Towards a Characterization of Truthful Combinatorial Auctions
Preliminary version – Comments welcome

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Abstract

This paper analyzes incentive compatible (truthful) mechanisms over restricted domains of preferences, the leading example being combinatorial auctions. Our work generalizes the characterization of Roberts (1979) who showed that truthful mechanisms over *unrestricted* domains with at least 3 possible outcomes must be "affine maximizers". We show that truthful mechanisms for combinatorial auctions (and related restricted domains) must be "almost affine maximizers" if they also satisfy an additional requirement of "independence of irrelevant alternatives". This requirement is without loss of generality for unrestricted domains as well as for auctions between two players where all goods must be allocated. This implies unconditional results for these cases, including a new proof of Roberts' theorem. The computational implications of this characterization are severe as reasonable "almost affine maximizers" are shown to be as computationally hard as exact optimization. This implies the near-helplessness of such truthful polynomial-time auctions in all cases where exact optimization is computationally intractable.

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1 Introduction

1.1 Background

In recent years we have seen much research aimed at designing algorithms that are intended to function in environments where the inputs to the algorithm are distributed among players who have different goals. Such environments become increasingly common on the Internet, in communication networks, and in many electronic commerce situations. In such cases, the algorithms must be designed in a way that motivates the participants to reveal the true inputs rather than some carefully designed lies intended to produce results favored by the particular participant.

The most commonly studied model is taken from the economic field of mechanism design (see e.g. [22, 25, 17, 21, 6, 2, 10, 9, 28]). In this model each player has a private valuation function that assigns real values to each possible outcome of the algorithm. The algorithm motivates the players by handing payments to them. Each player is assumed to be rational in the sense of attempting to maximize his net utility, i.e. the sum of his valuation of the algorithms' outcome and his payment. Such algorithms with attached payment functions are called mechanisms. The payments should be carefully designed to ensure that each rational player will report his input – his valuation function – truthfully. Mechanisms for which reporting the truth is a dominant strategy for each player are called incentive compatible, or truthful[1]. The basic challenge in this field is that of designing *computationally efficient incentive compatible* mechanisms for the relevant computational problems.

A particularly central problem of this kind is that of combinatorial auctions. In a combinatorial auction, k items are simultaneously auctioned among n bidders. Bidders value bundle of items in a way that may depend on the combination they win, i.e. each bidder has a valuation function v_i that assigns a real value $v_i(S)$ for each possible subset of items that he may win. The goal is to find a partition $S_1 ... S_n$ of the items in a way that maximizes the total social welfare $\sum_i v_i(S_i)$. This problem has many interesting sub-cases, abstracts many complex allocation problems, and has been extensively studied. The combinatorial auction problem is NP-complete for even very simple valuation functions v_i, and may be approximated to within a factor of $O(\sqrt{k})$ (but no better [18, 14]). Experimental results have shown that many reasonable heuristics can quickly obtain optimal or approximately optimal solutions for combinatorial auctions with up-to thousands of items [32, 27, 12].

Unfortunately, it is not known how to turn such non-fully-optimal heuristics or approximation algorithms into incentive compatible mechanisms! There is only one known general method for designing incentive compatible mechanisms: the VCG payment scheme [34, 8, 13]. The key difficulty is the fact that VCG-payments ensure truthfulness only when the chosen outcome optimizes the social welfare; thus attaching VCG payments to an approximation algorithm or heuristic does *not* ensure incentive compatibility! This problem was noticed in [25, 18] and was shown to be essentially universal in [24]. Additionally, there are many computational scenarios where the optimization goal is different than optimizing the social welfare [25]. Thus, the main positive tool of mechanism design does not address the basic challenge of algorithmic mechanism design for combinatorial auctions as well as most other computationally interesting scenarios! Indeed, almost all mechanisms designed so far for computational scenarios apply only to very restricted "single dimensional" domains. Such mechanisms include e.g. scheduling to minimize the makespan [2], revenue maximization (for digital and other types of goods) [10, 30], auctioning with bounded communication [6], as well as combinatorial auctions with very restrictive bidders (see below).

[1]This model makes several assumptions that may all be relaxed. The most significant ones are the existence of "money" (quasi-linearity), the independence of the valuations (private values), and using the notion of dominant strategies rather than weaker notions of equilibria.

1.2 Characterizing Incentive Compatibility

A general approach to this central task of designing computationally efficient incentive compatible mechanisms would be to obtain a characterization of their powers. To do this, let us get slightly more formal about the basic model.

There is a set A of possible outcomes of the mechanism, and each player has a valuation function $v_i : A \rightarrow \mathcal{R}$ that specifies his value $v_i(a)$ for each possible outcome $a \in A$, where v_i is chosen from some possible domain of valuations V_i. For each n-tuple of valuations $v = (v_1, \ldots, v_n)$, the mechanism produces some outcome $f(v)$ that may be viewed as aggregating the preferences v_i of the n players. The function f is called the *social choice function.* Additionally, the mechanism hands out payments to the players. For example, in the case of combinatorial auctions, A is the set of all possible partitions $(a_1, \ldots, a_n)$ of the items, and each V_i is the set of valuations that depend only on a_i ("no externalities") and are monotone in a_i ("free disposal"). It turns out that for each social choice function f that may be the obtained by an incentive compatible mechanism there is essentially a single way to "implement it", i.e. to set the payments needed as to ensure incentive compatibility[2]. The basic question is *what social choice functions are implementable?*

The VCG mechanism mentioned above implements the social choice function that maximizes the social welfare, i.e. the social choice function $f(v) = \mathrm{argmax}_{a\in A} \sum_i v_i(a)$. Three generalizations may be applied to the VCG payment scheme, yielding generalizations to the implemented social choice function: (a) the range may be restricted to an arbitrary $A' \subset A$; (b) different non-negative weights ω_i can be given to the different players; (c) different additive weights γ_a can be given to different outcomes. All three generalizations can be combined, yielding an implementation for any social choice function that is an *affine maximizer*:

Definition: *A social choice function f is an affine maximizer if for some $A' \subset A$, non-negative $\{\omega_i\}$, and $\{\gamma_a\}$, for all $v_1 \in V_1, \ldots, v_n \in V_n$ we have $f(v_1, \ldots, v_n) \in argmax_{a\in A'}(\sum_i \omega_i v_i(a) + \gamma_a)$.*

What other social choice functions can be implemented? A classic negative result of Roberts [29] shows that if the domain of players' valuations is unrestricted, and the range is non-trivial, then nothing more:

Theorem (Roberts, 1979): *If there are at least 3 possible outcomes, and players' valuations are unrestricted ($V_i = \mathcal{R}^{|A|}$), then any implementable[3] social choice function is an affine maximizer.*

The requirement that the valuations are unrestricted is very restrictive. In almost all interesting computational applications the domain of valuations is restricted. E.g., as mentioned, for the combinatorial auction problem the valuations are restricted in two ways: "free disposal" and no "externalities", and thus $V_i \neq \mathcal{R}^{|A|}$. Indeed, some assumption about the space of valuations is also necessary: In the extreme opposite case, the domain is so restricted as to become single dimensional, for which truthful non affine maximizers exist. A trivial example is given by an auction of a single item, where v_i is a scalar, and a truthful auction may assign the auctioned item to the bidder i with highest value of $(v_i)^i$ (for the price of $(v_j)^{j/i}$ where j is the player with second highest value of $(v_j)^j$). More interesting examples in the context of combinatorial auctions involve "single-minded" bidders, where the valuation function is given by a single value v_i offered for a single set of items S_i [18]. While the optimization problem in this case is still NP-complete and thus affine maximization is not

[2] Imprecisely stated, when outcome a is chosen, each player i is charged the minimum value of $v_i(a)$ that would still result in a being chosen.

[3] Roberts, as we do here, only discusses implementation in dominant strategies in private-value environments. See [19] for a generalization to environments with inter-dependent valuations.

efficiently computable, [18] presented computationally efficient truthful approximation mechanisms for it. Additional mechanisms for this single-minded case were presented in [21, 1].

However, most interesting computational problems are not single dimensional either – they lie somewhere between the two extremes of "unrestricted" and "single dimensional". This intermediate range includes combinatorial auctions and many of their interesting special cases such as, multi-unit (homogeneous) auctions, or unit-demand auctions (matching). It also includes most examples of other combinatorial optimization problems such as various variants of scheduling and routing problems. Almost nothing is known about this intermediate range. The only positive example of a non-VCG mechanism for non-single-dimensional domains is for a special case of multi-unit combinatorial auctions where each bidder is restricted to demand at most a fraction of the number of units of each type [5].

It is interesting to draw parallels with the non-quasi-linear case, i.e. the model where player preferences are given by order relations $\succeq_i$ over the possible outcomes. The classic Gibbard-Satterthwaite result [11, 33] shows that, in this case, no non-trivial social choice function over an unrestricted domain is implementable. The proof shows that any implementable social choice function must essentially satisfy Arrow's condition of "Independence of Irrelevant Alternatives", and thus Arrow's impossibility result [4] applies. On the other hand, in this non-quasi-linear case, there exists much literature implementing non-trivial social choice functions over various interesting restricted domains, e.g. "single peaked domains" [7, 20].

1.3 Our results

In this paper we initiate an analysis of implementable social choice functions over restricted domains in quasi-linear environments. It is widely known that certain monotonicity requirements characterize implementable social choice functions. E.g. Roberts starts by defining a condition of "positive association of differences" (PAD) that characterizes implementable social choice functions over unrestricted domains. It turns out that this condition is usually meaningless for restricted domains. We start with a formulation of a "weak monotonicity" condition (W-MON), that provides this characterization for "usual" restricted domains (exact definitions of all terms appear in the body of the paper). We demonstrate that other natural notions are not appropriate.

Lemma: *Every implementable social choice function over every domain must satisfy W-MON. Over "usual" domains, W-MON is also a sufficient condition.*

As opposed to the case of unrestricted domains, it turns out that, for restricted domains, W-MON by itself does not imply affine maximization! A key contribution of this paper is the identification of a key additional property, *Independence of Irrelevant Alternatives (IIA)*, that will provide this implication. This property is a natural analog, in the quasi-linear setting, of Arrow's similarly named property in the non-quasi-linear setting. This condition states that if the social choice function changes its value from one outcome a to another outcome b, then this is due to a change in some player's preference between a and b.

Definition: *A social choice function f satisfies IIA if for any $v, u \in V$, if $f(v) = a$ and $f(u) = b \neq a$ then there exists a player i such that $u_i(a) - u_i(b) \neq v_i(a) - v_i(b)$.*

We show that the IIA property is equivalent to a slight, but significant, strengthening of the W-MON condition, termed "strong monotonicity". We further show that in unrestricted domains IIA may be assumed without loss of generality. This is also true in a class of domains that includes the case of combinatorial auctions with two players in which all items are always allocated. In other domains we demonstrate that IIA may not be assumed without loss of generality.

We then get to our main result: incentive compatible mechanisms that also satisfy IIA must be "almost" affine maximizers. The theorem is proved in a general setting and requires certain technical conditions.

Main Theorem: *In "certain reasonable domains", an implementable social choice function that additionally satisfies IIA and certain technical conditions must be an "almost" affine maximizer.*

The proof of this theorem is different from the one Roberts provides for unrestricted domains, and uses ideas suggested, in a somewhat different context, by Archer and Tardos [3]. This theorem applies to combinatorial auctions as well as to multi-unit (non-combinatorial) auctions. It even applies to the case of "known double minded bidders", i.e. where each bidder has only two bundles on which he may bid – showing that the mechanisms of [18, 21] regarding single-minded bidders cannot be generalized this way. For unrestricted domains, the IIA condition may be assumed without loss of generality, and therefore this yields a new proof of Roberts' theorem (the qualifications in the theorem statement all disappear in this case). For two-player auctions where all items must always be allocated, the IIA condition can similarly be dropped. We also show that in this two-player case, the requirement that all items must always be allocated is necessary – without it, there exist implementable social choice functions that are not almost affine maximizers (and do not satisfy IIA)[4].

The major open problem we leave is whether the IIA condition is necessary:

Main Open Problem: *Are there incentive compatible combinatorial auctions that are not "essentially" affine maximizers?*

The meaning of "essentially" in this open problem is soft, as we demonstrate that various "minor" variations from affine maximization are possible. The question is really whether anything *useful* is possible, e.g. can any non-trivial approximation be achieved.

We close the circle by returning to our original motivation, that of the existence of computationally efficient incentive compatible approximation mechanisms. We observe that essentially any affine maximizer is as computationally hard as exact social welfare maximization. This implies that if exact computation of the optimal allocation is computationally hard, then incentive compatible mechanisms that satisfy IIA are essentially powerless. For an exact statement of computational hardness we must first fix an input format, i.e. a "bidding language" [23] that is powerful enough to make the exact optimization problem computationally intractable[5]. We say that a combinatorial auction mechanism is *unanimity-respecting* if whenever every bidder values only a single bundle, and furthermore, these bundles compose a valid allocation, then this allocation is chosen[6]. This condition essentially ensures that all allocations are possible outputs, ruling out "bundling" auctions[7].

[4]We note that the incentive compatible mechanism of [5] also does not always allocate all items.

[5]E.g.: for general combinatorial auctions any complete bidding language that can succinctly express single-minded bids is enough; if the number of players is a fixed constant, the language must allow OR-bids; for multi-unit (non-combinatorial) auctions, the bidding language must allow specifying the number of items in binary.

[6]This is essentially equivalent to the property of a "reasonable" auction of [24].

[7]E.g., where all items are sold as a single bundle in a simple auction – this clearly gives a factor $\min(n, k)$-approximation. Slightly better approximations in polynomial time are possible by partitioning the items into a constant number of bundles [15].

Theorem: *(Assuming $P \neq NP$ and a sufficiently powerful bidding language) Any unanimity-respecting truthful polynomial-time combinatorial (or multi-unit) auction that satisfies IIA cannot obtain any polynomially-bounded approximation ratio.*

Note that without the incentive-compatibility requirement, an $O(\sqrt{k})$ and n approximations can be obtained for combinatorial auctions, and an FPAS for multi-unit auctions. Note also that for the two-player case where all items are always allocated, the IIA condition can be dropped.

An especially crisp result is obtained for the case of two-player multi-unit auctions. This case is computationally equivalent to a weighted knapsack problem and is thus NP-complete to solve exactly, but has a fully polynomial time approximation scheme. However, this approximation is not incentive compatible. Indeed, [16] who considered this problem were only able to show "almost incentive compatibility" [8]. Our results show that this is no accident. Exact incentive compatibility directly collides with an approximation scheme.

