Mass-Spring Systems – Part 2

Schedule for the next few weeks

A1 will be out tonight

January	14	19	21	26	28
February	2	4	9	11	16

In-class test

A1 due Feb 15 at 11:59pm.
Demos and brief presentation during class on Feb 16. Cool prize for contest winner.

Start thinking of topics for the final project. Form teams. Talk to us as soon as possible.

Backward Euler – from last class

- Boils down to solving systems of linear equations:

$$\underbrace{\left(M - h\frac{\partial F}{\partial v} - h^2\frac{\partial F}{\partial x}\right)}_{\mathrm{A}}\underbrace{\Delta v}_{x} = \underbrace{M\left(v_n - v^k\right) + hF}_{\mathrm{b}}$$

- Matrix A is large, sparse, symmetric, (sometimes positive definite)
    - these characteristics will inform the choice of algorithm we can/should use to solve the systems of equations

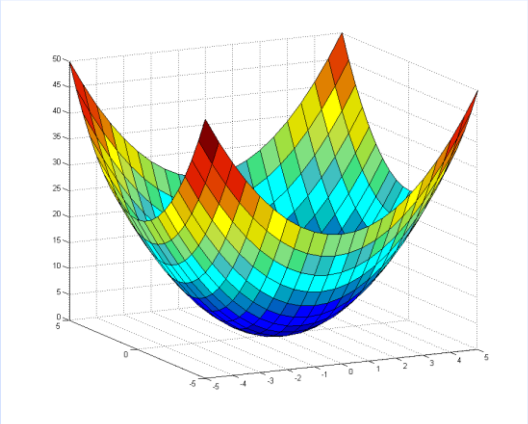
Symmetric Positive Definiteness

- Some solvers only work if **A** is symmetric positive definite:

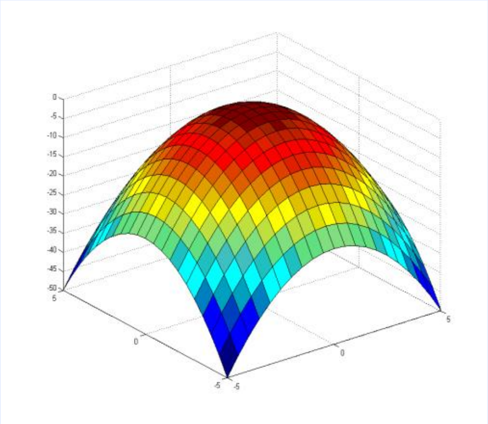
$$\mathbf{v}^{t}\mathbf{A}\mathbf{v} > 0 \; \forall \; \mathbf{v} \neq \mathbf{0}$$

- Think of a quadratic energy function (e.g. potential energy stored in spring):

