial s}{\partial x_1}, \ldots, \frac{\partial s}{\partial x_d} \right)^t$
  - Laplace operator $\Delta = \nabla \cdot \nabla = \nabla^2 = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$

(in *d* dimensions)

PDE classification

- 2^{nd} order linear PDEs are of highest practical relevance ⇨ *dedicated classification*

- A 2^{nd} order linear PDE in 2 variables (x,y) has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

- A 2^{nd} order linear PDE in 2 variables is
  - **Hyperbolic** $B^2 - AC > 0$ *(Wave equation)*
  - **Parabolic** $B^2 - AC = 0$ *(Heat equation)*
  - **Elliptic** $B^2 - AC < 0$ *(Laplace equation)*

Elliptic PDEs

Elliptic 2^{nd} –order PDEs ($B^2\text{-}AC < 0$)

- describe static problems (*systems in equilibrium*)

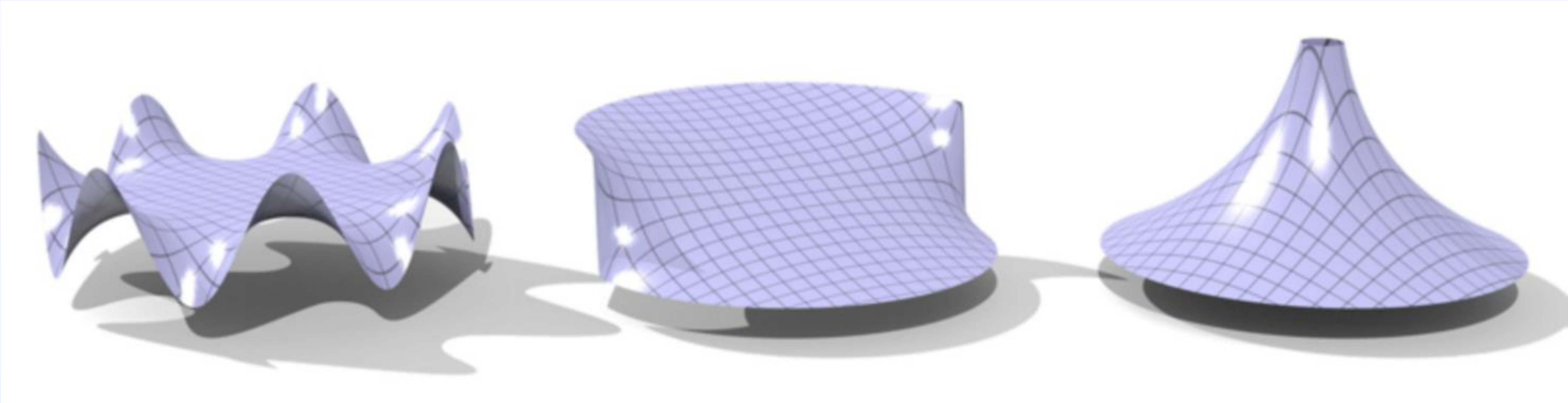
Prototype: **Laplace Equation** $\Delta u(\mathbf{x}) = 0$

Applications

- Steady-state solutions to parabolic PDEs
- Equilibrium problems

Laplace Equation

- What is the smoothest function given boundary data? Think elastic membrane clamped at boundaries.



- How do we solve it?

Laplace Equation

$$\Delta u(x,y) = \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = 0$$

Discretizing on a grid with cell size h:

$$\Delta u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2} = 0$$

- Solution: each point must be equal to average of “neighbors”
- Can iteratively keep setting to average of neighbor values (Jacobi method!) or set up a linear system. Very easy!