Corollary: *(Assuming $P \neq NP$ and a sufficiently powerful bidding language) No polynomial time incentive compatible mechanism for a multi-unit auction between two players that always allocates all units can achieve an approximation factor better than 2.*

1.4 Paper Organization

The rest of the paper is organized as follows. In section 2 we describe our model. In section 3 we discuss the connection between monotonicity and truthfulness, with proofs and examples appearing in appendix A. Section 4 gives an exposition of the main theorem and its proof, where the full details appear in appendix B. Section 5 discusses the implications to computationally efficient combinatorial auctions, with full details appearing in appendix C. Appendix D provides our alternative proof of Roberts' theorem.

2 Setting and Notations

2.1 Social choice functions on restricted domains

Social Choice Function. We study a general model of a social choice function $f : V_1 \times ... \times V_n \rightarrow A$. The interpretation is that f gets as its input a vector of *players' preferences* and chooses an alternative among a finite set of possible alternatives A. We denote $|A| = m$, and assume w.l.o.g that f is onto A.

The Domain (player types). Each player i ($1 \leq i \leq n$) assigns a real value $v_i(a)$ to each possible alternative from A. The vector $v_i \in R^m$ is called the player's *type* and is interpreted as specifying the player's preferences. The set $V_i \subseteq R^m$ is the set of possible valuations v_i. We denote $V = V_1 \times ... \times V_n$. We use the notation $v = (v_1, ..., v_n) \in R^{nm}$, and $v(a) = (v_1(a), ..., v_n(a)) \in R^n$. We also use the notation $v_{-i} = (v_1 ... v_{i-1}, v_{i+1} ... v_n) \in R^{n-1}$.

The main point in this paper is that V_i may be a proper subset of R^m. Here are some of the domains that we are concerned with in this paper:

- **Unrestricted Domains.** We say that the domain is *unrestricted* if $V_i = R^m$. In other words, the value of alternative a for player i does not place any restrictions upon i's values for the other alternatives.

[8] A somewhat similar notion of "almost incentive compatibility" for an approximation scheme for a different problem was also obtained in [1].

- **Combinatorial Auctions (CA).** In a combinatorial auction, a set Ω of k items are auctioned between n bidders. The "alternatives" that the auction chooses among are allocations of items to bidders. That is, an alternative a is an allocation $a = (a_1 ... a_n)$, where $a_i \subseteq \Omega$ is the set of items allocated to player i, and $a_i \cap a_j = \emptyset$ for $i \neq j$ (each item can be allocated to at most one player). The valuations are assumed to satisfy three conditions:
  1. No externalities: v_i should only depend on i's allocated bundle a_i. I.e. $v_i(a) = v_i(a_i)$.
  2. Free disposal: v_i should be non-decreasing with the set of allocated items. I.e. For every $a_i \subseteq b_i$, we have that $v_i(a_i) \leq v_i(b_i)$.
  3. Normalization: $v_i(\emptyset) = 0$.
- **Multi Unit Auctions.** This is the special case of combinatorial auctions where the items are homogeneous. In this case an allocation $(a_1 ... a_n)$ is simply a vector of nonnegative integers, subject to the restriction that $\sum_i a_i \leq k$, and the valuation functions v_i can be represented as non-decreasing non-negative functions $v_i : \{1...k\} \to R_+$.
- **Order-Based Domains.** We will phrase our results in this paper in terms of a general family of domains termed "order-based", which contains all the previous examples, as well as others. These are domains where each V_i is defined by a (finite) family of inequalities and equalities of the form $v_i(a) \leq v_i(b)$, $v_i(a) < v_i(b)$, $v_i(a) = v_i(b)$ or $v_i(a) = 0$. Thus for example an unrestricted domain is defined by the empty family, while the domain of valuations for combinatorial auctions is defined by the following set of inequalities: for all $a, b \in A$ such that $a_i = b_i : v_i(a) = v_i(b)$ (no externalities); for all $a, b \in A$ such that $a_i \subseteq b_i$: $v_i(a) \leq v_i(b)$ (free disposal); for all $a \in A$ such that $a_i = \emptyset$: $v_i(a) = 0$.
- **Strict Order-Based Domains.** A subset of order-based domains for which we can prove strong statements is those defined only by strict inequalities $v_i(a) < v_i(b)$, as well as at most a single equality of the form $v_i(a) = 0$. Examples of strict order-based domains are two-players combinatorial auctions, or two-player multi-unit auctions, where all items must be allocated, i.e. $a_1 \cup a_2 = \Omega$ (this is discussed in details in section 5). Trivially, unrestricted domains are also strict order based.

2.2 Implementation and Truthfulness

Mechanisms. We assume that players' valuations are private information. Thus, a player might be motivated to declare a different type than his true type, in order to shift the social choice in some direction desirable for him. One solution is to construct a *mechanism*, which is allowed to charge payments ($p_i : V \to \mathcal{R}$) from the players, in addition to producing the chosen alternative. We assume that players are quasi-linear and rational in the sense of maximizing their total utility: $u_i = v_i(f(v)) - p_i(v)$. In *truthful* mechanisms, a player is motivated to be truthful and declare his true type, v_i, rather than a different type, u_i:

Definition 1 (Truthfulness) [9] *A mechanism $(f, p_1 ... p_n)$, where $f : V \to A$ and $p_i : V \to \mathcal{R}$ is called* truthful *if for any player i, any $v_{-i} \in V_{-i}$, and any $v_i, u_i \in V_i$: $v_i(f(v)) - p_i(v) \geq v_i(f(u_i, v_{-i})) - p_i(u_i, v_{-i})$. We say that such a mechanism* implements *the social choice function f. We say that the social choice function f is* implementable *or* truthful *if there exists some mechanism that implements it.*

[9] In this paper we only discuss direct revelation mechanisms with dominant strategy implementations in quasi-linear private value domains.

The only known general class of truthful social choice functions over multi-dimensional domains are *affine maximizers*, which can be implemented using the VCG mechanism:

Affine maximization. A social choice function f is an *affine maximizer* if there exist constants $\omega_1, \dots, \omega_n \geq 0$ and $\{\gamma_a\}_{a\in A}$ such that for any $v \in V$: $f(v) \in \operatorname{argmax}_{a\in A}\{\sum_{i=1}^n \omega_i v_i(a) + \gamma_a\}$. It can be verified that, in this case, f is implemented by the payments $p_i = -\omega_i^{-1}(\sum_{j\neq i}^n \omega_j v_j(a) + \gamma_a)$.

3 Truthfulness and Monotonicity

It is well known that truthfulness is related to some notions of monotonicity. In this section we derive these relationships which serve as the embarking point towards our main characterization. All proofs are given in Appendix A.

3.1 Weak monotonicity

In simple "one parameter" domains, monotonicity is usually the property of "still winning when raising my value". In general domains, we must examine value *differences* . Roberts [29] used a definition of monotonicity called PAD: f satisfies PAD if for every $v \in V$, $f(v) = a$, then $u_i(a) - v_i(a) > u_i(b) - v_i(b)$ for all $i = 1, \dots, n$ and all $b \in A$ implies that $f(u) = a$. However, PAD has no real meaning for most restricted domains: Suppose there exists a player i and two alternatives a, b s.t. $v_i(a) = v_i(b)$ for all $v_i \in V_i$ (e.g. in CA, when i gets the same bundle in a and b). Then the condition of PAD is never satisfied if $f(v) = a$. One can make several attempts to "fix" this. In the appendix we describe several natural "candidates" for a more general monotonicity condition, and demonstrate that they fail to be necessary for truthfulness. We identify the "correct" notion of monotonicity:

Definition 2 (Weak Monotonicity (W-MON)) *A social choice function f satisfies W-MON if for any $v \in V$, player i, and $u_i \in V_i$: $f(v) = a$ and $f(u_i, v_{-i}) = b$ imply that $u_i(b) - v_i(b) \geq u_i(a) - v_i(a)$.*

In other words, if player i caused the outcome of f to change from a to b by changing his valuation from v_i to u_i, then it must be that i's value for b has not increased less than i's value for a. W-MON implies PAD on every domain (see appendix A.1), but makes sense also in domains where PAD does not. It is essentially equivalent to truthfulness:

Theorem 1 *Every implementable social choice function in any domain satisfies W-MON. If V is an order based domain then W-MON is also a sufficient condition for truthfulness.*

The condition that the domain is order-based is needed (although it may be relaxed) to ensure that W-MON is a sufficient condition, as is shown by example 2 in Appendix A.1. The example is inspired by [31].

3.2 Strong monotonicity and IIA

So far we have seen that weak monotonicity is almost equivalent to truthfulness. We identify the following slightly stronger monotonicity condition, where the inequality in the definition is strict, as being of particular importance. We require this stronger condition for our main result.

Definition 3 (Strong Monotonicity (S-MON)) *A social choice function f satisfies S-MON if for any $v \in V$, player i, and $u_i \in V_i$: $f(v) = a$ and $f(u_i, v_{-i}) = b \neq a$ imply that $u_i(b) - v_i(b) > u_i(a) - v_i(a)$.*

In both definitions, we have the situation that i's valuation changed from v_i to u_i and this caused the outcome of f to change from a to b. S-MON asserts that this implies that i's valuation of b had to increase more than did the valuation of a. W-MON only requires that it did not increase less. While this seems like a slight change, it is in fact crucial. S-MON is not a necessary condition for truthfulness, and we give several counter examples in section 5, in the context of Combinatorial Auctions. The following definition, inspired by Arrow's notion for non-quasi-linear environments [4], essentially characterizes the difference between W-MON and S-MON:

Definition 4 (Independence of Irrelevant Alternatives (IIA)) *f satisfies IIA if for any $v, u \in V$, if $f(v) = a$ and $f(u) = b \neq a$ then there exists a player i such that $u_i(a) - u_i(b) \neq v_i(a) - v_i(b)$.*

In other words, if the social choice function on some valuations prefers a over b (i.e. a is chosen), and no player changes his preference of a with respect to b then it cannot be the case that the social choice function would prefer b over a.

Proposition 1 *W-MON + IIA $\Rightarrow$ S-MON. For order based domains, S-MON $\Rightarrow$ W-MON + IIA.*

Remark: We actually show that, for order based domains, S-MON implies the following "generalized S-MON": $f(v) = a$ and $f(u) = b \;\Rightarrow\; \exists i:\; u_i(b) - u_i(a) > v_i(b) - v_i(a)$. (Clearly this implies IIA). We also give some degenerate domain for which S-MON holds but IIA does not hold.

For certain domains, the notions of W-MON and S-MON are essentially equivalent. This equivalence is true for unrestricted domains and may be viewed as implicit in the proof of Roberts.

Theorem 2 *If V is strict order based then for every social choice function f there exists a function $\tilde{f} : V \to A$ such that, if f satisfies W-MON then $\tilde{f}$ satisfies S-MON, and then, if $\tilde{f}$ is an affine maximizer then f is an affine maximizer as well.*

4 Main Theorem

In what follows we give an exposition of our main theorem and its proof. A full description is given in Appendix B.

Our main theorem shows that, under certain conditions, social choice functions that satisfy S-MON are "almost" affine maximizers. Let us first explain these conditions and qualifications:

- **The Domain:** The theorem holds for some family of domains which we call *order-based domains with conflicting preferences* – basically domains that satisfy various technical requirements in the proof. Combinatorial auctions, multi-unit auctions, and unrestricted domains all satisfy these requirements.

- **The Range:** The actual range of the social choice function must be *non-degenerate*. For combinatorial auctions or multi-unit auctions this means that there exists some player (w.l.o.g player 1) such that, for every other player i, the range includes an allocation a with $a_1 \neq \emptyset$ and $a_i \neq \emptyset$. For unrestricted domains this means that the range is of size at least 3. Without this condition, the problem may essentially be reduced to a single-dimensional setting, in which case many non affine maximizer truthful choice functions exist.

- **The Social Choice Function:** We require *player decisiveness.* This means that a player can ensure that his top alternative is chosen if he bids high enough on it. Formally, that for every i there exists an outcome c such that for all v_{-i} there exists u_i with $f(u_i, v_{-i}) = c$.
- **Almost Affine Maximizer:** The theorem only shows that the social choice function must be an affine maximizer for large enough input valuations. I.e. there exists a threshold M s.t. the function is an affine maximizer if $v_i(a) \geq M$ for all a and i (except from inherently zero alternatives). We believe that this restriction is just a technical artifact of the current proof.

Theorem 3 *Every social choice function over an order-based domain with conflicting preferences and onto a non-degenerate range, that is player decisive and satisfies S-MON, must be almost affine maximizer.*

We now provide an intuitive outline of the proof. Full details appear in appendix B. We start by attempting to infer some order that f induces on the domain. Specifically, if for some vector v of preferences the choice is $a = f(v)$ then we may say that the vector of values $v(a) = (v_1(a), \ldots, v_n(a))$ has more weight than the vector $v(b)$. This leads us to the following definition:

Definition 5 *("x at a" is larger than "y at b") For $a, b \in A$ and $x, y \in \mathcal{R}^n$ we say that $x@a > y@b$ if there exists $v \in V$ such that $v(a) = x,\ v(b) = y,$ and $f(v) = a$.*

This notation certainly suggests that ">" is an order. In unrestricted domains this is indeed the case. However, in restricted domains, it is not generally so. The requirements of the theorem imply "just enough" of the properties of an order to proceed with the proof.

Once such a "near-order" is defined, we can compare every $x@a$ to multiples of some fixed reference $z@c$. This is inspired by the "min-function" model of Archer and Tardos [2]. We would expect that for small values of α we would have $x@a > (\alpha z)@c$, while for large values of α we would have $x@a < (\alpha z)@c$. The value of α where the change happens somehow summarizes the "size" of $x@a$. To proceed we need to find such c and z where this holds for "enough" x and a. From now on, let such appropriate c and z be fixed.

Definition 6 *The "measure of x at a" is defined as:*

$$m(x@a) = inf\ \{\ \alpha \mid x@a < (\alpha \cdot z)@c\ \}$$

This measure captures the choice function, as the following property shows:

Claim: *Under the conditions of the theorem, if $m(v(a)@a) < m(v(b)@b)$ then $f(v) \neq a$.*

This claim basically shows that $f(v) \in \text{argmax}_a\{m(v(a)@a)\}$. What remains to show is that $m(x@a)$ is in fact a linear function (in x) on R^n! (And, that it does not depend on a, up to an additive constant.) To get this result let us, informally, consider the partial derivative $\partial m(x@a)/\partial x_i$. A key observation is that this partial derivative must be equal to $\partial m(y@b)/\partial y_i$ for any other "compatible" y and b. Let us see the intuition for this: consider some v such that $v(a) = x$ and $v(b) = y$. Since the S-MON requirement only looks at differences $v_i(a) - v_i(b)$ when "choosing between a and b", we would expect that adding a constant δ to both $x_i = v_i(a)$ and to $y_i = v_i(b)$ will also leave $m(x@a) - m(y@b)$ unchanged. This is indeed the case:

Claim: *Under the conditions of the theorem, for all (appropriate) a, b, x, y and δ we have that $m(x@a) - m(y@b) = m((x + \delta \cdot e_i)@a) - m((y + \delta \cdot e_i)@b)$. (Here e_i is the i'th unit vector).*

But now we claim that this means that $\partial m(x@a)/\partial x_i$ is independent of x! To see this, fix some y and denote $h_i(\delta) = m((y + \delta \cdot e_i)@b) - m(y@b)$. The previous claim states that $m((x + \delta \cdot e_i)@a) - m(x@a) = h_i(\delta)$. It is a simple exercise to verify that such a condition on $m(x@a)$ implies that it is linear in x. Specifically it must have the form $m(x@a) = \sum_i h_i(1) \cdot x_i + \gamma_a$, where γ_a is an arbitrary constant. This is (almost) the required result. (Delicate difficulties enter when we are unable to choose a single $y@b$ in an appropriate way for all $x@a$ – those are treated in the full proof.)