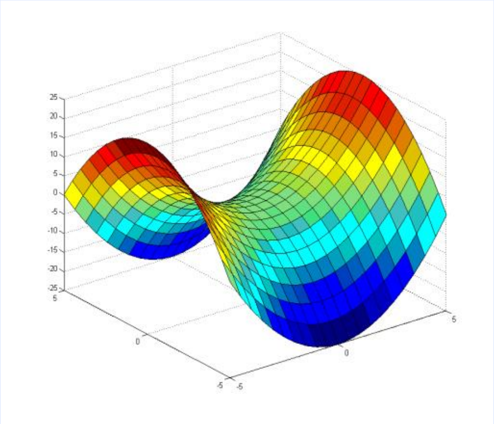
Positive Definite



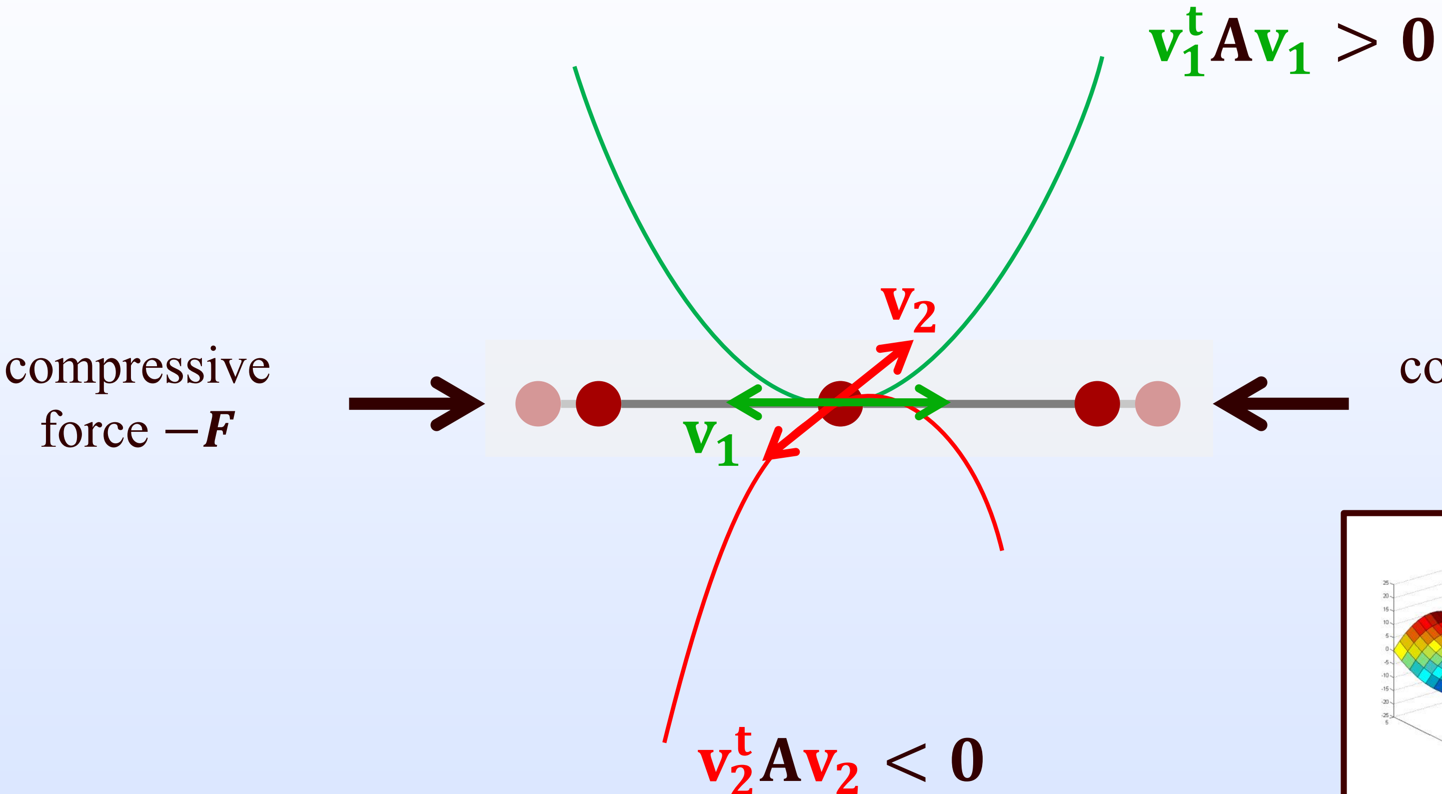
Negative Definite

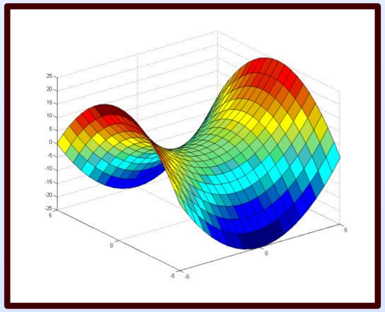


Indefinite



Analogy: Compressed Springs





⇨ $\mathbf{A}$ is indefinite, we are at a saddle point!
How can you tell which way particle should go?

