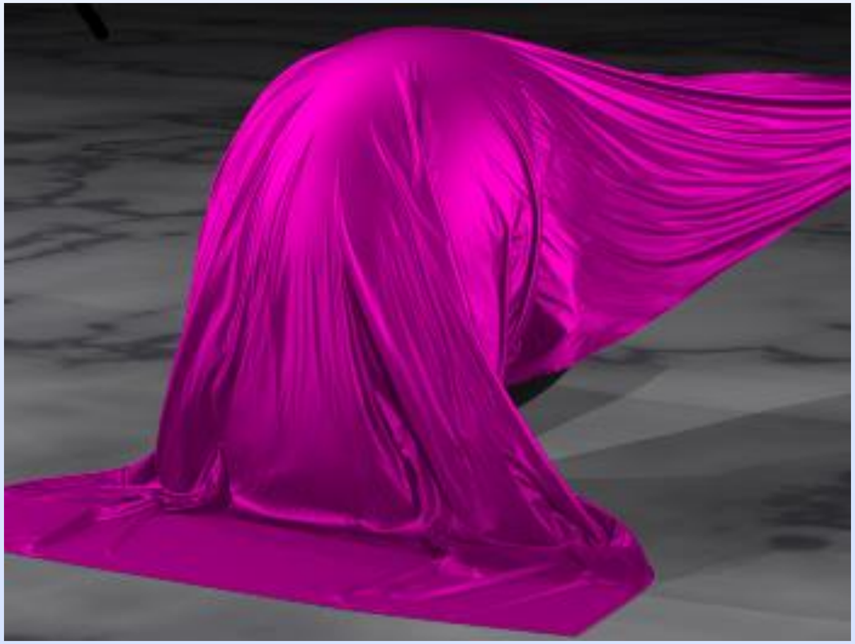
Forces and Practical Guide to Numerical Integration



Brief Recap

- What we've seen so far:
  - System of p particles, each with state (x^i, v^i)
    - Note: $v = \dot{x}, a = \ddot{x}$
  - Arbitrary number of forces acting on each particle
  - For each particle, acceleration is:

$$a^i = \frac{1}{m^i} \sum f^i(x^i, v^i)$$

Forces

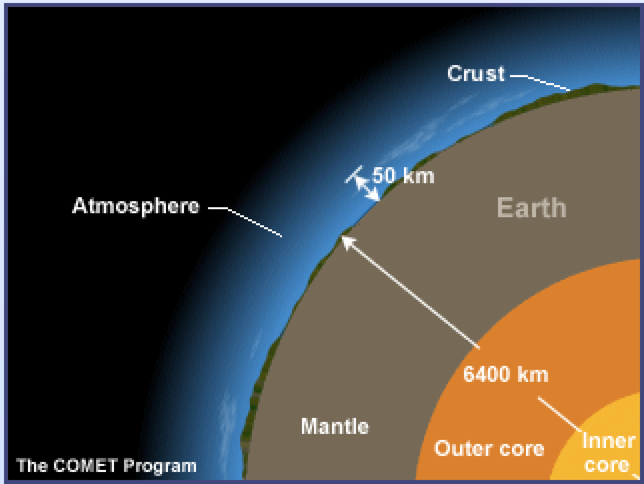
- We've seen: Gravitational forces

$$F_{gravity} = -GmM_0 \frac{x - x_0}{|x - x_0|^3}$$

- When $M_0 >> m$, and vector $x\text{-}x_0$ is approximately constant, this reduces to the familiar:

$$F_{gravity} = mg$$

$|g|=9.8m/s^2$, g points down

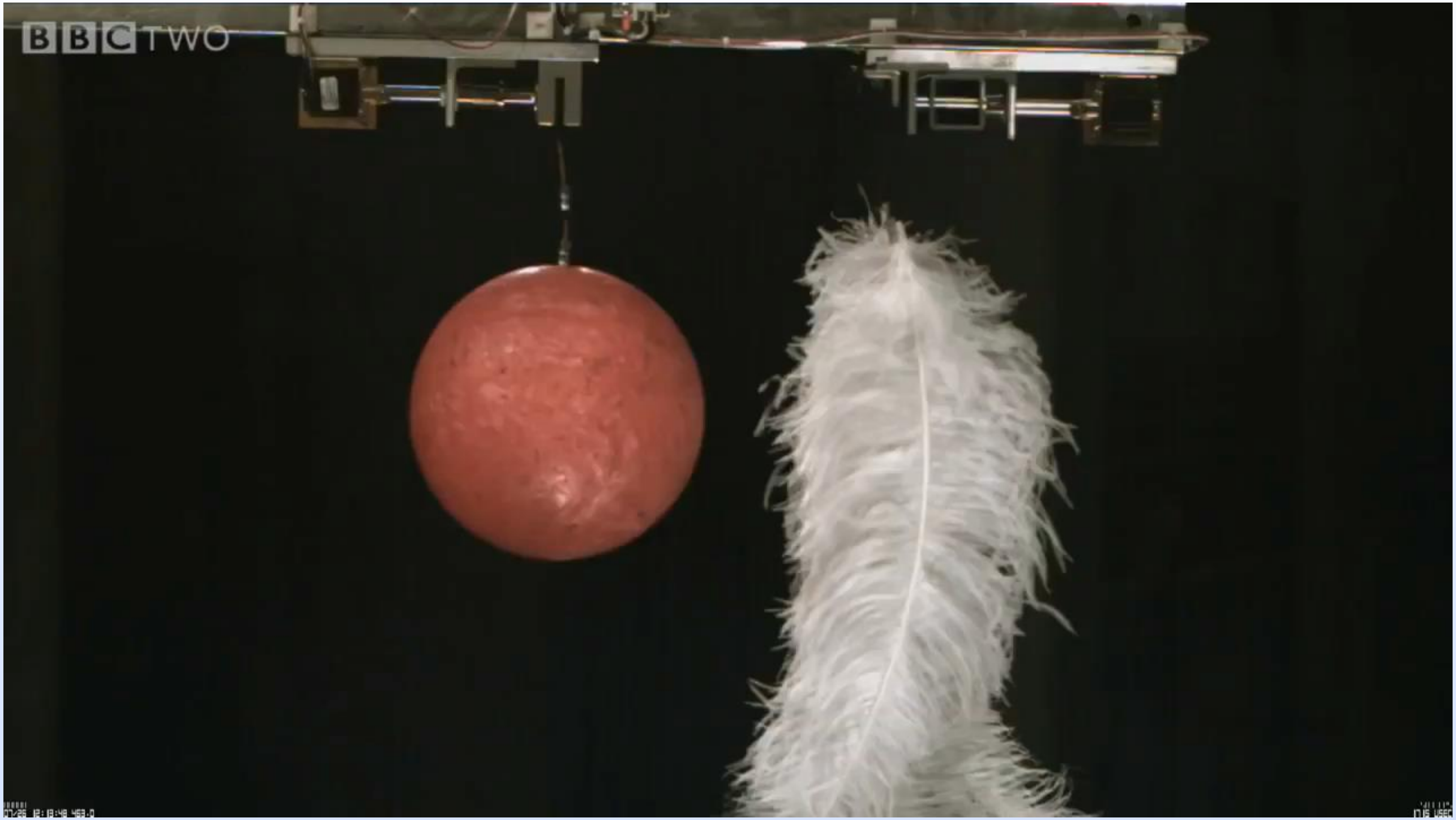


Forces

- We’ve seen: Air drag
  - Aka Stokes' drag: objects moving relatively slowly through low Reynold number fluids