5 Computational Implications for Combinatorial Auctions

In this section we discuss the applicability of the main theorem to the main motivating problem: computationally efficient truthful mechanisms for approximating the optimal allocation in combinatorial auctions (CAs) and multi-unit auctions (MUAs). For this application most of the technical issues in the main theorem can be dropped. We start dealing with the general issues, proceed with those implied by approximation factors, and conclude with the computational ones.

5.1 General Issues

CAs and MUAs satisfy all the requirements on the domain of theorem 3. Thus any CA or MUA that satisfies S-MON and player decisiveness, onto a non-degenerate domain, must be almost affine maximizer. In fact, a non-degenerate domain captures even the case where each bidder is interested in only two, known in advance, bundles ("known double minded bidders"), where one of bundles is the set of all goods.

Let us now look at the different requirements of the theorem. First notice that if the range is degenerate, then as discussed above, the social choice function need not be an almost affine maximizer[10]. As for the strong monotonicity, in example 4 in appendix C we show that there are truthful CAs that do not satisfy S-MON, and indeed are *not* affine maximizers. (But are still "close" to affine maximizers).

When there are two players, and all the goods are always allocated, then S-MON is no longer a burden: in this case, for any distinct allocations a and b we have that $a_i \neq b_i$. Thus, V is very close to being strictly ordered, so we expect to be able to use Theorem 2 to reduce S-MON to W-MON. Specifically, define the *interior* of V to be $\mathring{V} = \{v \in V \mid v_i(a) < v_i(b) \text{ for all } a, b \in A \text{ s.t. } a_i \subsetneq b_i\}$ and define $\mathring{f} : \mathring{V} \to A$ by $\mathring{f}(v) = f(v)$.

Theorem 4 *Fix any truthful CA or MUA f for two players, that always allocates all the goods. Suppose that $\mathring{f}$ is player decisive and onto a non-degenerate range. Then:*

1. *f must be almost affine maximizer in the interior of V.*

2. *If the $\{\gamma_a\}_{a \in A}$'s are all zero then f is almost affine maximizer in all of V.*

If we drop the assumption of always allocating all goods, then S-MON cannot be assumed w.l.o.g. In Appendix C we give an example of a specific CA that sometimes leaves unallocated goods, is player decisive and onto a full range, but does not satisfy S-MON, and is not almost affine maximizer.

[10] A simple class of examples is the "bundled auction" that allocates all items to the player with maximum value of $t_i(v_i(\Omega))$, where each t_i is an arbitrary monotone real function.

5.2 Approximation

Since exact welfare optimization in CAs is computationally hard (see also below), we ask whether there exist truthful welfare approximations. A social choice function is a c-approximation of the optimal welfare if, for any type v, the alternative $f(v)$ has welfare of at least $1/c$ times the optimal welfare for v. For this class of functions, we are able to show that most of the qualifiers of the main theorem can be dropped.

Specifically, we define an auction to be *unanimity-respecting* (essentially equivalent to the notion of "reasonable" in [25]) if, whenever every player values only a single bundle a_i, and $a_i \cap a_j = \emptyset$ for all i, j, then f chooses the allocation $a = (a_1, \ldots, a_n)$. Using these, the "almost" qualifier and the player decisiveness property are dropped from the main theorem:

Lemma 1 *Any unanimity-respecting truthful CA or MUA that satisfies IIA and achieves a c-approximation must be an affine maximizer. Furthermore, the weights must satisfy $\gamma_a = 0$ for all alternatives a and and $(1/c) \leq (\omega_i/\omega_j) \leq c$ for all players i, j.*

For two players, where all the goods are always allocated, we can drop even the remaining qualifiers:

Lemma 2 *Any truthful CA or MUA for two players that always allocates all items and achieves an approximation factor of $c < 2$ must be an affine maximizer. Furthermore, it must have a full range, and the weights must satisfy $\gamma_a = 0$ far all a and $0.5 < (\omega_i/\omega_j) < 2$ for all i, j.*

5.3 Polynomial-Time Computation

All treatment of mechanisms so far assumed a fixed number of players n and a fixed number of items k. When formalizing the notion of computational running time we must let these parameters (or at least the number of items k) grow, and consider the running time as a function of them. A mechanism whose running time we wish to analyze would apply to all k and, if n is not fixed, for all n, i.e. would really be a uniform family of mechanisms. The characterization as affine maximizer above would then only apply to each mechanism in the family separately (with no explicit relationship across the different values of n and k.)

In order to represent the mechanisms' running time as a function of its input size, we must fix an input representation for the valuations, i.e. a *bidding language* [23]. Our results apply to any such choice of a bidding language as long as it is complete (i.e. can represent all valuation) and sufficiently powerful. Our interest is in cases where the bidding language is sufficiently powerful as to make exact welfare maximization NP-complete. In particular, [18] show that as long as even single-minded bids are possible then the CA problem with n players is NP-complete (where n is not fixed). In appendix C.4 we observe that this is true for MUAs as well, as long as the number of desired items may be given in binary (rather than unary). When the number of players is fixed, then single-minded bids (as well as XOR-bids) may be handled in polynomial time, but we show that allowing OR bids results in an NP-complete optimization problem.

Our basic result is that in all such cases, truthful polynomial-time auctions that satisfy IIA can do essentially nothing except bundling of the items.

Theorem 5 *(Assuming $P \neq NP$) Fix a sufficiently powerful bidding language. Any Unanimity-respecting truthful polynomial-time combinatorial (or multi-unit) auction that satisfies IIA cannot obtain $poly(n, k)$ welfare approximation.*

For the case of two-player auctions, we can omit the "unanimity-respecting" and "IIA" assumptions:

Corollary 1 *(Assuming $P \neq NP$) Fix a sufficiently powerful bidding language. Any truthful polynomial-time multi-unit (or combinatorial) auction between two players cannot obtain a welfare approximation better than* 2*.*

In contrast, for MUA without the truthfulness requirement there exists an FPAS [26]! Also notice that a truthful 2-approximation can be easily obtained using a simple auction of the bundle of all goods.

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A Truthfulness and Monotonicity

We start with some basic notations, and a formal definition of order based domains:

Notation: For any $v_i \in V_i$, we denote by $u_i = v_i|^{a+=\delta}$ the following type: $u_i(a) = v_i(a) + \delta$, and for all $b \neq a$, $u_i(b) = v_i(b)$. Similarly, $u_i = v_i|^{a=\delta}$ denotes the type $u_i(a) = \delta$, and for all $b \neq a$, $u_i(b) = v_i(b)$.
We use 1^m to denote the vector $(1, \ldots, 1) \in \mathcal{R}^m$.

Definition 7 (An order-based domain) *A domain V_i of player i is "order-based" if there exists some set $0_i \subset A$ and binary relations $R_i(a,b) \in \{ >, \geq, <, \leq, =, null \}$ (for all pairs $a, b \in A$), such that, for any $v_i \in \mathcal{R}^m$, $v_i \in V_i$ if and only if:*

1. *$v_i(a) = 0$ for every $a \in 0_i$.*
2. *$v_i(a)\ R_i(a,b)\ v_i(b)$ for all pairs $a, b \in A$ such that $R_i(a,b) \neq null$.*

A domain $V = V_1 \times \ldots \times V_n$ is order based if all the V_i's are order based. [11]

For example, a matching of n people to n chairs is an order based domain. Here, every alternative is a permutation of the chairs. For any two alternatives a, b, if i has the same position in a and b then $R_i(a,b) =$ " $=$ ", otherwise $R_i(a,b) = null$. As discussed in the paper body, our main example of order based domains are combinatorial auctions and multi-unit auctions.

A.1 Weak monotonicity and truthfulness

Roberts [29] uses PAD as his basic monotonicity condition:

Definition (PAD): f satisfies PAD if for every $v, u \in V$, where $f(v) = a$, if for all $i = 1, \ldots, n$ and all $b \in A$: $u_i(a) - v_i(a) > u_i(b) - v_i(b)$ then $f(u) = a$.

However, we note that PAD has no real meaning for many interesting domains: Suppose there exist a player i and two alternatives a, b such that $v_i(a) = v_i(b)$ for all $v_i \in V_i$ (e.g. in CA, when i is allocated the same bundle in a and b). Then the condition of PAD is never satisfied if $f(v) = a$. One can make several straight forward attempts to "fix" this. We next describe two natural "candidates" for a more general monotonicity condition, and show by an example that they fail to be necessary for truthfulness.

Strong PAD: For every $v, u \in V$, where $f(v) = a$), if for all $i = 1, \ldots, n$ and $b \in A$: $u_i(a) - v_i(a) \geq u_i(b) - v_i(b)$ then $f(u) = a$.

Generalized W-MON: For every $v, u \in V$, if $f(v) = a$ and $f(u) = b$ then there exists a player i such that: $u_i(b) - v_i(b) \geq u_i(a) - v_i(a)$.

Example 1 *Suppose there are two players, and four alternatives: $A = \{YY, YN, NY, NN\}$. A player type is determined by one positive value v_i, as follows. For any $a \in A$ (denote $a = a_1 a_2$ where $a_i \in \{Y, N\}$): if $a_i = N$ then $v_i(a) = 0$, and if $a_i = Y$ then $v_i(a) = v_i$. Define $f(v) = a_1 a_2$, where $a_i = Y$ if $v_i > 2v_j - 10$, otherwise $a_i = N$. It is easy to verify that f is truthful, with the payments $p_i(N, v_j) = 0$ and $p_i(Y, v_j) = 2v_j - 10$.*

[11] We assume that $R_i(a,b) =$ " $<$ " if and only if $R_i(b,a) =$ " $>$ " (and similarly for the other relations), although, for example, if $R_i(a,b) =$ " $<$ ", one can set $R_i(b,a) = null$ without changing the domain.

To contradict both types of monotonicity, suppose $v_1 = v_2 = 9$, and $u_1 = u_2 = 11$. Then $f(v) = YY$, but $f(u) = NN$! (Note that PAD trivially holds – its condition is never satisfied). [12]

We are left with a more straight-forward condition for truthfulness:

Definition (Weak Monotonicity (W-MON)): A social choice function f satisfies W-MON if for any $v \in V$, player i, and $u_i \in V_i$: $f(v) = a$ and $f(u_i, v_{-i}) = b$ imply that $u_i(b) - v_i(b) \geq u_i(a) - v_i(a)$.

We first argue that W-MON implies PAD, for every domain:

Claim 1 *If f satisfies W-MON then f satisfies PAD.*

proof: Fix any $v, u \in V$. Suppose $f(v) = a$, and $u_i(a) - v_i(a) > u_i(b) - v_i(b)$ for all $i = 1, \ldots, n$ and $b \in A$. Let $v^0 = v$, $v^1 = (u_1, v_2, \ldots, v_n)$, $v^2 = (u_1, u_2, v_3, \ldots, v_n)$, $v^n = (u_1, \ldots, u_n) = u$. Now, $f(v^0) = a$ and $f(v^i) = a$ implies by W-MON that $f(v^{i+1}) = a$. ∎

We now argue that W-MON is, in fact, equivalent to truthfulness:

Theorem 1 *Every truthful social choice function satisfies W-MON. If V is order based then W-MON is also a sufficient condition for truthfulness.*

proof: We start with a basic claim:

Claim 2 *Any truthful function f has (price) functions $p_i : A \times V_{-i} \to \mathcal{R} \cup \{\infty\}$ such that, for any $v \in V$ and any player i, $f(v) \in argmax_{a \in A}\{v_i(a) - p_i(a, v_{-i})\}$.*

proof: Since f is truthful it has price functions $\tilde{p}_i : V \to \mathcal{R}$. Suppose by contradiction that there exists $v \in V$ and $u_i \in V_i$ such that $f(v) = f(u_i, v_{-i}) = a$, but $\tilde{p}_i(v) \neq \tilde{p}_i(u_i, v_{-i})$. W.l.o.g $\tilde{p}_i(v) > \tilde{p}_i(u_i, v_{-i})$. Thus when the other players declare v_{-i}, and the true type of player i is v_i, she will increase her utility by declaring u_i, a contradiction. Therefore we can define the price functions $p_i : A \times V_{-i} \to \mathcal{R} \cup \{\infty\}$, as follows. For any i, $v_{-i} \in V_{-i}$, and $a \in A$, if there exists $v_i \in V_i$ such that $f(v) = a$ we set $p_i(a, v_{-i}) = \tilde{p}_i(v)$, otherwise $p_i(a, v_{-i}) = \infty$.

To see that $f(v) \in argmax_{a \in A}\{v_i(a) - p_i(a, v_{-i})\}$, suppose by contradiction that there exists $v \in V$ such that $f(v) = a$, and $v_i(a) - p_i(a, v_{-i}) < v_i(b) - p_i(b, v_{-i})$. Let $u_i \in V_i$ be the type that determined $p_i(b, v_{-i})$. Therefore if i will declare u_i instead of v_i, when his true valuation is v_i, she will increase his utility, a contradiction. ∎

Claim 3 *Every truthful social choice function satisfies W-MON.*

proof: Let $p_i : A \times V_{-i} \to \mathcal{R} \cup \{\infty\}$ be the price functions according to claim 2. Suppose that $f(v) = a$ and $f(u_i, v_{-i}) = b$. Therefore $v_i(a) - p_i(a, v_{-i}) \geq v_i(b) - p_i(b, v_{-i})$, and $u_i(b) - p_i(b, v_{-i}) \geq u_i(a) - p_i(a, v_{-i})$. Thus $u_i(b) - u_i(a) \geq p_i(b, v_{-i}) - p_i(a, v_{-i}) \geq v_i(b) - v_i(a)$, and the claim follows. ∎

This proves the first direction of the theorem. For the second direction, we assume that V is ordered based, and use the following definitions. Fix any player i. For any $a, b \in A$, let $E_i(a) = \{d \in A \mid R_i(a, d) = \text{“} = \text{”} \text{ or } d = a\}$, and define:

$$\delta_{ab}(v_{-i}) = \inf \{v_i(a) - v_i(b) \mid v_i \in V_i \text{ and } f(v_i, v_{-i}) \in E_i(a)\}$$

[12] This example is also “far from S-MON” and “far from affine maximization”, and can be extended to more than two players.