Solving linear systems

$$Ax = b$$

- Direct Methods:
  - Explicitly compute inverse (e.g. via Gaussian Elimination)
  - decompose A (LU, LDL', etc), solve by exploiting structure

Solving linear systems

- LU decomposition:

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

$$Ax = LUx = Ly = b$$

$$Ux = y$$

Solving linear systems

- If A is symmetric, LU decomposition is unique, and is called Cholesky decomposition:

$$A = LDL^T$$

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ -4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 4 & & \\ & 1 & \\ & & 9 \end{pmatrix} \begin{pmatrix} 1 & 3 & -4 \\ & 1 & 5 \\ & & 1 \end{pmatrix}$$

```
//compute x = A^-1 * b
Eigen::SimplicialLDLT<SparseMatrix> solver;
solver.compute(A);
x = solver.solve(b);
```

Solving linear systems

$$Ax = b$$

- Direct Methods:
  - Gaussian Elimination
  - decompose A (LU, LDL', etc), solve by exploiting structure
  - Exact solution ~O(n^3) for dense matrices, constant varies

Solving linear systems

$$Ax = b$$

- Indirect Methods:
  - Iteratively improve approximate solution x^{k+1}
    - Can terminate when result is “good enough”
  - Gauss-Seidel & the Jacobi Method

Solving linear systems

- Gauss-Seidel

$$A = L_* + U \qquad \text{where} \qquad L_* = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$L_* \mathbf{x}^{(k+1)} = \mathbf{b} - U \mathbf{x}^{(k)},$$

- x^{k+1} can be computed in place, only one storage vector required

Solving linear systems

- Gauss-Seidel

$$A = L_* + U \qquad \text{where} \qquad L_* = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$L_* \mathbf{x}^{(k+1)} = \mathbf{b} - U \mathbf{x}^{(k)},$$

- x^{k+1} can be computed in place, only one storage vector required
- converges if A is symmetric positive-definite
- think of it as an iterative constraint solver

Solving linear systems

- Jacobi Method

$$A = D + R \qquad \text{where} \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ and } R = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}.$$

$$\mathbf{x}^{(k+1)} = D^{-1}(\mathbf{b} - R\mathbf{x}^{(k)}),$$

- x^{k+1} cannot be computed in place
- equivalent to solving each equation independently
- parallelizable

Solving linear systems

$$Ax = b$$

- Indirect Methods:
  - Iteratively improve approximate solution x^{k+1}
    - Can terminate when result is "good enough"
  - Gauss-Seidel & the Jacobi Method
  - Gradient Descent & Conjugate Gradient Method

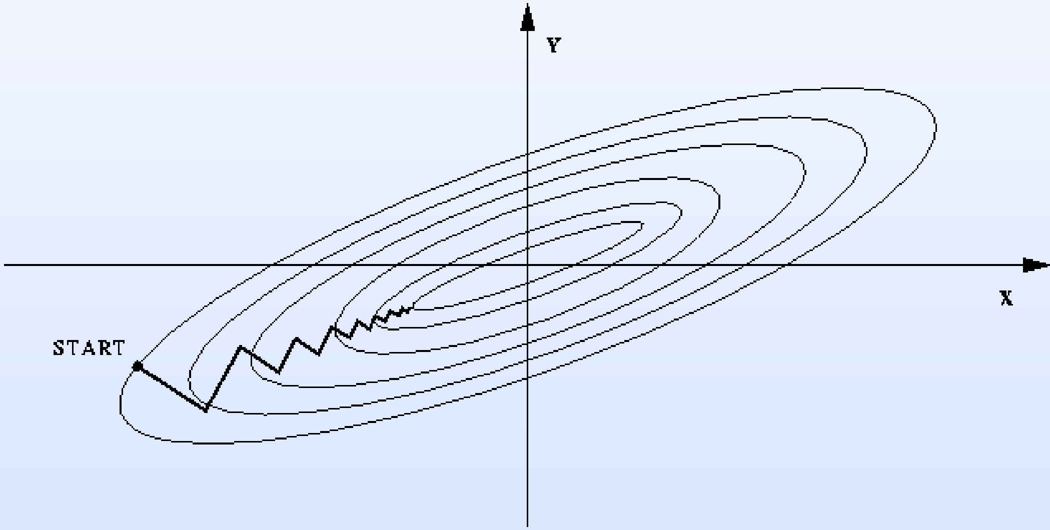
Solving linear systems

- Gradient Descent:

$$\min_x \frac{1}{2} x^T A x - x^T b$$

$$r^k = b - Ax^k$$

$$x^{k+1} = x^k + \alpha r^k$$



- Slow convergence, too much backtracking…

Solving linear systems

- The Conjugate Gradient Method

Main idea:

- find basis (p_1, p_2,…) of conjugate search directions (orthogonal with respect to generalized dot product $a^T A b$=0)