$$F_{drag} = -Dv$$

Drag forces

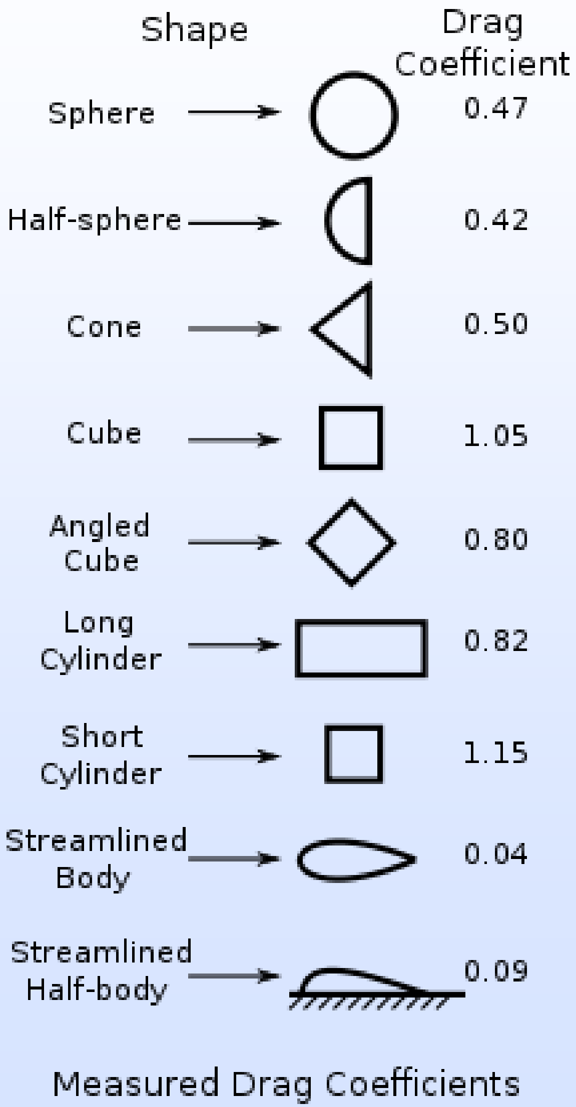


Forces

- We've seen: Air drag
  - Aka Stokes' drag: objects moving relatively slowly through low Reynold number fluids

$$F_{drag} = -Dv$$

- D is a complex function of shape



Spring Forces

- Forces governed by Hooke's law:
  - *Ut tensio, sic vis.* (Hooke, 1678)
  - "as the extension, so the force"

$$F_{spring} = -K\overbrace{(x - x_0)}^{\Delta x}$$

- x_0 is the attachment point of the spring
- Could be a fixed point in the world, or the mouse cursor, or another particle (but add equal and opposite force!)
- When is this force zero?
  - Zero rest length spring!

Nonzero Rest Length Spring

- Better to measure "strain":
  - stretch/deformation *relative* to rest length L
  - K is a material property, rather than an object property!

$$F_{spring} = -K\left(\frac{|x - x_0|}{L} - 1\right)\frac{x - x_0}{|x - x_0|}$$

Spring Damping

- Simple spring damping:

$$F_{damp} = -D(v - v_0)$$

  - But this damps rotation too!
- Better spring damping:

$$F_{damp} = -Du \bullet (v - v_0)u$$

  - Here u is $(x-x_0)/|x-x_0|$, the spring direction

Forces, Forces

- What we've seen so far:
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    - Note: $v = \dot{x}, a = \ddot{x}$
  - Arbitrary number of forces acting on each particle
  - For each particle, acceleration is:

$$a^i = \frac{1}{m^i}\sum f^i(x^i, v^i) = \frac{1}{m^i}F^i(x^i, v^i)$$

  - Too tedious to consider one particle at a time. Need a concise formulation!

Matrix form

$$a = M^{-1}F(x, v)$$

Symplectic Euler

- Easy!

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ M^{-1}F(x,v) \end{pmatrix}$$

1) Compute & sum up forces: $F(x_n, v_n)$
2) Update velocities: $v_{n+1} = v_n + hM^{-1}F(x_n, v_n)$
3) Update positions: $x_{n+1} = x_n + hv_{n+1}$

Backward Euler

- A bit more interesting…
- Recall need to solve:

$$x_{n+1} = x_n + hv_{n+1}$$
$$v_{n+1} = v_n + hM^{-1}F(x_{n+1}, v_{n+1})$$

- In general, forces are non-linear functions of positions/velocities

Setting Up Backward Euler – Newton's Method for root finding

- Eliminate positions, solve for velocities:

$$v_{n+1} - v_n - hM^{-1}F\left(x_n + hv_{n+1}, v_{n+1}\right) = 0$$

- Linearize around guess $\tilde{v}$ ($g(\tilde{v} + \Delta v) \approx g(\tilde{v}) + dg/dv(\tilde{v})$):

$$\tilde{v} + \Delta v - v_n - hM^{-1}\left(F + h\frac{\partial F}{\partial x}\Delta v + \frac{\partial F}{\partial v}\Delta v\right) = 0$$

- Where force F and Jacobians $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial v}$ are evaluated at:

$$x = x_n + h\tilde{v}, v = \tilde{v}$$

- Re-arrange to get: $\left(M - h\frac{\partial F}{\partial v} - h^2\frac{\partial F}{\partial x}\right)\Delta v = M\left(v_n - v^k\right) + hF$

Setting Up Backward Euler

- Start with initial guess $\tilde{v} = v_n$
- Evaluate $F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial v}$ at $x = x_n + h\tilde{v}, v = \tilde{v}$
- Solve for Δv (linear system of equations)

$$\underbrace{\left(M - h\frac{\partial F}{\partial v} - h^2\frac{\partial F}{\partial x}\right)}_{\mathbf{A}}\Delta v = \underbrace{M\left(v_n - v^k\right) + hF}_{\mathbf{b}}$$

- Update guess $\tilde{v} = \tilde{v} + \Delta v$, repeat until $|\Delta v| \approx 0$
- Set $v_{n+1} = \tilde{v}$, $x_{n+1} = x_n + hv_{n+1}$