Claim 4 *For any $a, b, c \in A$, and $v_{-i} \in V_{-i}$:*

1. *W-MON implies that $\delta_{ab}(v_{-i}) \geq -\delta_{ba}(v_{-i})$.*
2. *If $R_i(a,b) \in \{=, \leq\}$ then $\delta_{cb}(v_{-i}) \leq \delta_{ca}(v_{-i})$.*

proof: Suppose by contradiction that $\delta_{ab}(v_{-i}) < -\delta_{ba}(v_{-i})$. Take $v_i \in V_i$ such that $v_i(a) - v_i(b) = \delta_{ab}(v_{-i}) + \epsilon$ and $f(v) = \tilde{a}$, where $\tilde{a} \in E_i(a)$, and $u_i \in V_i$ such that $u_i(b) - u_i(a) = \delta_{ba}(v_{-i}) + \epsilon$ and $f(u_i, v_{-i}) = \tilde{b}(\tilde{b} \in E_i(b))$. Since $R_i(a, \tilde{a}) = R_i(b, \tilde{b}) =$ " $=$ " it follows that $v_i(\tilde{a}) - v_i(\tilde{b}) < u_i(\tilde{a}) - u_i(\tilde{b})$. But by W-MON, since $f(v) = \tilde{a}$ it follows that $f(u_i, v_{-i}) \neq \tilde{b}$, a contradiction.

For the second part, assume by contradiction that $\delta_{cb}(v_{-i}) > \delta_{ca}(v_{-i})$, and choose some v_i such that $v_i(c) - v_i(a) = \delta_{ca}(v_{-i}) + \epsilon < \delta_{cb}(v_{-i})$ and $f(v) \in E_i(c)$. Since $v_i(a) \leq v_i(b)$ it follows that $v_i(c) - v_i(b) \leq v_i(c) - v_i(a) < \delta_{cb}(v_{-i})$, contradicting the definition of δ_{cb}. ∎

To prove that W-MON is sufficient for truthfulness, we describe a price function $p_i : A \times V_{-i} \to \mathcal{R}$ such that, for all $v \in V$: $f(v) \in argmax_{a \in A}\{v_i(a) - p_i(a, v_{-i})\}$. For this, fix some alternative $c \in A$ such that for any other $d \in A$, $R_i(c,d) \notin \{\leq, <\}$ (there always exists such alternative since the R_i relations depict partial order over A) [13], and set:

$$p_i(a, v_{-i}) = \begin{cases} 0 & a \in E_i(c) \\ -\delta_{ca}(v_{-i}) & \text{otherwise} \end{cases}$$

Claim 5 *For any $a \in A$, $\tilde{c} \in E_i(c)$, and $v \in V$:*

1. *If $v_i(a) - p_i(a, v_{-i}) < v_i(\tilde{c}) - p_i(\tilde{c}, v_{-i})$ then $f(v) \neq a$.*
2. *If $v_i(a) - p_i(a, v_{-i}) > v_i(\tilde{c}) - p_i(\tilde{c}, v_{-i})$ then $f(v) \neq \tilde{c}$.*

proof: By definition, $p_i(\tilde{c}, v_{-i}) = 0$ and $v_i(\tilde{c}) = v_i(c)$. First suppose that $v_i(a) - p_i(a, v_{-i}) < v_i(c)$. By definition and by claim 4, $v_i(a) - v_i(c) < -\delta_{ca}(v_{-i}) \leq \delta_{ac}(v_{-i})$, and therefore $f(v) \neq a$. In the other direction, $v_i(c) - v_i(a) < -p_i(a, v_{-i}) = \delta_{ca}(v_{-i})$, and therefore $f(v) \neq \tilde{c}$. ∎

We can now finish the proof. Suppose by contradiction that there exists $v \in V$ such that $f(v) = a$, and $v_i(a) - p_i(a, v_{-i}) < v_i(b) - p_i(b, v_{-i})$. By claim 5 it follows that $a, b \notin E_i(c)$, and that $v_i(c) - p_i(c, v_{-i}) \leq v_i(a) - p_i(a, v_{-i})$. Choose some small enough $\epsilon > 0$ and some δ such that $v_i(a) + \epsilon - p_i(a, v_{-i}) < v_i(c) + \delta - p_i(c, v_{-i}) < v_i(b) - p_i(b, v_{-i})$. Define $T_i = \{a\} \cup \{d \in A \mid v_i(d) = v_i(a)$ and $R_i(a,d) \in \{\leq, =\}\ \}$, and let $u_i = v_i|^{E_i(c) += \delta,\ T_i += \epsilon}$. Notice that $u_i \in V_i$ (we can raise $E_i(c)$ as we wish, and raise T_i by some small enough ϵ) [14].

By claim 4, for any $d \in T_i$, $p_i(a, v_{-i}) \leq p_i(d, v_{-i})$, and therefore $v_i(d) - p_i(d, v_{-i}) \leq v_i(a) - p_i(a, v_{-i})$. From this we conclude that $b \notin T_i$, and also that for any $d \in T_i$, $u_i(d) - p_i(d, v_{-i}) < u_i(c) - p_i(c, v_{-i})$. Thus, by claim 5, $f(u_i, v_{-i}) \neq d$. Similarly, for any $\tilde{c} \in E_i(c)$, $u_i(\tilde{c}) - p_i(\tilde{c}, v_{-i}) < u_i(b) - p_i(b, v_{-i})$, and so $f(u_i, v_{-i}) \neq \tilde{c}$. But, by W-MON, since $f(v) = a$ it must be the case that $f(u_i, v_{-i}) \in E_i(c) \cup T_i$, a contradiction. ∎

It is now natural to ask whether W-MON is always a sufficient condition for truthfulness. Unfortunately, as the following example demonstrates, the answer is no.

[13] We also assume that $c \notin 0_i$. This is w.l.o.g since we can always "normalize" the domain with respect to any other alternative a, as follows: we convert any original type v_i to a new type $u_i = v_i - v_i(a) \cdot 1^m$. It is not hard to verify that this maintains truthfulness.

[14] It is possible that there exists some $\tilde{c} \in E_i(c) \cap T_i$. In this case we raise $\tilde{c}$ by δ

Example 2 *Consider a single player with* $A = \{a, b, c\}$ *and a domain of three possible types* v_a, v_b, v_c, *as follows:* $v_a = (0, 1, -2)$; $v_b = (-2, 0, 1)$; $v_c = (1, -2, 0)$ *(the first coordinate in each type is a's value, the second is b's value, and the third c's value). The function* f *has* $f(v_x) = x$ $(x \in \{a, b, c\})$. f *satisfies W-MON since* $v_x(x) - v_y(x) > v_x(y) - v_y(y)$ *for any* $x, y \in \{a, b, c\}$ $(v_x(y) + v_y(x) = -1 < 0 = v_x(x) + v_y(y))$.

Suppose by contradiction that there are truthful prices. Therefore: $-1 = v_c(c) - v_c(a) \geq p(c) - p(a)$. *Similarly,* $-1 = v_a(a) - v_a(b) \geq p(a) - p(b)$, *and* $-1 = v_b(b) - v_b(c) \geq p(b) - p(c)$. *But the last two inequalities imply* $p(c) - p(a) \geq 2$, *a contradiction.*

We note that this example is inspired by a necessary and sufficient condition for truthfulness described in [31].

A.2 Strong monotonicity and IIA

Proposition 1 *If* f *satisfies W-MON and IIA then it satisfies S-MON. In the other direction, if* V *is order based and* f *satisfies S-MON then* f *satisfies W-MON and IIA.*

proof: We prove this using several claims:

Claim 6 *If* f *satisfies W-MON and IIA then* f *satisfies S-MON.*

proof: Fix any $v \in V$, player i, and $u_i \in V_i$. Suppose $f(v) = a$ and $f(u_i, v_{-i}) = b$. We need to show that $u_i(b) - v_i(b) > u_i(a) - v_i(a)$. By W-MON it follows that $u_i(b) - v_i(b) \geq u_i(a) - v_i(a)$. Suppose by contradiction that $u_i(b) - v_i(b) = u_i(a) - v_i(a)$. But then, denote $u = (u_i, v_{-i})$, and we have $f(v) = a$, $f(u) = b$, and for any player j, $v_j(a) - v_j(b) = u_j(a) - u_j(b)$, thus contradicting IIA. ■

For the other direction, we first claim that we can assume w.l.o.g that V is not normalized, i.e. $0_i = \emptyset$ for all i:

Claim 7 *If* V *is normalized then there exists a non-normalized order based domain* $\tilde{V}$ *and a function* $\tilde{f} : \tilde{V} \to A$ *such that:*

1. *If* f *satisfies S-MON then* $\tilde{f}$ *satisfies S-MON as well.*
2. $V \subseteq \tilde{V}$, *and for any* $v \in V$, $f(v) = \tilde{f}(v)$.
3. *If* $\tilde{f}$ *satisfies IIA then* f *satisfies IIA as well.*

proof: Define $\tilde{V}$ as the order based domain defined by exactly the same relations $R_i(a, b)$ but with $0_i = \emptyset$, for all i. $V \subseteq \tilde{V}$ since for any $v \in V$, all the relations $R_i(a, b)$ hold, and therefore $v \in \tilde{V}$. Define $\tilde{f} : \tilde{V} \to A$ as follows: For every i, choose some $a^i \in 0_i$. For any $\tilde{v} \in \tilde{V}$, let $v_i = \tilde{v_i} - \tilde{v_i}(a^i)$, and define $\tilde{f}(\tilde{v}) = f(v)$ ($v \in V$ since all inequalities hold after a translation, and for any $b \in 0_i$, $\tilde{v_i}(b) - \tilde{v_i}(a) = 0$ since $R_i(a, b) =$ " $=$ ").

To see that $\tilde{f}$ satisfies S-MON, suppose $\tilde{f}(\tilde{v}) = a$ and $\tilde{f}(\tilde{u}_i, \tilde{v}_{-i}) = b$. Let $v_j = \tilde{v_j} - \tilde{v_j}(a^j)$ (for $j = 1, \ldots, n$), and $u_i = \tilde{u_i} - \tilde{u_i}(a^i)$. By definition, $f(v) = a$ and $f(u_i, v_{-i}) = b$. Since f satisfies S-MON, $u_i(b) - v_i(b) > u_i(a) - v_i(a)$. Therefore $\tilde{u}_i(b) - \tilde{v}_i(b) > \tilde{u}_i(a) - \tilde{v}_i(a)$, and thus $\tilde{f}$ satisfies S-MON.

Since $V \subseteq \tilde{V}$, contradicting IIA for f implies contradicting IIA for $\tilde{f}$, and the claim follows. ■

Claim 8 (Dependence on Differences (DOD)) *Suppose* V *is order based and non normalized, and* f *satisfies S-MON. Then for any* $v \in V$ *and* $\delta \in \mathcal{R}$: $v_i + \delta \cdot 1^m \in V_i$, *and* $f(v) = f(v_i + \delta \cdot 1^m, v_{-i})$.

proof: $v_i + \delta \cdot 1^m \in V_i$ since all inequalities hold after a translation. Since $[v_i(b) + \delta] - v_i(b) = [v_i(a) + \delta] - v_i(a)$ for any $a, b \in A$, it follows from S-MON that $f(v) = f(v_i + \delta \cdot 1^m, v_{-i})$. ∎

Claim 9 (Generalized S-MON) *Suppose V is order based, and f satisfies S-MON. Then for any $u, v \in V$, if $f(v) = a$ and $f(u) = b$ then there exists a player i such that $u_i(b) - v_i(b) > u_i(a) - v_i(a)$.*

proof: By claim 7 we can assume w.l.o.g that V is not normalized: otherwise, move to $\tilde{f}$, and then, contradicting generalized S-MON for f implies contradicting generalized S-MON for $\tilde{f}$. By claim 8 we can assume w.l.o.g that $u_i(a) = v_i(a)$: otherwise let $\tilde{v}_i = v_i + [u_i(a) - v_i(a)] \cdot 1^m$, then $f(\tilde{v}) = a$, and finding i such that $u_i(b) - \tilde{v}_i(b) > u_i(a) - \tilde{v}_i(a) = 0$ implies that $u_i(b) - v_i(b) > u_i(a) - v_i(a)$.

Now, we "move" from v to u by L "elementary steps" $v = v^1, v^2, \ldots, v^L = u$, such that: (1) for any index j there exists a player i and $d \in A$ such that $v_i^{j+1} = v_i^j|^{d+=u_i(d)-v_i(d)}$, (2) every pair (i, d) appears only once in the sequence, and (3) there exists an index l^* such that for any $l \leq l^*$, $u_i(d) - v_i(d) < 0$, and for any $l > l^*$, $u_i(d) - v_i(d) > 0$ (since V is order based, we can construct such a sequence of types). By S-MON, $f(v^{l^*}) = a$, and for any $l > l^*$, if $f(v^l) = c$ then $f(v^{l+1}) \in \{c, d\}$ (where d is the alternative that changes from v^l to v^{l+1}). Therefore, if $f(v^L) = b$ it follows that there exists i such that $u_i(b) - v_i(b) > 0$, as claimed. ∎

Clearly, generalized S-MON implies IIA, and S-MON implies W-MON, hence the second direction of the proposition follows. ∎

The following example shows that S-MON does not always imply IIA:

Example 3 *Suppose there are four alternatives ($A = \{a, b, c, d\}$) and two players, each one with two possible types v_i, u_i such that: $u_1(c) - v_1(c) > u_1(a) - v_1(a) = u_1(b) - v_1(b) > u_1(d) - v_1(d)$, and $u_2(d) - v_2(d) > u_2(a) - v_2(a) = u_2(b) - v_2(b) > u_2(c) - v_2(c)$. Define f as follows: $f(v_1, v_2) = a$, $f(u_1, u_2) = b$, $f(u_1, v_2) = c$, and $f(v_1, u_2) = d$. It is not hard to verify that S-MON holds (there are four inequalities to check, all of them follow from the way the types are defined). IIA does not hold since $f(v) = a$, $f(u) = b$, but $u(a) - u(b) = v(a) - v(b)$.*

For the following sub-class of order based domains, W-MON implies (w.l.o.g) S-MON:

Definition 8 (Strict order based domains) *An order based domain V is "strict" if, for any $a, b \in A$ and any player i, $R_i(a, b) \in \{>, <, \text{"null"}\}$.*

Trivially, an unrestricted domain (i.e. $V_i = \mathcal{R}^m$ for any i) is a strict order based domain. Another important example is CA with two players, where all the goods must always be allocated (this is discussed in details in section 5). The following theorem implies that for strict order based domains, S-MON can be assumed w.l.o.g if W-MON holds:

Theorem 2 *If V is strict order based then there exists a function $\tilde{f} : V \to A$ such that:*

1. *If f satisfies W-MON then $\tilde{f}$ satisfies S-MON.*
2. *If $\tilde{f}$ is affine maximizer then f is affine maximizer.*

proof: We prove this using the following claims:

Claim 10 *If V is an open set* [15] *then there exists a function $\tilde{f} : V \to A$ such that:*

[15] V is open if for any $v \in V$ there exists $\epsilon_v > 0$ such that for any $u \in \mathcal{R}^{m \times n}$, if $|u_i(a) - v_i(a)| < \epsilon_v$ for all i, a then $u \in V$ as well.