- compute step α (independently!) along each direction such that

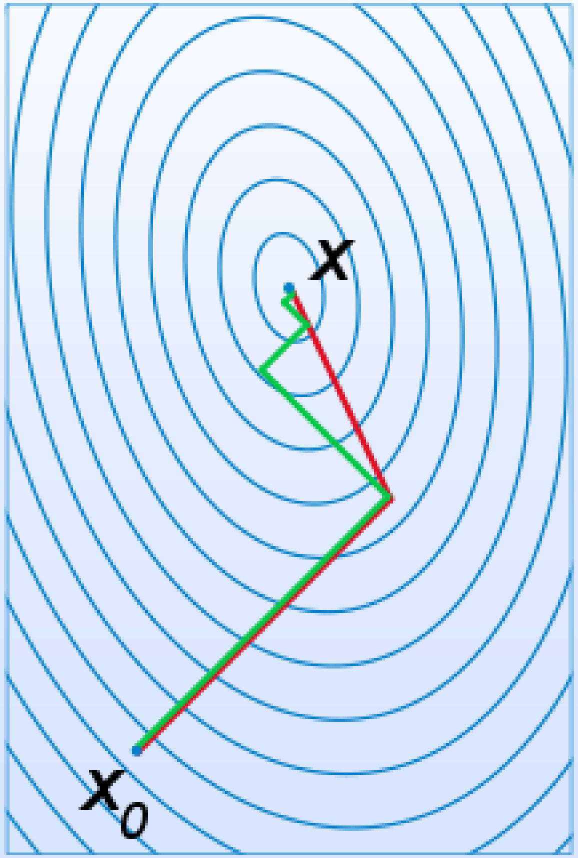
$$x = \sum \alpha_i p_i$$

- Build basis iteratively. E.g, if first step was along direction p_1 and gradient at step 2 is $r_2 = \alpha A p_1 - b$, direction for step 2 is:

$$p_2 = r_2 - \frac{p_1^T A r_2}{p_1^T A p_1}$$

Solving linear systems

- Gradient Descent vs Conjugate Gradients



"An Introduction to the Conjugate Gradient Method Without the Agonizing Pain"

- Jonathan Richard Shewchuk

Solving linear systems

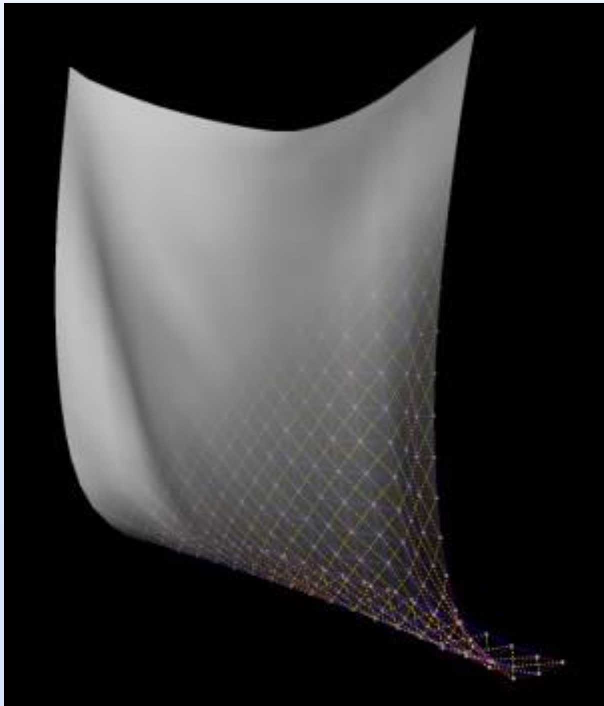
$$Ax = b$$

- Indirect Methods:
  - Iteratively improve approximate solution x^{k+1}
    - Can terminate when result is “good enough”
  - Gauss-Seidel & the Jacobi Method
  - Gradient Descent & Conjugate Gradient Method
  - Some methods do not require matrix to be explicitly built

Questions so far?

Assignment 1 – the fun part!