Newton's Method

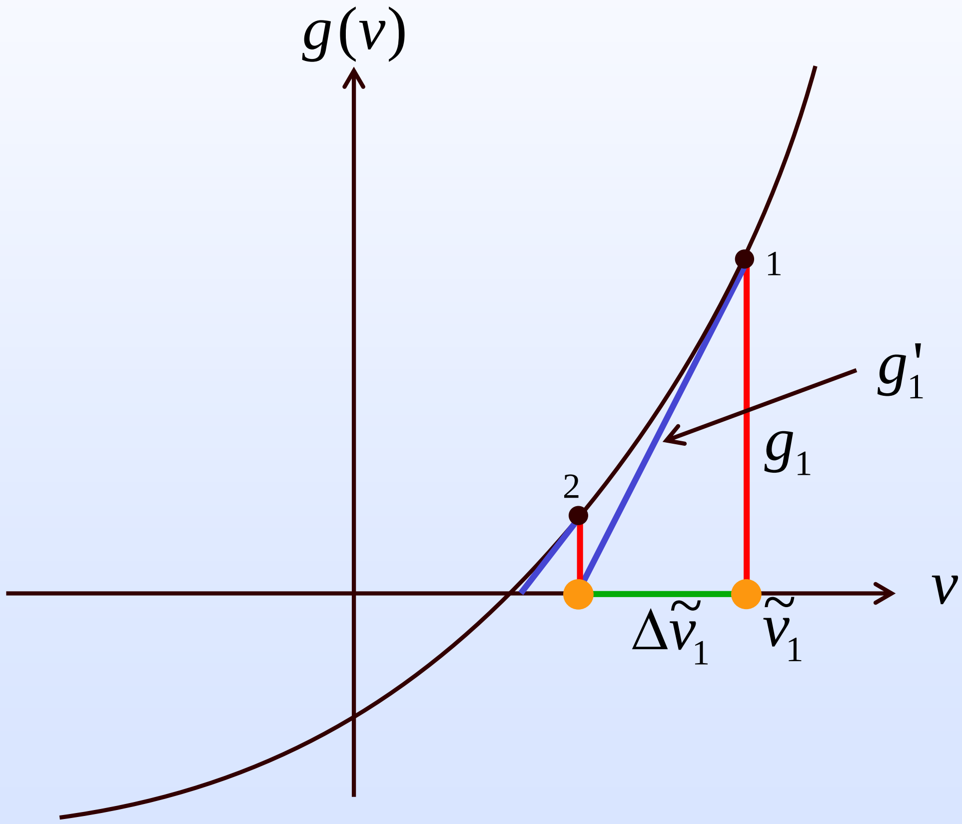
Solve for $g(v) = 0$

Solve for Δv

$$\frac{dg}{dv}(\tilde{v}) \cdot \Delta v = -g(\tilde{v})$$

Update

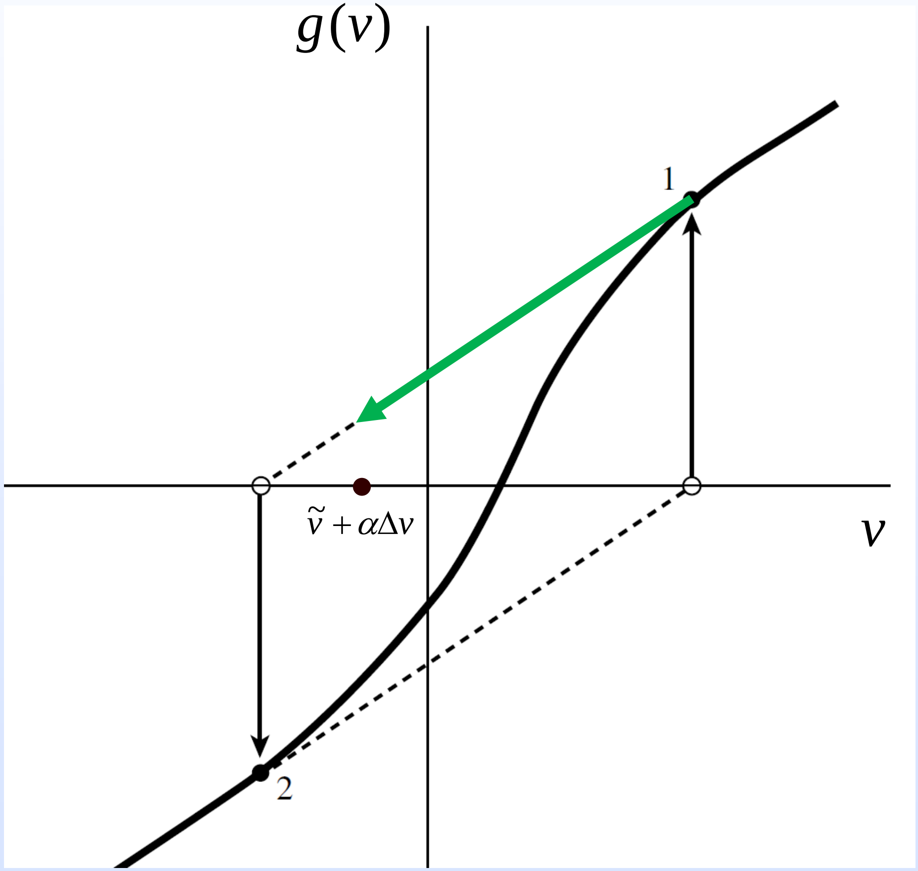
$$\tilde{v} = \tilde{v} + \Delta v$$



Newton's Method

- Newton's method is **great** when it works, but it might not always work ☹

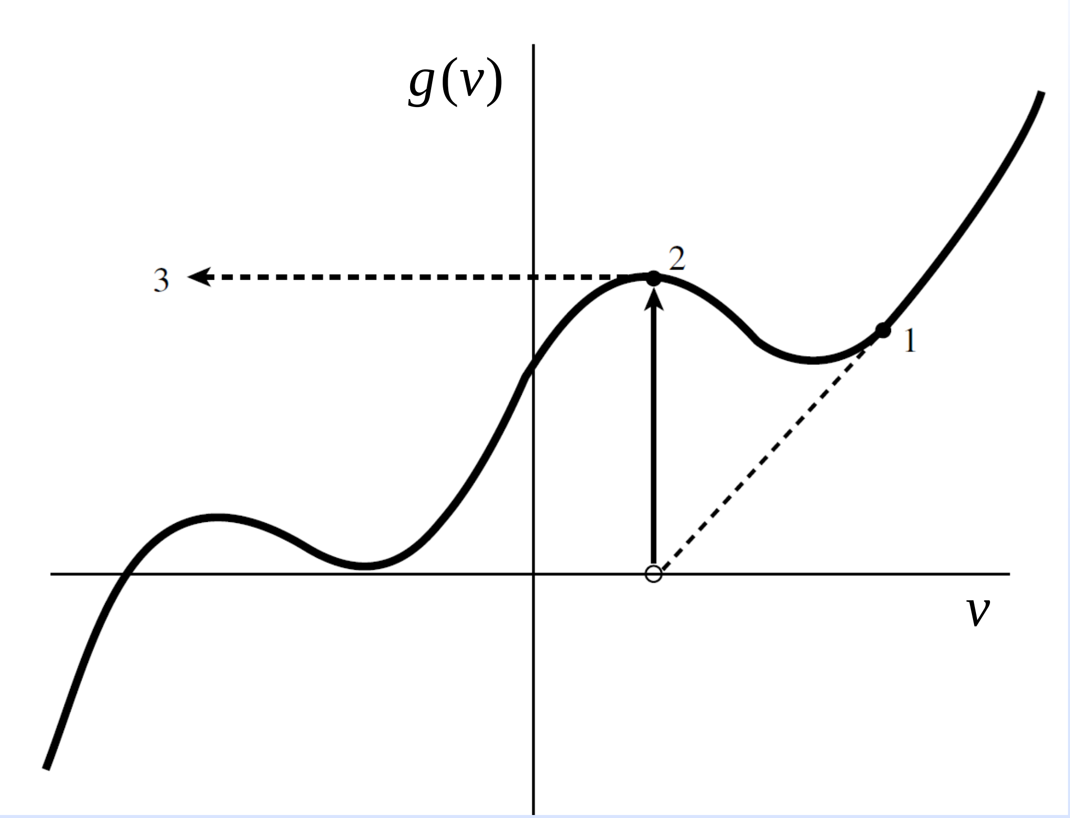
Newton's Method



Newton's Method

- Newton's method is **great** when it works, but it might not always work ☹
- Try using a line search
  - Recall, we are looking for Δv s.t. $g(\tilde{v} + \Delta v) = 0$
  - Take a step scaled by α s.t. $|g(\tilde{v} + \alpha \Delta v)| < |g(\tilde{v})|$
  - Bisection approach is a good start

Newton's Method



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  - Take a step scaled by α s.t. $|g(\tilde{v} + \alpha \Delta v)| < |g(\tilde{v})|$
  - Bisection approach is a good start
- Local extrema are bad!
  - Smaller time steps to make the problem easier to solve?
  - Change model?