1. *If f satisfies W-MON then $\tilde{f}$ satisfies S-MON.*

2. *If $\tilde{f}$ is affine maximizer then f is affine maximizer.*

proof: We use the notation $v + \epsilon 1_{i,b} = (v_i|^{b+=\epsilon}, v_{-i})$, and $v + \epsilon 1_b = v + \epsilon 1_{1b} + \ldots + \epsilon 1_{nb}$. For any $v \in V$, define:

$$T(v) \;=\; \{\; b \in A \mid \exists \epsilon^* > 0 \text{ s.t. } \forall \epsilon \in (0, \epsilon^*) : f(v + \epsilon 1_b) = b \;\}$$

Claim 11 *For any $v \in V$, i, and $u_i \in V_i$: if $a \in T(v)$, $b \in T(u_i, v_{-i})$, and $u_i(a) - v_i(a) \geq u_i(b) - v_i(b)$, then $a \in T(u_i, v_{-i})$.*

proof: For any (small enough) $\epsilon > 0$, since $a \in T(v)$, $f(v + \epsilon 1_a) = a$. By W-MON, it follows that:

$$f(v_i + \epsilon 1_{i,a},\ v_{-i} + 2\epsilon 1_{-i,b} + 4\epsilon 1_{-i,a}) = a \tag{1}$$

(this follows by changing the player types one at a time). Similarly, since $b \in T(u_i, v_{-i})$, we get that $f(u_i + \epsilon 1_{i,b},\ v_{-i} + \epsilon 1_{-i,b}) = b$. By W-MON (changing the player types one at a time):

$$f(u_i + 2\epsilon 1_{i,b} + 4\epsilon 1_{i,a},\ v_{-i} + 2\epsilon 1_{-i,b} + 4\epsilon 1_{-i,a}) \in \{a, b\} \tag{2}$$

Since $u_i(a) - v_i(a) \geq u_i(b) - v_i(b)$ it follows that $[u_i(a) + 4\epsilon] - [v_i(a) + \epsilon] > [u_i(b) + 2\epsilon] - v_i(b)$. Therefore, comparing Eq. 2 to Eq. 1, and by W-MON, we conclude that $f(u_i + 2\epsilon 1_{i,b} + 4\epsilon 1_{i,a},\ v_{-i} + 2\epsilon 1_{-i,b} + 4\epsilon 1_{-i,a}) = a$. Thus also $f(u_i + 5\epsilon 1_{i,a},\ v_{-i} + 5\epsilon 1_{-i,a}) = a$, hence $a \in T(u_i, v_{-i})$, and the claim follows. ∎

We can now define $\tilde{f}$. Fix any complete order $\succ$ on A, and then:

$$\tilde{f}(v) = \max_{\succ} T(v)$$

To see that $\tilde{f}$ satisfies S-MON, suppose that $\tilde{f}(v) = a$ and $\tilde{f}(u_i, v_{-i}) = b$. Therefore $a \in T(v)$ and $b \in T(u_i, v_{-i})$. Assume by contradiction that $u_i(a) - v_i(a) \geq u_i(b) - v_i(b)$. By claim 11 it follows that $a \in T(u_i, v_{-i})$, and thus $b \succ a$. On the other hand, it is also the case that $v_i(b) - u_i(b) \geq v_i(a) - u_i(a)$, and so, by claim 11 again (changing variable names), we get that $b \in T(v)$ and therefore $a \succ b$, a contradiction.

To see that, if $\tilde{f}$ is affine maximizer, then f is affine maximizer as well, assume that for any $v \in V$, $\tilde{f}(v) \in argmax_{a \in A}\{\sum_i \omega_i v_i(a) + \gamma_a\}$, and suppose that $\tilde{f}(v) = a$ but $f(v) = b$. We first claim that for any (small enough) $\epsilon > 0$, $\tilde{f}(v + \epsilon 1_b) = b$: otherwise, suppose it equals c. By definition, this implies that $f(v + \epsilon 1_b + (\epsilon/2) 1_c) = c$, contradicting PAD (claim 1), since $f(v) = b$ and b was raised strictly more than all other alternatives for all players. Since $\tilde{f}$ is affine maximizer it follows that $\sum_i \omega_i [v_i(b) + \epsilon] + \gamma_b \geq \sum_i \omega_i [v_i(a) + \epsilon] + \gamma_a$. This is true for any (small enough) $\epsilon > 0$, so it follows that $f(v)$ chooses a maximal alternative as well, as claimed.
This concludes the proof of claim 10. ∎

We can now conclude the proof of the theorem. We first claim that we can assume w.l.o.g that V is not normalized. Otherwise, we can expand V to a non normalized $\hat{V}$, exactly as in claim 7. Then $\hat{f}$ also satisfies W-MON, and if $\hat{f}$ is affine maximizer then f is affine maximizer as well. Since $\hat{V}$ is strict order based and is not normalized, it follows that it is an open set, and we can apply claim 10. Thus there exists $\tilde{f}$ that satisfies S-MON, and if $\tilde{f}$ is affine maximizer then $\hat{f}$ is affine maximizer, which in turn implies that f is affine maximizer as well.

To be completely exact, we note that all translations preserve player decisiveness (a requirement that will be defined later and will be used to prove affine maximization). Also, to prove affine maximization we will need the domain to have conflicting preferences, but $\tilde{V}$ does not have conflicting preferences - it lacks normalization. This is solved by defining $\tilde{V}|_0 = \{v \in V | \forall j \neq i : \; v_i(c^j) = 0\}$, and $\tilde{f}|_0 : \tilde{V}|_0 \to A$ such that $\tilde{f}|_0(v) = \tilde{f}(v)$. Now $\tilde{V}|_0$ has conflicting preferences (and satisfies S-MON and player decisiveness), and if $\tilde{f}|_0$ is affine maximizer then $\tilde{f}$ is affine maximizer as well (this follows since $\tilde{f}$ satisfies S-MON). ∎

B Proof of Main Theorem

Our main theorem is stated for order based domains with *conflicting preferences.* These are essentially order based domains in which the most preferred alternative of player i is the least preferred alternative of all other players:

Definition 9 (top and bottom alternatives of player i) *Suppose V_i is order based. The alternative $a \in A \setminus 0_i$ is a top alternative if for all other $b \in A$, $R_i(a,b) \in \{ > \, , \, \geq \, , \, null \}$. The alternative $a \in A$ is a bottom alternative if for all other $b \in A$, $R_i(a,b) \notin \{ > \, , \, \geq \}$.*

Definition 10 (Conflicting preferences) *An order based domain has conflicting preferences if:*

1. *Any player i has at least one top alternative (denoted c^i).*
2. *For all i and $j \neq i$, c^j is a bottom alternative for player i, and $c^j \in 0_i$* [16]*.*

Note that $c^1 \neq c^i$ for all $i \neq 1$ as $c^1 \notin 0_1$ and $c^i \in 0_1$. CAs and MUAs have conflicting preferences: the allocation of all the goods to player i is a top alternative for i, and is indeed a bottom alternative (with a value of zero) for all other players. Matching, however, does not have conflicting preferences, since there is no top alternative.

The theorem states that for order based domains with conflicting preferences, any strongly monotone function must be almost affine maximizer. For this, we need two additional conditions:

Definition 11 (Player decisiveness) *f is player-decisive if any player can always ensure that his top alternative will be chosen by bidding high enough on it. More formally, for any $v \in V$ and any player i there exist $u_i = v_i|^{c^i+=\delta}$ and $\delta > 0$ such that $f(u_i, v_{-i}) = c^i$.*

Definition 12 (Non-degenerate range) *A is non-degenerate if for any player $i > 1$ there exists $a \in A$ such that $a \notin 0_1$ and $a \notin 0_i$.*

The latter condition is essential: for example, if we limit the range to contain only the top alternatives of all players, then this becomes a one-parameter mechanism, which may be implemented in many diverse ways, as discussed in the paper body. The first condition is essential for welfare approximation.

B.1 Exposition of the main proof

In this sub-section, we describe the main proof in a somewhat abstract way, hiding many definitions and proofs in the sub-sections below. This abstraction also enables us to show that this proof, without most of the qualifiers, holds for unrestricted domains (this is described in section D below). In addition, the abstraction emphasizes what is really needed in order to derive the main proof, and so S-MON, player decisiveness and the other qualifiers do not appear in the abstraction itself. We

[16] This normalization is for convenience. We can instead just assume that for any i, j, l, $R_i(c^j, c^l)$ is "=", and use S-MON (definition in the sequel) to normalize the domain.

continue by describing several requirements for f, V, and A. We show in parallel that: (1) these requirements are satisfied for order based domains with conflicting preferences, and, (2) under these requirements, f must be almost affine maximizer, more specifically, f must be affine maximizer for any $v \in V^* \subseteq V$, for a specific "large" V^* defined below.

Consider the following notation: by denoting $x@a$ ("x *at* a"), for $x \in \mathcal{R}^n$ and $a \in A$, we state that there exists $v \in V$ such that $v(a) = x$.

Definition: *("x at a" is larger than "y at b") For any distinct $a, b \in A$ and $x, y \in \mathcal{R}^n$ we say that $x@a > y@b$ if there exists $v \in V$ such that $v(a) = x$, $v(b) = y$, and $f(v) = a$. (we also use $x@a < y@b$ to denote $y@b > x@a$).*

Definition 13 *(Comparable) $x@a$ and $y@b$ are comparable if either $x@a > y@b$ or $y@b > x@a$.*

Requirement 1: Closure under positive translation.

(the exact details of the requirements are less important for this exposition, and are described in detail in the next sub-section).

Proposition 2 *If V is order based with conflicting preferences, and f satisfies S-MON, then Requirement 1 is satisfied.*

An affine maximizer attaches a measure to every alternative (its weighted welfare), and chooses the one with the highest measure. Essentially, we show that every social choice function with certain characteristics must do the same. This is inspired by the "min-function" model of Archer and Tardos [2]. We then show that this measure must be an affine function. In order to actually calculate the measure of alternative a, for some type v, we use a *reference* alternative, c. Intuitively, we start with some type profile where $x@a$ is chosen, and raise c's welfare until c is chosen. If we do this "slowly", then at some point the measure of the two alternatives will become equal, and thus, by knowing the measure of c, we obtain the measure of a. For this to succeed, we need an alternative with several technical properties:

Requirement 2: A Transitive Reference $z@c$ (where $z \in \mathcal{R}^n_+$) for the set $M_c \subseteq A \setminus \{c\}$.

Proposition 3 *If V is order based with conflicting preferences, and f satisfies S-MON and player decisiveness, then Requirement 2 is satisfied: $e_1@c^1$ is a transitive reference for $A \setminus \{c^1\}$.*

We can now measure the "weighted welfare" of $x@a$:

Definition: *The "measure of x at a", for $a \in M_c$ and $x \in V^a$, is defined as:*

$$m(x@a) = inf\ \{\ \alpha \in R \mid x@a < (\alpha \cdot z)@c\ \},$$

where $\alpha \cdot z = (\alpha z_1, \alpha z_2, \ldots, \alpha z_n)$.

The following claim specifies some properties of the measure function: the second property implies that, in a sense, f chooses an alternative with maximal measure, and thus the measure induces a partial order. The fourth property implies that, for some alternatives, we can "inflate" and control the measure in a very simple way. The other properties show that the measure function preserves monotonicity and translation:

Claim 12 *Requirements 1 and 2 imply that, for any $a, b \in M_c$, $x \in V^*|_a$, and $y \in V^*|_b$* [17]*,*

[17] $V^*|_a = \{\ v(a) \mid v \in V^*\ \}$

1. $m(x@a) = sup \{ \alpha \mid x@a > (\alpha \cdot z)@c \}$, *and, as a conclusion,* $-\infty < m(x@a) < \infty$.

2. $m(x@a) < m(y@b) \;\Rightarrow\; \neg(x@a > y@b)$.

3. *If* $a \notin 0_i$ *then* $m((x + \delta \cdot e_i)@a) \geq m(x@a)$ *for any* $\delta > 0$.

4. *Define the "support" of* $z@c$: $S_c = \{ a \in M_c \mid a \notin 0_i$ *for all* i *s.t.* $z_i > 0 \}$. *Then, if* $a \in S_c$ *and* $\beta > 0$: $m((x + \beta \cdot z)@a) = m(x@a) + \beta$.

5. *Suppose* $a, b \notin 0_i$, $x@a$ *and* $y@b$ *are comparable, and* $m(x@a) < m(y@b)$. *Then* $m((x + \delta \cdot e_i)@a) \leq m((y + \delta \cdot e_i)@b)$, *for any* $\delta > 0$.

Using these properties, we next show that $m(x@a)$ is affine.

Definition 14 $x@a$ *"calibrates"* $y@b$ *using* $w \in \mathcal{R}^n_+$ *if* $x@a < y@b$ *and if* $(x + \alpha \cdot w)@a$ *and* $y@b$ *are comparable for any* $\alpha > 0$.

In a descriptive manner, $x@a$ calibrates $y@b$ using w if we can "inflate" $x@a$ to $(x + \alpha \cdot w)@a$ while still keeping it comparable to $y@b$. The following claim implies that, if we can inflate the measure function as well, then the derivatives of $m(x@a)$ and $m(y@b)$ are identical:

Claim 13 *Suppose there exist* $\omega \in \mathcal{R}$, $\omega > 0$ *and* $w \in \mathcal{R}^n_+$ *such that* $m((x+\alpha\cdot w)@a) = m(x@a)+\omega\cdot\alpha$ *for all* $x \in V^a$ *and* $\alpha > 0$. *Fix any* $x \in V^a$ *and* $y \in V^b$ *such that* $x@a$ *calibrates* $y@b$ *using* w. *Then for any* i *such that* $a, b \notin 0_i$ *and any* $\delta > 0$: $m((x+\delta\cdot e_i)@a) - m(x@a) = m((y+\delta\cdot e_i)@b) - m(y@b)$.