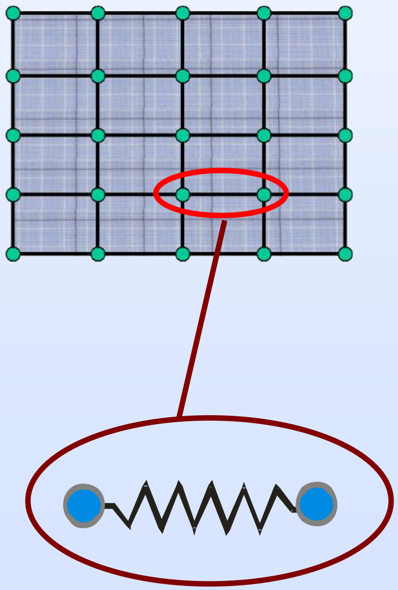
- How would you model…
  - cloth



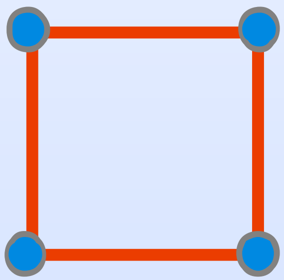
Assignment 1 – the fun part!

- How would you model…
  - cloth

    What types of springs are required?

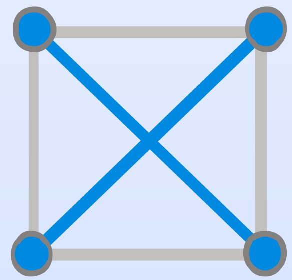


Structural Springs



Stretching

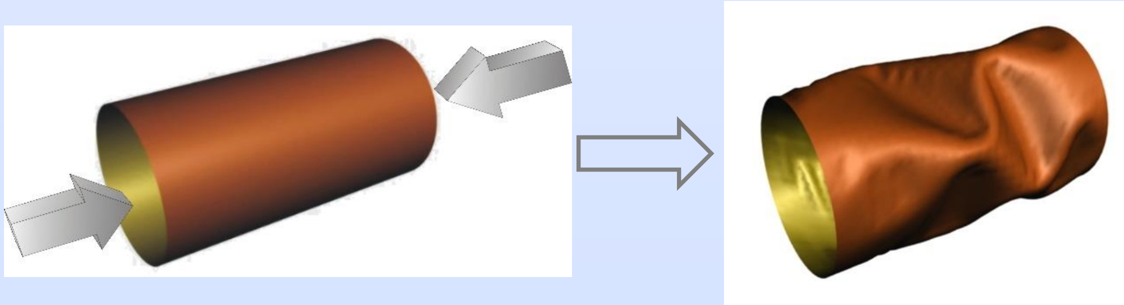
Diagonal Springs



Shearing

Assignment 1 – the fun part!

- How would you model…
  - shells



Assignment 1 – the fun part!

- How would you model…
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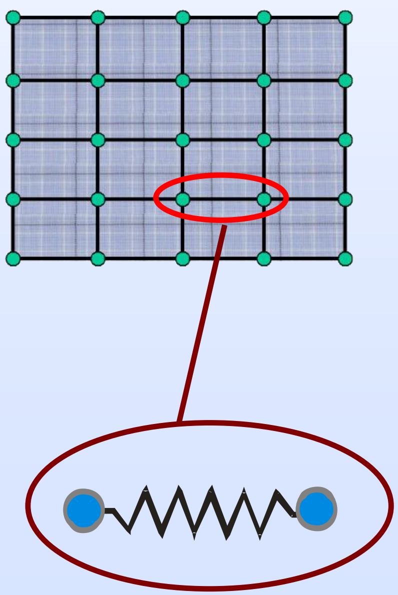
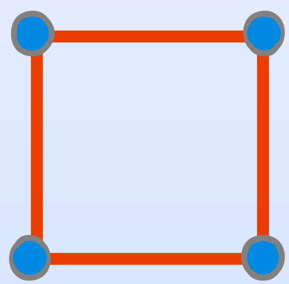
What types of springs are required?