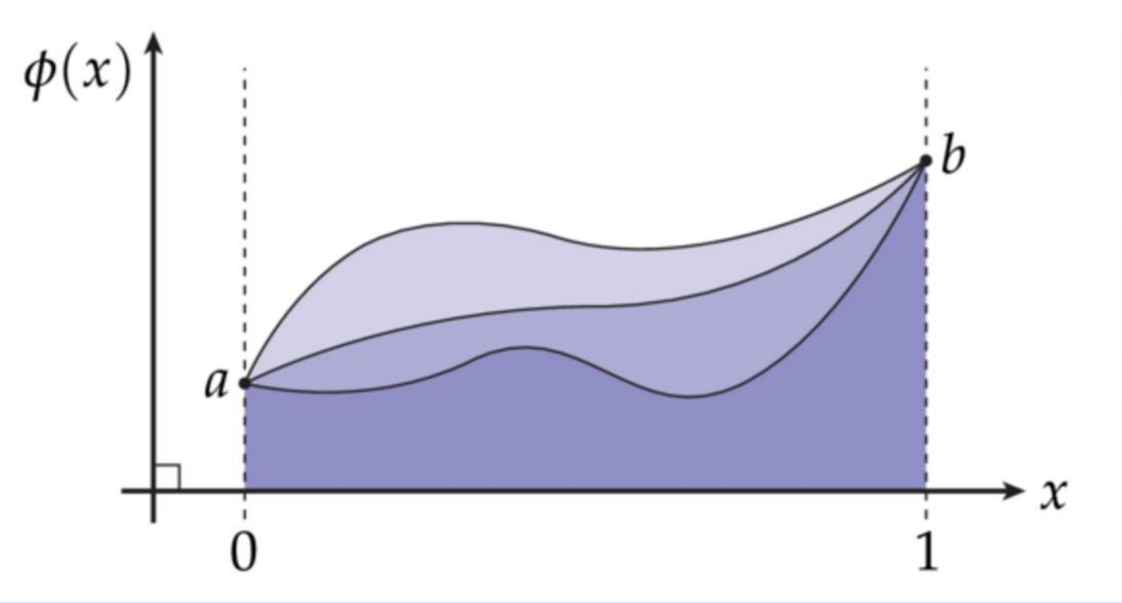
Boundary Conditions

- What happens at the “edges” of the simulation domain?
  - Need additional information
  - For ODEs, we had an initial value
  - For PDEs, we also need boundary conditions

- Two common choices:
  - Dirichlet – boundary data specifies values
  - Neumann – boundary data specifies derivatives

Boundary Conditions

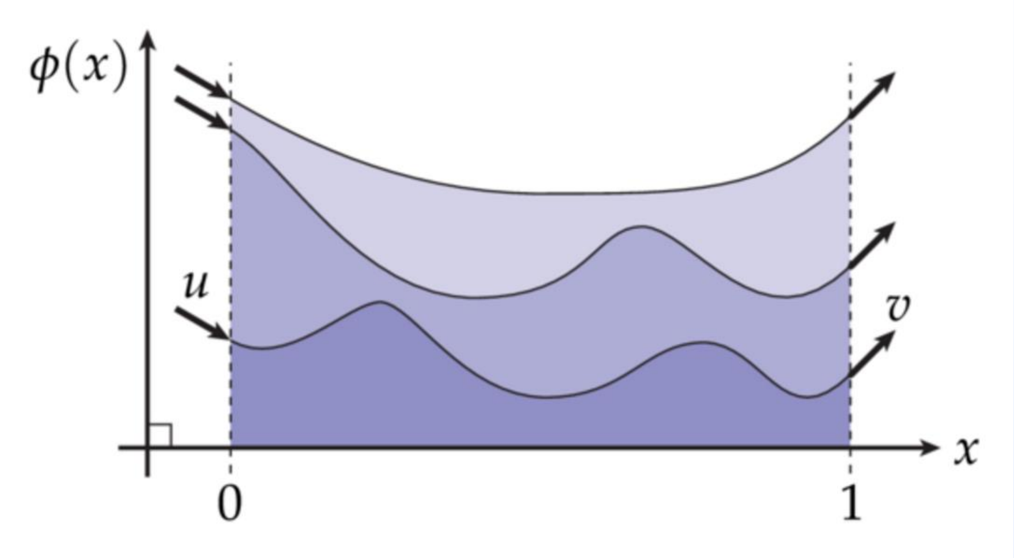
- *Dirichlet* Boundary Conditions: $\phi(0) = a, \phi(1) = b$



- Many possible functions! The Laplace equation gives the smoothest one.

Boundary Conditions

- *Neumann* Boundary Conditions: $\phi'(0) = u, \phi'(1) = v$



- Again, many possible functions!
- Some combinations of PDEs + boundary conditions may not have solutions!