In order to show affinity, we follow this basic technical fact:

Technical Fact: *Fix some* $m : X \to \mathcal{R}$, *where* $X \subseteq \mathcal{R}^n$ *has the property that* $x \in X$ *and* $y \geq x$ *implies* $y \in X$. *Suppose also that* m *is monotonically non-decreasing.*

1. *If there exists* $h_i : \mathcal{R}_+ \to \mathcal{R}_+$ *such that* $m(x + \delta \cdot e_i) - m(x) = h_i(\delta)$ *for any* $x \in X$ *and* $\delta > 0$, *then there exist* $\omega_i \in \mathcal{R}$ *such that* $h_i(\delta) = \omega_i \cdot \delta$.

2. *If there exist* $\omega_1, \ldots, \omega_n$ *such that* $m(x + \delta \cdot e_i) - m(x) = \omega_i \cdot \delta$ *for any* i, $x \in X$, *and* $\delta > 0$ *then* $m(x) = \sum_{i=1}^{n} \omega_i \cdot x_i \; + \gamma$ *(for some constant* $\gamma \in \mathcal{R}$*)*.

Now, suppose there is some alternative d such that, for any $y@d$ and $\tilde{y}@d$, there exists $x@a$ that calibrates both. Then, by claim 13 and by the technical fact, we can conclude that $m(y@d)$ is affine. If, in addition, for any $x@a$ there exists $y@d$ that calibrates it, we can simply conclude that all the measure functions are affine. We call such an alternative a calibrator:

Requirement 3: A calibrator for player i (for any $i > 1$).

Proposition 4 *If* V *is order based with conflicting preferences,* f *satisfies S-MON and player decisiveness, and* A *is non-degenerate, then alternative* c^i *is a calibrator for* i.

As described above, the consequence of a calibrator for i is:

Claim 14 *If there exists a calibrator for player* i *then there exists* $\omega_i \in \mathcal{R}$, $\omega_i > 0$ *such that for any* $b \in M_c \setminus 0_i$, *for any* $x \in V^b$ *and for any* $\delta > 0$: $m((x + \delta \cdot e_i)@b) - m(x@b) = \omega_i \cdot \delta$.

From all the above, we can now conclude our main theorem:

Theorem 3 *Suppose V is order based with conflicting preferences, f is strongly monotone and player-decisive, and A is non-degenerate. Then f is affine maximizer for any $v \in V^*$.*

Corollary 2 *If V is order based with conflicting preferences, f satisfies S-MON and player-decisiveness, and A is non-degenerate, then there exist $\omega_1, \ldots, \omega_n$, $\{\gamma_a\}_{a \in A}$, and a constant $M \in \mathcal{R}$ s.t.:*

$$f(v) \in argmax_{a \in A}\{ \sum_{i=1}^{n} \omega_i \cdot v_i(a) + \gamma_a \}$$

for all $v \in V$ such that $v_i(a) > M$ for all i and $a \notin 0_i$.

The next corollary attempts to drop the "almost" qualifier. E.g., for a CA of 2 players, where all the goods are always allocated, this corollary implies that either one of the players receives all the goods, or the allocation with the maximal weighted welfare among all other allocations is chosen.

Corollary 3 *Suppose V is an order based domain with conflicting preferences, f is strongly monotone and player-decisive, and A is non-degenerate. For any $I' \subseteq \{1, \ldots, n\}$, denote $B_{I'} = \{a \in A \mid a \in 0_i$ if and only if $i \in I'\}$. Then there exist $\omega_1, \ldots, \omega_n$ and $\{\gamma_a\}_{a \in A}$ such that:*

$$f(v) \in \bigcup_{I' \subseteq \{1,\ldots,n\}} argmax_{b \in B_{I'}} \{ \sum_{i=1}^{n} \omega_i v_i(b) + \gamma_b \}$$

Among the three conditions on f, it seems that the crucial one is the strong monotonicity (indeed, in section 5 we show examples of truthful CAs with non-degenerate range, that satisfy player decisiveness, but do not satisfy S-MON, and are not affine maximizers). On the other hand, for one parameter domains, it is not hard to construct strongly monotone functions that are not (almost) affine maximizers. The main question remained is, for exactly what domains is S-MON the main characterization of affine maximization:

Open Problem 1: *Is there a weaker condition than S-MON that implies affine maximization for combinatorial auctions?*

Open Problem 2: *Does S-MON imply affine maximization for order based domains that do not have conflicting preferences (e.g. matching) ?*

B.2 Requirements 1 and 2

As we cannot measure a general assignment $x@a$ if there is no $v \in V$ such that $f(v) = a$ and $v(a) = x$, we refer to the abstract sets $V^a \subseteq \{x \mid \exists v \in V \text{ such that } v(a) = x\}$, for all $a \in A$, and $V^* = \{ v \in V \mid \forall a : v(a) \in V^a \}$.

This enables us to describe the proof in an abstract context, somewhat disconnected from its main application – order based domains with conflicting preferences. For this specific application, we define the V^a's as follows:

Definition of V^a, for all $a \in A$: Denote $T = \{c^1, c^2, \ldots, c^n\}$ and $S = \{a \in A \mid a_1 \notin 0_1 \text{ and } a \neq c^1\}$, then

1. $V^{c^1} = \{ v(c^1) \mid v \in V \}$.
2. For any $a \in S : V^a = \{ x \mid \exists v \in V \text{ s.t. } v(a) = x \text{ and } f(v) = a \}$.

3. For any $c^i, i > 1\ :\ V^{c^i} = \{\ y \mid \exists a \in S,\ a \notin 0_i,\ \text{and}\ \exists x \in V^a \text{ s.t. } y@c^i > x@a\ \}$.
4. For any $a \in 0_1 \setminus T\ :\ V^a = \{\ x \mid\ \text{for all } i \text{ s.t. } a \notin 0_i \text{ there exists } y \in V^{c^i} \text{ s.t. } x@a > y@c^i\ \}$.

Note that f is on A, and this implies e.g. that V^a, for all $a \in S$, is not empty. We shall show that all the V^a's are not empty as well, and thus V^* is well defined.

We next describe exactly requirements 1 and 2 and show that they are satisfied in order based domains with conflicting preferences.

Requirement 1: Closure under positive Translation. For any $a, b \in A$, player i such that $a, b \notin 0_i$, and $\delta > 0$:

1. $x@a > y@b$ implies $(x + \delta \cdot e_i)@a > (y + \delta \cdot e_i)@b$ [18].
2. $x \in V^a$ implies $(x + \delta \cdot e_i) \in V^a$.

Proposition 5 *If V is order based with conflicting preferences, and f satisfies S-MON, then Requirement 1 is satisfied.*

proof: We prove this in several steps, as follows:

Definition 15 (Closure under minimum) *A domain V is "closed under minimum" if for any $v, v' \in V\ :\ min(v, v') \in V$, where the minimum is taken componentwise.*

Claim 15 *Any order based domain is closed under minimum.*

proof: Suppose $v_i, v_i' \in V_i$, and let $u_i = min(v_i, v_i')$. For any $a \in 0_i$, since $v_i(a) = v_i'(a) = 0$ then $u_i(a) = 0$. For any $a, b \in A$, if $R_i(a, b) \in \{ = ,\ \text{null}\ \}$ then trivially "$u_i(a)\ R_i(a, b)\ u_i(b)$". Suppose $R_i(a, b)$ is "$\geq$", and suppose $u_i(a) = v_i(a)$. Then $u_i(b) \leq v_i(b) \leq v_i(a) = u_i(a)$, as needed. The other cases are similar. ∎

Claim 16 *Suppose V is closed under minimum and f is strongly monotone. Then for any $x@a$, $y@b$, $\tilde{x} \geq x$, and $\tilde{y} \leq y$: $x@a > y@b \Rightarrow \neg(\tilde{x}@a < \tilde{y}@b)$.*

proof: Since $x@a > y@b$, $\exists v \in V$ such that $v(a) = x$, $v(b) = y$, and $f(v) = a$. Suppose by contradiction that $\tilde{x}@a < \tilde{y}@b$. Therefore $\exists v' \in V$ such that $v'(a) = \tilde{x}$, $v'(b) = \tilde{y}$, and $f(v') = b$. Let $u = min(v, v')$. By S-MON, since $u(a) = v(a)$ and $u \leq v$, $f(v) = a$ implies $f(u) = a$. And since $u(b) = v'(b)$ and $u \leq v'$, $f(v') = b$ implies $f(u) = b$, a contradiction. ∎

Claim 17 *Suppose V is order based with conflicting preferences, and f is strongly monotone. Then for any $a, b \in A$, if $x@a > y@b$ and $a, b \notin 0_i$ then $(x + \delta \cdot e_i)@a > (y + \delta \cdot e_i)@b$ for any $\delta > 0$.*

proof: Since $x@a > y@b$ there exists $v \in V$ with $v(a) = x$, $v(b) = y$, and $f(v) = a$. Define u_i as follows: $u_i(d) = v_i(d) + \delta$ for any $d \in A \setminus 0_i$, and $u_i(d) = v_i(d)$ for any $d \in 0_i$. Since V is order based, and the alternatives in 0_i are all bottom alternatives (this is a side-effect of the second condition of conflicting preferences), raising all non-zero coordinates by a constant does not violate any relation, and so $u_i \in V$. Since $u_i(a) - v_i(a) \geq u_i(d) - v_i(d)$ for any $d \neq a$ it follows from S-MON that $f(u_i, v_{-i}) = a$, so $(x + \delta \cdot e_i)@a > (y + \delta \cdot e_i)@b$. ∎

[18] $x \in \mathcal{R}^n$ is termed e_i (the unit vector) if $x_j = 0$ for all $j \neq i$ and $x_i = 1$.

Claim 18 *For any $a \in A$, V^a is non-empty, and $x \in V^a \Rightarrow (x + \delta \cdot e_i) \in V^a$, for all i such that $a \notin 0_i$, and for any $\delta > 0$.*

proof: For any $a \in S$ take any $v \in V$ such that $f(v) = a$, thus $v(a) \in V^a$, and, similarly to claim 17, $(v(a) + \delta \cdot e_i) \in V^a$, whenever $\delta > 0$ and $a \notin 0_i$.

For any c^i, choose some $a \in S \setminus 0_i$ (it is not empty since A is non-degenerate) and $x \in V^a$. By player decisiveness there exists some $y@c^i$ such that $y@c^i > x@a$, and thus $y \in V^{c^i}$. By S-MON and since c^i is a top alternative, $(y + \delta \cdot e_i)@c^i > x@a$, and thus $(y + \delta \cdot e_i) \in V^{c^i}$ as well.

For any $a \in 0_1 \setminus T$, take any $v \in V$ such that $f(v) = a$, thus $x@a > y@c^i$ (where $y = v(c^i)$). Notice that, since V is normalized, $y = y_i \cdot e_i$. Therefore there exists $\delta \geq 0$ such that $(y + \delta \cdot e_i) \in V^{c^i}$, and, by claim 17, $(x + \delta \cdot e_i)@a > (y + \delta \cdot e_i)@c^i$, thus $(x + \delta \cdot e_i) \in V^a$. If $x \in V^a$ then $x@a > y@c^i$ and, similarly to claim 17, $(x + \delta \cdot e_j)@a > y@c^i$ for any distinct i, j ($a \notin 0_i, 0_j$). ∎

We can therefore conclude that the closure under positive translation requirement follows from claims 17 and 18. ∎

Requirement 2: A Transitive Reference $z@c$ for the set $M_c \subseteq A \setminus \{c\}$.

$z@c$ (where $c \in A$ and $z \in \mathcal{R}^n_+, z \neq 0$) is called a transitive reference for the set $M_c \subseteq A \setminus \{c\}$ if, for any $a, b \in M_c$, $x \in V^a$, and $y \in V^b$:

1. Measurability: There exist $\alpha, \beta \in \mathcal{R}$, $\alpha < \beta$, such that $(\alpha \cdot z)@c < x@a < (\beta \cdot z)@c$, and for any $\alpha' \in (\alpha, \beta)$: $x@a$ and $(\alpha' \cdot z)@c$ are comparable.
2. (semi) Transitivity: If $x@a < (\alpha \cdot z)@c$ and $\neg(y@b < (\alpha \cdot z)@c)$ then $\neg(x@a > y@b)$.
3. R-monotonicity: $x@a > (\alpha \cdot z)@c \Rightarrow \neg(x@a < (\beta \cdot z)@c)$ for any $\beta \leq \alpha$.
4. L-monotonicity: For $\delta > 0$, $(x + \delta \cdot e_i)@a < (\alpha \cdot z)@c \Rightarrow x@a < (\alpha \cdot z)@c$.

Proposition 6 *If V is order based with conflicting preferences, and f satisfies S-MON and player decisiveness, then Requirement 2 is satisfied by taking $e_1@c^1$ to be a transitive reference for $A \setminus \{c^1\}$.*

proof: We show that $e_1@c^1$ is a transitive reference by several claims:

Claim 19 *If $x@a < (\alpha \cdot e_1)@c^1$ and $\neg(y@b < (\alpha \cdot e_1)@c^1)$ then $\neg(x@a > y@b)$.*

proof: Suppose by contradiction that $x@a > y@b$, and take $v \in V$ such that $v(a) = x$, $v(b) = y$, and $f(v) = a$. Since V is normalized, $v(c^1) = \beta \cdot e_1$. Thus $x@a > (\beta \cdot e_1)@c^1$ and by claim 16 it follows that $\beta \leq \alpha$. Denote $u_1 = v_1|^{c^1 = \alpha}$. By S-MON, $f(u_1, v_{-1}) \in \{a, c^1\}$. If it is a then $x@a > (\alpha \cdot e_1)@c^1$, contradicting the assumption $x@a < (\alpha \cdot e_1)@c^1$ (by claim 16). Thus it is c^1, and therefore $y@b < (\alpha \cdot e_1)@c^1$, a contradiction. ∎

Claim 20 *$\forall a \in A$ and $x \in V^a$, if $(x + \delta \cdot e_i)@a < (\alpha \cdot e_1)@c^1$, for some $\delta > 0$, then $x@a < (\alpha \cdot e_1)@c^1$.*

proof: Choose some v such that $v(a) = (x + \delta \cdot e_i)$, $v(c^1) = \alpha \cdot e_1$, and $f(v) = c^1$, and v' such that $v'(a) = x$ (there is one since $x \in V^a$). Let $u = min(v, v')$ (thus $u(a) = x$ and $u_1(c^1) \leq \alpha$) and $u' = u|^{c^1 = \alpha \cdot e_1}$. Thus $u'(a) = x$, $u'(c^1) = \alpha \cdot e_1$, and $u' \leq v$. By S-MON, $f(u') = c^1$, as needed. ∎

Definition 16 (Compatible) *$x@a$ and $y@b$ are "compatible" if there exists $v \in V$ such that $v(a) = x$ and $v(b) = y$.*

Claim 21 *For any $a \in A$, $c^i \in T$, $x \in V^a$, and $(\alpha \cdot e_i)@c^i$, if $x@a$ and $(\alpha \cdot e_i)@c^i$ are compatible then they are comparable.*

proof: Since $x \in V^a$ there exists $v \in V$ such that $v(a) = x$ and $f(v) = a$. Since $x@a$ and $(\alpha \cdot e_i)@c^i$ are compatible then there exists $v' \in V$ such that $v'(a) = x$ and $v'(c^i) = \alpha \cdot e_i$. Let $u = min(v, v')$. Thus $f(u) = a$ and $u_i(c^i) \leq \alpha$. Therefore $f(u|^{c^i=\alpha \cdot e_i}) \in \{a, c^i\}$, and the claim follows. ∎

We can now conclude that $e_1@c^1$ is a transitive reference for $A \backslash \{c^1\}$: Transitivity is by claim 19, R-monotonicity follows from claim 16, and L-monotonicity is by claim 20. Measurability follows since, by the definition of the V^a's, there exists $(\alpha \cdot z)@c$ such that $x@a > (\alpha \cdot z)@c$, by player decisiveness there exists $(\beta \cdot z)@c$ such that $x@a < (\beta \cdot z)@c$, and by claim 21, since for any $\alpha' \in (\alpha, \beta)$, $x@a$ and $(\alpha' \cdot e_1)@c^1$ are compatible, they are comparable. ∎

B.3 The measure of $x@a$

Definition: *The "measure of x at a", for $a \in M_c$ and $x \in V^a$, is defined as:*

$$m(x@a) = inf \ \{ \ \alpha \mid x@a < (\alpha \cdot z)@c \ \}$$

and also: $m((\alpha \cdot z)@c) = \alpha$.