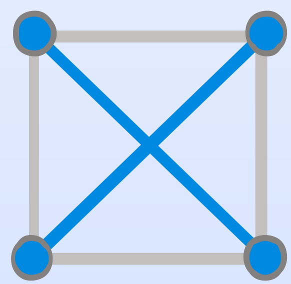


Structural Springs



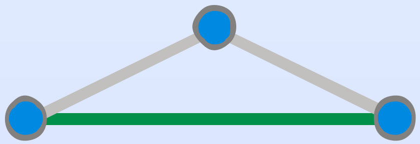
Stretching

Diagonal Springs



Shearing

Interleaved Springs



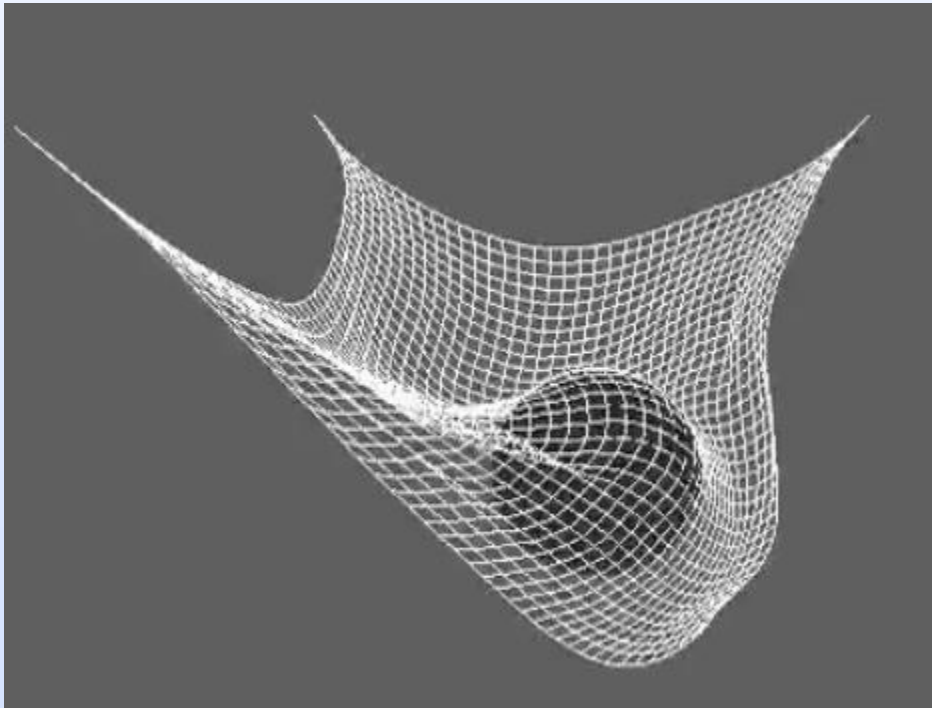
Bending

Assignment 1 – the fun part!

- How would you model…
  - fur and hairs

Assignment 1 – the fun part!

- How would you model…
  - contacts and friction



Simple Collision Response

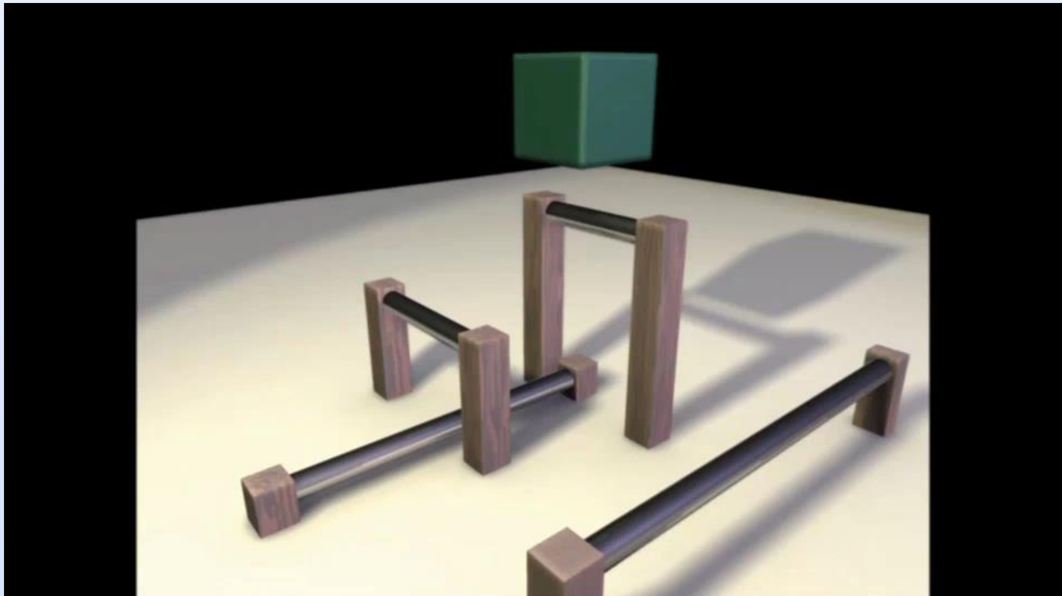
- If in contact, project back on surface, find normal n
  - For ground, n=(0,1,0)
- Filter velocities. First, decompose into
  - normal component $v_N=(v \cdot n)n$ and
  - tangential component $v_T=v-v_N$
- Normal response: $v_N^{after} = -\varepsilon v_N^{before}, \quad \varepsilon \in [0,1]$
  - $\varepsilon=0$ is fully inelastic
  - $\varepsilon=1$ is elastic
- Tangential response
  - Simple model of friction: $v_T^{after} = \alpha v_T^{before}, \alpha \in [0,1]$
- Then reassemble velocity $v=v_N+v_T$