Parabolic PDEs

Parabolic 2nd –order PDEs ($B^2\text{-}AC = 0$)

- Are typically time dependent problems
- Solutions *smooth out* as time increases

Prototype:

Heat Equation $$\frac{\partial u(\mathbf{x},t)}{\partial t} = c^2 \Delta u(\mathbf{x},t)$$

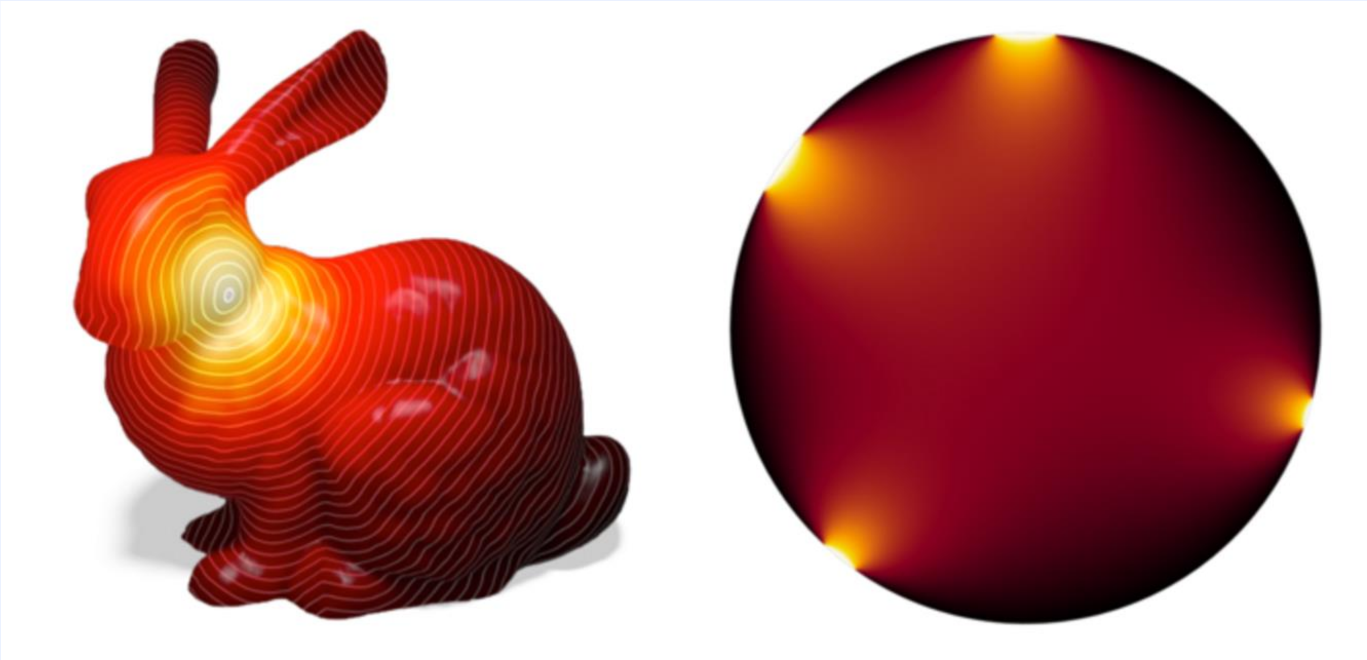
u temperature, α thermal diffusivity

Applications

- Heat conduction and general diffusion processes

The Heat Equation

- How does an initial distribution of heat spread out over time? After a long enough time, solution is the same as the Laplace equation.



The Heat Equation

- Know how to discretize the Laplacian

$$\Delta u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

- Know how to discretize time (e.g. forward Euler)

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t \Delta u_{i,j}^{n}$$

- Can easily put the two together…

Hyperbolic PDEs

Hyperbolic 2nd –order PDEs ($B^2-AC > 0$)

- time dependent problems
- Retain & propagate disturbances present in initial data

Prototype: **Wave Equation** $$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} = c^2 \Delta u(\mathbf{x},t)$$

u amplitude, c propagation speed

Applications

- Simulate wave propagation for sound, light, and water
- Mechanics (*oscillatory motion, vibrating strings*)

The Wave Equation

- How does a wave front propagate over time?