The following claims relate to any $a, b \in M_c$, $x \in V^a$, and $y \in V^b$:

Claim 22 *$m(x@a) = sup \ \{ \ \alpha \mid x@a > (\alpha \cdot z)@c \ \}$, and, as a conclusion, $-\infty < m(x@a) < \infty$.*

proof: By measurability, there exist $\alpha, \beta \in \mathcal{R}$ such that $(\alpha \cdot z)@c < x@a < (\beta \cdot z)@c$. Hence the infimum is at most β, and the supremum is at least α. By R-monotonicity, the infimum is not smaller than the supremum. Therefore they both reside in the interval $[\alpha, \beta]$. Since for any $\alpha' \in [\alpha, \beta]$, $x@a$ and $(\alpha' \cdot z)@c$ are comparable, the claim follows. ∎

Claim 23 *$m(x@a) < m(y@b) \Rightarrow \neg(x@a > y@b)$ (and thus $x@a > y@b \Rightarrow m(x@a) \geq m(y@b)$).*

proof: Choose some α, $m(x@a) \leq \alpha < m(y@b)$, such that $x@a < (\alpha \cdot z)@c$. Since $\neg(y@b < (\alpha \cdot z)@c)$ it follows, by Transitivity, that $\neg(x@a > y@b)$. ∎

Claim 24 *If $a \notin 0_i$ then $m((x + \delta \cdot e_i)@a) \geq m(x@a)$ for any $\delta > 0$.*

proof: Suppose by contradiction that $m(x@a) > m((x + \delta \cdot e_i)@a)$, and choose some α, $m(x@a) > \alpha \geq m((x + \delta \cdot e_i)@a)$, such that $(x + \delta \cdot e_i)@a < (\alpha \cdot z)@c$. But since $\neg(x@a < (\alpha \cdot z)@c)$, this contradicts L-monotonicity. ∎

Definition 17 (The "support" of $z@c$) $S_c = \{ \ a \in M_c \mid a \notin 0_i$ *for all i s.t.* $z_i > 0 \ \}$.

Claim 25 *If $a \in S_c$ then for any $\beta > 0$: $m((x + \beta \cdot z)@a) = m(x@a) + \beta$.*

proof: Suppose by contradiction that $m((x + \beta \cdot z)@a) > m(x@a) + \beta$. Choose some α, $m((x + \beta \cdot z)@a) - \beta > \alpha \geq m(x@a)$, such that $x@a < (\alpha \cdot z)@c$. By positive translation, $x@a < (\alpha \cdot z)@c \Rightarrow (x + \beta \cdot z)@a < (\alpha + \beta) \cdot z@c$. But since $\alpha + \beta < m((x + \beta \cdot z)@a)$, we have a contradiction.

Similarly, suppose by contradiction that $m((x + \beta \cdot z)@a) < m(x@a) + \beta$. Choose some α, $m((x + \beta \cdot z)@a) - \beta < \alpha \leq m(x@a)$, such that $x@a > (\alpha \cdot z)@c$ (such α exists by claim 22). By positive translation, $x@a > (\alpha \cdot z)@c \Rightarrow (x + \beta \cdot z)@a > (\alpha + \beta) \cdot z@c$. But since $\alpha + \beta > m((x + \beta \cdot z)@a)$, we have a contradiction to claim 22. ∎

Claim 26 *Suppose $a, b \notin 0_i$, $x@a$ and $y@b$ are comparable, and $m(x@a) < m(y@b)$. Then $m((x + \delta \cdot e_i)@a) \leq m((y + \delta \cdot e_i)@b)$, for any $\delta > 0$.*

proof: $m(x@a) < m(y@b)$ implies $\neg(x@a > y@b)$ by claim 23. Since they are comparable it follows that $x@a < y@b$. Thus $(x + \delta \cdot e_i)@a < (y + \delta \cdot e_i)@b$ by the closure under positive translation property, and thus by claim 23 again, $m((x + \delta \cdot e_i)@a) \leq m((y + \delta \cdot e_i)@b)$, as claimed. ∎

The next claim argues that, in some sense, f will choose an alternative with maximal measure:

Claim 27 *For any $a, b \in M_c$, and $v \in V$ such that $v(a) = x \in V^a$,*

1. *If $v(b) = y \in V^b$, and $m(x@a) < m(y@b)$, then $f(v) \neq a$.*
2. *If $v(c) = \alpha \cdot z$ (for some $\alpha \in \mathcal{R}$) and $m(x@a) < \alpha$, then $f(v) \neq a$.*
3. *If $v(c) = \alpha \cdot z$ (for some $\alpha \in \mathcal{R}$) and $m(x@a) > \alpha$, then $f(v) \neq c$.*

proof: (1) If, by contradiction, $f(v) = a$ then $x@a > y@b$ by definition, contradicting claim 23.
(2) Suppose $f(v) = a$, thus $x@a > (\alpha \cdot z)@c$, and by claim 22, $m(x@a) \geq \alpha$, a contradiction.
(3) Suppose $f(v) = c$, thus $x@a < (\alpha \cdot z)@c$ and, by definition, $m(x@a) \leq \alpha$, a contradiction. ∎

B.4 Two technical facts

Claim 28 *Suppose $m : \mathcal{R}_+ \to \mathcal{R}$ is monotonically non-decreasing and there exists $h : \mathcal{R}_+ \to \mathcal{R}_+$ such that $m(x+\delta) - m(x) = h(\delta)$ for any $x, \delta \in \mathcal{R}_+$. Then there exist $\omega \in \mathcal{R}_+$ such that $h(\delta) = \omega \cdot \delta$.*

proof: Let $\omega = h(1)$ (note that $\omega \geq 0$ since m is non-decreasing). First we claim that for any two integers p, q, $h(p/q) = \omega \cdot (p/q)$. Note that $h(1) = m(1) - m(0) = \sum_{i=0}^{q-1} m((i+1)/q) - m(i/q) = q \cdot h(1/q)$. Thus $h(1/q) = (1/q) \cdot h(1)$. Similarly, $h(p/q) = m(p/q) - m(0) = \sum_{i=0}^{p-1} m((i+1)/q) - m(i/q) = p \cdot h(1/q) = (p/q) \cdot h(1) = (p/q) \cdot \omega$. Now we claim that for any real δ, $h(\delta) = \delta \cdot \omega$. Notice that since m is monotonically non-decreasing then h must be monotonically non-decreasing as well. Suppose by contradiction that $h(\delta) > \delta \cdot \omega$. Choose some rational $r > \delta$ close enough to δ such that $h(\delta) > r \cdot \omega$. Since h is monotone and $r > \delta$ then $h(r) \geq h(\delta)$, but since r is rational, $h(r) = r \cdot \omega < h(\delta)$, a contradiction. A similar argument holds if $h(\delta) < \delta \cdot \omega$. ∎

Claim 29 *Suppose that $X \subseteq \mathcal{R}^n$ has the property that $x \in X$ and $y \geq x$ implies $y \in X$. Let $m : X \to \mathcal{R}$ be monotonically non-decreasing, and suppose there exist $\omega_1, \ldots, \omega_n$ such that $m(x + \delta \cdot e_i) - m(x) = \omega_i \cdot \delta$ for any i, $x \in X$, and $\delta > 0$. Then there exist $\gamma \in \mathcal{R}$ such that $m(x) = \sum_{i=1}^{n} \omega_i \cdot x_i \ + \gamma$.*

proof: First we claim that for any $x, y \in X$ such that $y_i \geq x_i$ for all i, it is the case that $m(y) = m(x) + \sum_{i=1}^{n} \omega_i \cdot (y_i - x_i)$. Notice that $(y_1, x_2, \ldots, x_n) \in X$, and $m(y_1, x_2, \ldots, x_n) = m(x) + h_1(y_1 - x_1)$. Repeating this step n times we get $m(y) = m(x) + \sum_{i=1}^{n} \omega_i \cdot (y_i - x_i)$.

Now fix some $x^* \in X$. We claim that for any $x \in X$, $m(x) = m(x^*) + \sum_{i=1}^{n} \omega_i \cdot (x_i - x_i^*)$. To see this, choose some y such that $y_i \geq x_i, x_i^*$ for all i. Thus $m(y) = m(x) + \sum_{i=1}^{n} \omega_i \cdot (y_i - x_i)$ and also $m(y) = m(x^*) + \sum_{i=1}^{n} \omega_i \cdot (y_i - x_i^*)$, therefore the claim follows. ∎

B.5 Calibration

Definition: *$x@a$ "calibrates" $y@b$ using $w \in \mathcal{R}^n_+$ if $x@a < y@b$, and, for any $\alpha > 0$, $(x+\alpha\cdot w)@a$ and $y@b$ are comparable.*

Claim 30 *Suppose there exist $\omega \in \mathcal{R}$, $\omega > 0$ and $w \in \mathcal{R}^n_+$ such that $m((x+\alpha\cdot w)@a) = m(x@a)+\omega\cdot\alpha$ for all $x \in V^a$ and $\alpha > 0$. Fix any $x \in V^a$ and $y \in V^b$ such that $x@a$ calibrates $y@b$ using w. Then for any i such that $a, b \notin 0_i$ and any $\delta > 0$: $m((x+\delta\cdot e_i)@a) - m(x@a) = m((y+\delta\cdot e_i)@b) - m(y@b)$.*

proof: Since $x@a < y@b$ then $m(x@a) \leq m(y@b)$. Let β be such that $\omega \cdot \beta = m(y@b) - m(x@a)$ (thus $\beta \geq 0$). If $\beta > 0$, then for any $0 < \alpha < \beta$, $m((x+\alpha\cdot w)@a) = m(x@a) + \omega\cdot\alpha < m(y@b)$, and thus by claim 26, $m((x+\delta\cdot e_i+\alpha\cdot w)@a) \leq m((y+\delta\cdot e_i)@b)$. Therefore $m((x+\delta\cdot e_i+\beta\cdot w)@a) \leq m((y+\delta\cdot e_i)@b)$. If $\beta = 0$ then since $x@a < y@b$, $(x+\delta\cdot e_i)@a < (y+\delta\cdot e_i)@b$ and so $m((x+\delta\cdot e_i)@a) \leq m((y+\delta\cdot e_i)@b)$.

For any $\alpha > \beta$, $m((x+\alpha\cdot w)@a) = m(x@a) + \omega\cdot\alpha > m(y@b)$, and thus by claim 26, $m((x+\delta\cdot e_i+\alpha\cdot w)@a) \geq m((y+\delta\cdot e_i)@b)$. Therefore $m((x+\delta\cdot e_i+\beta\cdot w)@a) = m((y+\delta\cdot e_i)@b)$. Since $m((x+\delta\cdot e_i+\beta\cdot w)@a) = m((x+\delta\cdot e_i)@a) + \omega\cdot\beta$ and $\omega\cdot\beta = m(y@b) - m(x@a)$, the claim follows. ∎

Requirement 3: A calibrator. An alternative $d \in M_c$ is a "calibrator" for i if,

1. $d \notin 0_i$, and there exist $y \in V^d$, $(\alpha\cdot z)@c$, and $\epsilon, \delta > 0$ such that $y@d < (\alpha\cdot z)@c$, and $(y+\delta\cdot e_i)@d > ((\alpha+\epsilon)\cdot z)@c$.
2. For all $y, \tilde{y} \in V^d$ there exists $a \in S_c \setminus 0_i$ and $x \in V^a$ such that $x@a$ calibrates both $y@d$ and $\tilde{y}@d$ using z.
3. For all $a \in S_c \setminus 0_i$ and $x \in V^a$ there exists $y \in V^d$ such that $x@a$ calibrates $y@d$ using z.
4. For all $b \in (M_c \setminus S_c) \setminus 0_i$, and $x \in V^b$, there exists $y \in V^d$ such that $y@d$ calibrates $x@b$ using e_i.

Proposition 7 *If V is order based with conflicting preferences, f satisfies S-MON and player decisiveness, and A is non-degenerate, then Requirement 3 is satisfied: alternative c^i is a calibrator for i (for any $i > 1$).*

proof: The first calibrator requirement follows immediately from player decisiveness. For the second requirement, notice that S is the support of $e_1@c^1$. Now fix any $y, \tilde{y} \in V^{c^i}$, and suppose $y \leq \tilde{y}$. Since A is non-degenerate, there exists some $a \in S \setminus 0_i$. By definition of V^{c^i}, there exists $x \in V^a$ such that $y@c^i > x@a$. Thus also $\tilde{y}@c^i > x@a$. Since $i > 1$, for any $\alpha > 0$, $y@c^i$ and $(x+\alpha\cdot e_1)@a$ are compatible. Since $(x+\alpha\cdot e_1) \in V^a$ it follows from claim 21 that they are comparable (the same is true for $\tilde{y}@c^i$), and so the second requirement holds. The third and fourth requirements follows from essentially the same arguments. ∎

Claim 31 *If there exists a calibrator d for player i then there exists some $\omega_i \in \mathcal{R}$, $\omega_i > 0$ such that for any $b \in M_c \setminus 0_i$, for any $x \in V^b$ and for any $\delta > 0$: $m((x+\delta\cdot e_i)@b) - m(x@b) = \omega_i\cdot\delta$.*

proof: By the second calibrator property, for any $y, \tilde{y} \in V^d$ there exists $x@a$ that satisfies the conditions of claim 30 (recall that by claim 25, $m((x+\alpha\cdot z)@a) = m(x@a)+\alpha$). Thus:

$$m((y+\delta\cdot e_i)@d) - m(y@d) = m((x+\delta\cdot e_i)@a) - m(x@a) = m((\tilde{y}+\delta\cdot e_i)@d) - m(\tilde{y}@d)$$

so we conclude that there exists some $h : R_+ \to R_+$ such that $m((y + \delta \cdot e_i)@d) - m(y@d) = h(\delta)$ for all $y \in V^d$ and $\delta > 0$. By claim 28 there exists some $\omega_i \in \mathcal{R}_+$ such that $h(\delta) = \omega_i \cdot \delta$.