Assignment 1 – the fun part!

- How would you model…
  - a squishy object

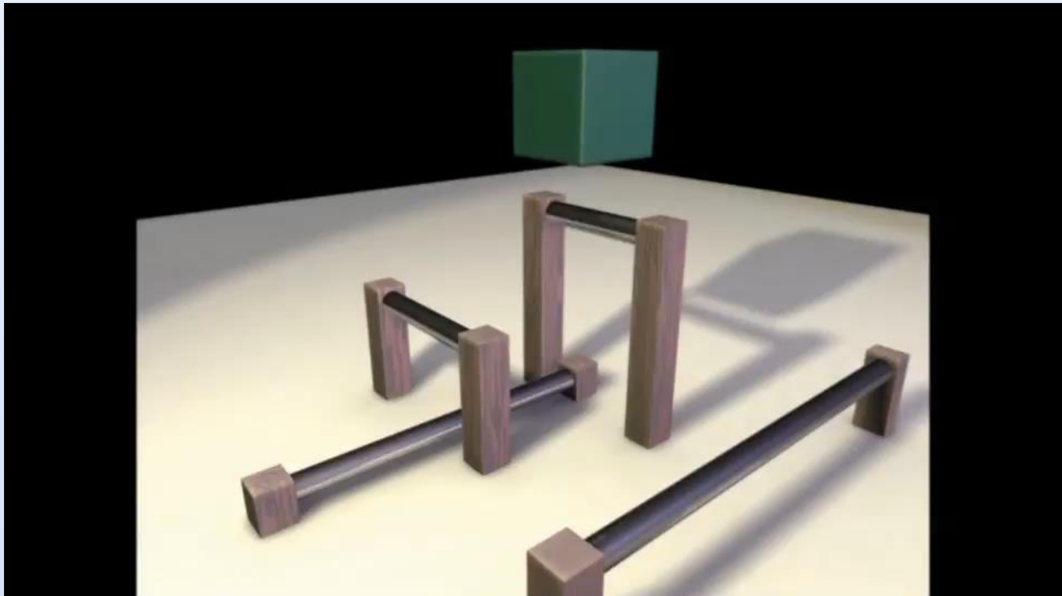
Assignment 1 – the fun part!

- How would you model…
  - plastic deformations



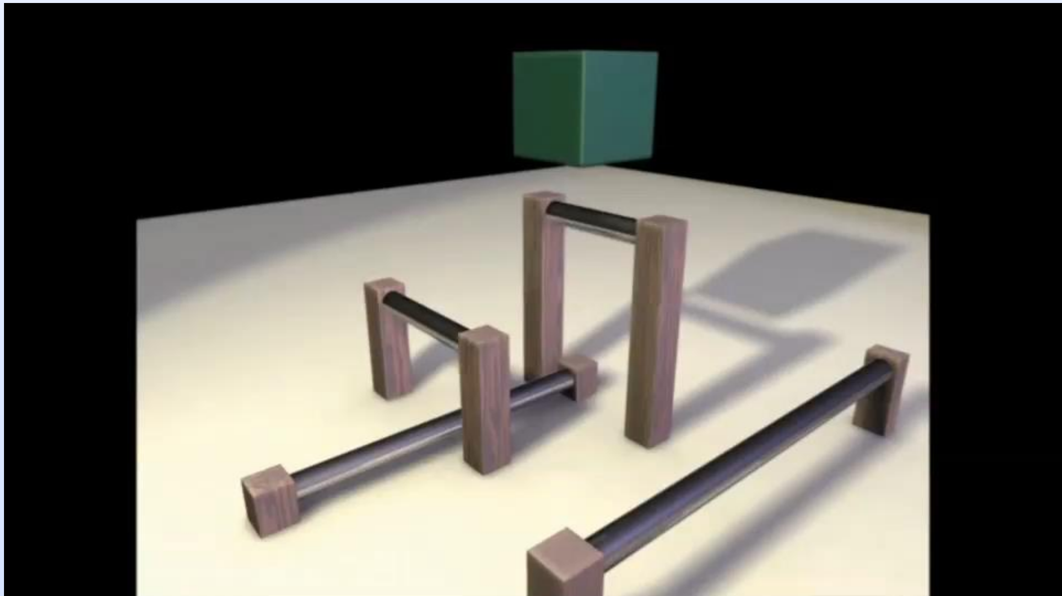
Assignment 1 – the fun part!

- How would you model…
  - viscous materials



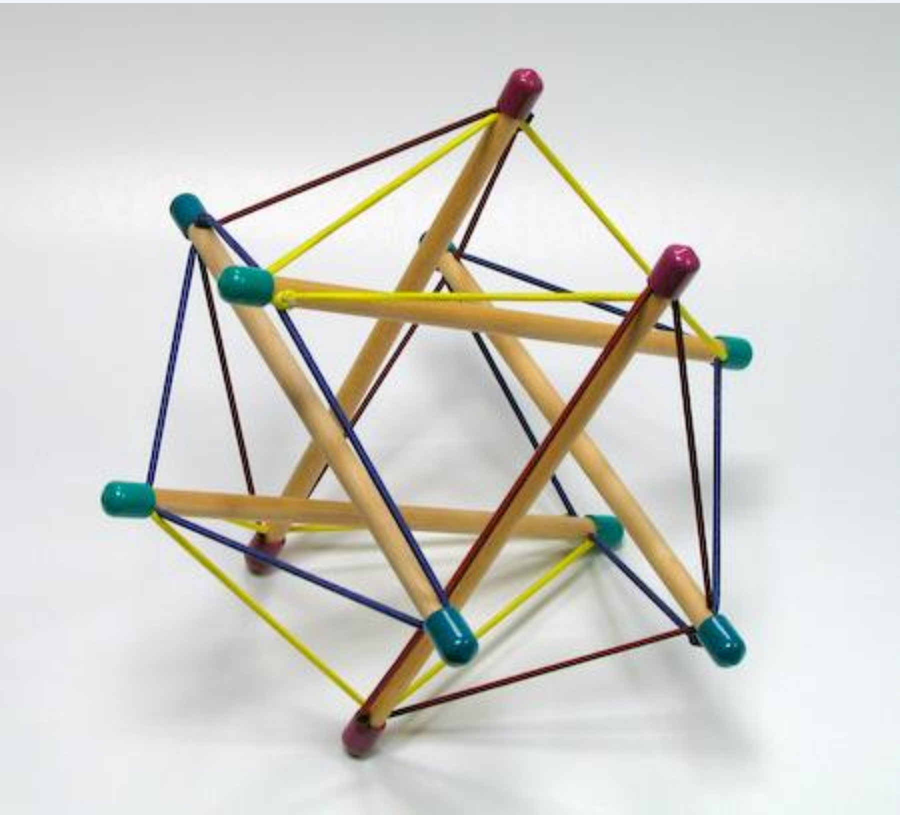
Assignment 1 – the fun part!

- How would you model…
  - a rigid body
  - an articulated rigid body structure



Assignment 1 – the fun part!

- How would you model…
  - a tensegrity structure



Assignment 1 – the fun part!

- How would you model…
  - Fracture, cutting, etc

Start early. Ask questions. Have fun!!!