By the first calibrator property we can conclude that $\omega_i > 0$, since, for $y \in V^d$ that is specified by the property, $m(y@d) \leq \alpha$, $m((y+\delta \cdot e_i)@d) \geq \alpha + \epsilon$, and thus $\omega_i \cdot \delta = m((y+\delta \cdot e_i)@d) - m(y@d) > 0$.

By the third and fourth calibrator properties, for any $b \in M_c \setminus 0_i$, for any $x \in V^b$ and for any $\delta > 0$, there exists $y \in V^d$ that satisfies the conditions of claim 30 (recall we already know that $m((y + \delta \cdot e_i)@d) = m(y@d) + \omega_i \cdot \delta$). Thus:

$$m((x + \delta \cdot e_i)@b) - m(x@b) = m((y + \delta \cdot e_i)@d) - m(y@d) = \omega_i \cdot \delta$$

and the claim follows. ∎

B.6 Affine maximization

Claim 32 *Requirements 1 to 3 imply that there exist constants $\omega_1, \ldots, \omega_n$ and $\{\gamma_a\}_{a \in A}$ such that:*

1. $m(x@a) = \sum_{i=1}^{n} \omega_i \cdot x_i + \gamma_a$ *for all $a \in A$ and $x \in V^a$.*
2. $\forall a, b \in A,\ v \in V :$ *if $v(a) \in V^a$, $v(b) \in V^b$, and $\sum_{i=1}^{n} \omega_i \cdot v_i(a) + \gamma_a < \sum_{i=1}^{n} \omega_i \cdot v_i(b) + \gamma_b$, then $f(v) \neq a$.*

proof: For any $i \neq 1$, $a \neq c$ and $x \in V^a$ there exists $\omega_i \in \mathcal{R}$ such that $m((x+\delta \cdot e_i)@a) - m(x@a) = \omega_i \cdot \delta$ for any $\delta > 0$ (by claim 31 if $a \notin 0_i$, and trivially if $a \in 0_i$). For $i = 1$, if $a \notin 0_i$ then $a \in S_c$, and thus by claim 25 $m((x + \delta \cdot e_1)@a) - m(x@a) = \delta$ so we take $\omega_1 = 1$. By claim 29 we conclude that $m(x@a) = \sum_{i=1}^{n} \omega_i \cdot x_i + \gamma_a$ for $a \neq c$. For c, since $x = v(c) = \alpha \cdot e_1$ then $m(x@c) = \alpha = \sum_{i=1}^{n} \omega_i \cdot x_i$ (so we take $\gamma_c = 0$). Therefore the first part of the claim follows. The second part is exactly claim 27, when replacing $m(x@a) = \sum_{i=1}^{n} \omega_i \cdot x_i + \gamma_a$ as shown in the first part of the proof. ∎

We can now immediately conclude:

Theorem 3 *Suppose V is order based with conflicting preferences, f is strongly monotone and player-decisive, and A is non-degenerate. Then f is affine maximizer for any $v \in V^*$.*

Corollary 4 *Suppose V is order based with conflicting preferences, f is strongly monotone and player-decisive, and A is non-degenerate. Then there exist $\omega_1, \ldots, \omega_n$, $\{\gamma_a\}_{a \in A}$, and a constant $M \in \mathcal{R}$ such that:*

$$f(v) \in argmax_{a \in A}\{ \sum_{i=1}^{n} \omega_i \cdot v_i(a) + \gamma_a \}$$

for all $v \in V$ such that $v_i(a) > M$ for all i and $a \notin 0_i$.

proof: (of corollary) Take representatives $v^a \in V^a$ and denote $M = max_{i,a}\{v_i(a)\}$. By the closure under positive translation of the V^a's, if $v_i(a) > M$ for all i and $a \notin 0_i$ then $v \in V^*$. ∎

Corollary 5 *Suppose V is order based with conflicting preferences, f is strongly monotone and player-decisive, and A is non-degenerate. For any $I' \subseteq \{1, \ldots, n\}$, denote $B_{I'} = \{a \in A | a \in 0_i \Leftrightarrow i \in I'\}$. Then there exist $\omega_1, \ldots, \omega_n$ and $\{\gamma_a\}_{a \in A}$ such that:*

$$f(v) \in \cup_{I' \subseteq \{1,\ldots,n\}} argmax_{b \in B_{I'}}\ \{ \sum_{i=1}^{n} \omega_i v_i(b) + \gamma_b \}$$

proof: Fix the constants implied by claim 32. Notice that $\cup_{I' \subseteq \{1,\ldots,n\}} B_{I'} = A$. Fix any $v \in V$. Suppose $f(v) = b \in B_{I'}$, but, by contradiction, there exists $a \in B_{I'}$ such that $\sum_{i=1}^{n} \omega_i v_i(b) + \gamma_b < \sum_{i=1}^{n} \omega_i v_i(a) + \gamma_a$ Choose some large enough δ so that $v(b) + \delta \cdot 1^n \in V^b$ (this is somewhat an abuse of notation, since only the non-zero coordinates are raised) and $v(a) + \delta \cdot 1^n \in V^a$. For every player $i \notin I'$ (i.e. $a \notin 0_i$) let $u_i = v_i + \delta \cdot 1^m$, and for player $i \in I'$, $u_i = v_i$. By S-MON, $f(u) = f(v) = a$, contradicting claim 32. ∎

C Proofs: Combinatorial Auctions and Multi-Unit Auctions

The following example describes a CA for three players, where all goods are always allocated, but S-MON is not satisfied, and indeed it is not affine maximizer:

Example 4 *Assume at least three players. Define A to be all possible allocations (where all the goods are allocated). Define constants $\gamma_a = 0$ if $a_1 \neq \emptyset$, and $\gamma_a = 1$ if $a_1 = \emptyset$. The function f is as follows. For player 1, choose some allocation a that maximizes $\sum_{i=1}^{n} v_i(a) + \gamma_a$ and allocate a_1 to player 1. (clearly this is truthful for 1, e.g. with a price $\sum_{i=2}^{n} v_i(a) + \gamma_a$). For the others, if $v_1(c^1) \geq 1$, choose the allocation a from before. If $v_1(c^1) < 1$, choose the allocation that maximizes $\sum_{i=2}^{n} \omega_i v_i(a)$ (for some fixed ω_i's) (clearly this is also truthful for the other players, from the same reason as before, and since the choice between the two different affine maximizers depends only on player 1's declaration). Notice that f always chooses a feasible allocation: if $v_1(c^1) < 1$ then it must be the case that player 1 gets the empty set.*

Notice that f is player decisive and A is non-degenerate. To see that f does not satisfy S-MON, consider the following two types of 1: at first, $v_1(c^1) = 1 + \epsilon$, but the others declare high enough, so 1 gets nothing. Now, if 1 lowers all his values by ϵ, the allocation changes since now, f maximizes $\sum_{i=2}^{n} \omega_i v_i(a)$.

C.1 Two Players

Define the *interior* of V to be:

$$\overset{\circ}{V} = \{v \in V | v_i(a) < v_i(b) \text{ for all } a, b \in A \text{ s.t. } a_i \subset b_i\}$$

and define $\overset{\circ}{f} : \overset{\circ}{V} \to A$ by $\overset{\circ}{f}(v) = f(v)$.

Theorem 4 *Fix any truthful CA or MUA f for two players, that always allocate all the goods. Suppose that $\overset{\circ}{f}$ is player decisive, onto a non-degenerate range. Then:*

1. *f must be almost affine maximizer in the interior of V.*
2. *If the constants $\{\gamma_a\}_{a \in A}$ are all zero then f is almost affine maximizer (i.e. also on the boundary of V).*

proof: (1) $\overset{\circ}{V}$ is strict order based, with conflicting preferences. Since f is truthful, it satisfies W-MON. Thus $\overset{\circ}{f}$ satisfies W-MON as well. Therefore, by theorem 2, we can assume w.l.o.g that $\overset{\circ}{f}$ satisfies S-MON. Therefore, by theorem 3, $\overset{\circ}{f}$ is almost affine maximizer, and, therefore, so is f.

(2) We show that, if there exists $v \in V$ such that $f(v) \notin argmax_{a \in A}\{\sum_i \omega_i v_i(a)\}$ then there exists $u \in \overset{\circ}{V}$ such that $f(u) \notin argmax_{a \in A}\{\sum_i \omega_i u_i(a)\}$, thus a contradiction. We do this in two steps, moving from v_i to $u_i \in \overset{\circ}{V}_i$ (so suppose w.l.o.g that $i = 1$).

Let $f(v) = d$. Take any $v_1' \in \overset{\circ}{V}_1$ such that $|v_1'(a) - v_1(a)| < \epsilon$ for all $a \in A$. Define $D = \{a \in A \mid d_1 \subseteq a_1 \text{ and } v_1(d) = v_1(a)\ \}$. Let $u_1 = v_1'|^{D+=2\epsilon}$. Choose ϵ small enough so that $u_1 \in \overset{\circ}{V}_1$ and $argmax_{a\in A}\{\omega_1 u_1(a) + \omega_2 v_2(a)\} \subseteq argmax_{a\in A}\{\sum_i \omega_i v_i(a)\}$. By W-MON, $f(u_1, v_2) \in D$ since $d \in D$, and all alternatives not in D were raised by at most ϵ, while d was raised by 2ϵ. We claim that for any $b \in argmax_{a\in A}\{\omega_1 u_1(a) + \omega_2 v_2(a)\}$, $b \notin D$ and the claim follows. To see this suppose by contradiction that $b \in D$. Therefore $d_1 \subset b_1$, and so $b_2 \subset d_2$. But also $v_1(d) = v_1(b)$, and since $\omega_1 v_1(d) + \omega_2 v_2(d) < \omega_1 v_1(b) + \omega_2 v_2(b)$ (since $b \in argmax_{a\in A}\{\sum_i \omega_i v_i(a)\}$ and $d \notin argmax_{a\in A}\{\sum_i \omega_i v_i(a)\}$) it follows that $v_2(d) < v_2(b)$, contradicting $b_2 \subset d_2$. ∎

The following example shows that it is not enough to require that f has a non-degenerate domain (i.e. we must require that $\overset{\circ}{f}$ has that):

Example 4 *Suppose a CA for two players, who considers three alternatives: c^i: allocating all the goods to player i, and a: allocating half the goods to player 1 and half to player 2. The allocation rule is as follows:*

$$f(v) = \begin{cases} c^2 & v_2(c^2) > \sqrt{v_1(c^1)} \\ c^1 & v_2(c^2) < \sqrt{v_1(c^1)} \\ a & v_2(c^2) = \sqrt{v_1(c^1)} \text{ and } v_2(c^2) = v_2(a) \text{ and } v_1(c^1) = v_1(a) \\ c^1 & \text{otherwise} \end{cases}$$

If a player gets nothing he pays zero. If player 1 is allocated some non-empty bundle he pays $(v_2(c^2))^2$, and if player 2 is allocated some non-empty bundle he pays $\sqrt{v_1(c^1)}$. To verify that this is truthful, notice that the prices of each player does not depend on his deceleration, and that a player always receives an allocation that maximizes his utility under these prices.

This auction is player decisive and has a non-degenerate domain, but is not almost affine maximizer - indeed, it does not satisfy S-MON: if a is chosen and player i raises his value for every non empty bundle by some $\delta > 0$, then c^i is chosen, thus contradicting S-MON.

In the interior of the domain, however, the situation changes. First, $\overset{\circ}{f}$ satisfies S-MON (and this is not accidental, as theorem 2 implies). But, clearly, $\overset{\circ}{f}$ is not almost affine maximizer, and this is since the range of $\overset{\circ}{f}$ becomes degenerate!

If we drop the assumption of always allocating all the goods, then S-MON cannot be assumed w.l.o.g, and, in fact, we can show a CA that is not almost affine maximizer:

Example 5 *For any $X \subseteq \Omega$, let $\tilde{p}_i(X, v_j) = v_j(\Omega) - v_j(\Omega \setminus X)$ (these are the Clarke prices). Suppose f allocates to each player the bundle that maximizes his utility under these prices (breaking ties for the two players in a consistent manner). It is not hard to verify that this is truthful and always chooses an optimal allocation. Now, change the prices of player 2 to be $p_2(X, v_1) = \tilde{p}_2(X, v_1) + v_1(\Omega)/2$ for any $X \neq \emptyset$, and $p_2(\emptyset, v_1) = \tilde{p}_2(\emptyset, v_1) = 0$. Clearly, f is still truthful. To see that it chooses a legal allocation, suppose that the Clarke function chooses $\tilde{a}$, and that $f(v) = a$. $a_1 = \tilde{a}_1$, since the prices of 1 did not change. a_2 is either the empty set or $\tilde{a}_2$: since we added a constant to the price of all the non-empty bundles, the difference between the utility of $\tilde{a}_2$ and any other non-empty bundle remains the same as in the Clarke function. Since we break ties in the same manner, f chooses either the empty set or $\tilde{a}_2$.*

To see that f is not almost affine maximizer (and does not satisfy S-MON), take any $g \in \Omega$, and fix $v_1(\Omega) = 16$, $v_1(\Omega \setminus \{g\}) = 10$, $v_2(\{g\}) = 9$ the rest of v is just small perturbations of the values, *so that v will be in the interior of V). The optimal allocation is $(\Omega \setminus \{g\}, \{g\})$, so, f allocates $\Omega \setminus \{g\}$ to 1. But $p_2(\{g\}, v_1) = 13 > v_2(\{g\})$, so 2 gets nothing. If player 1 lowers his value for Ω to 12, then $p_2(\{g\}, v_1) = 8$, and so the allocation changes, contradicting S-MON.*

C.2 Welfare approximation

We first show that, for welfare approximation mechanisms, the "almost" qualifier can be replaced with the following reasonable property:

Definition 18 (Unanimity-respecting) *Player i with a type v_i is "interested in a single bundle" Y if, for any other bundle X, if $Y \subseteq X$ then $v_i(X) = v_i(Y) > 0$, and otherwise $v_i(X) = 0$.*

A s.c.f f is **unanimity-respecting** *if for every $v \in V$ such that every player i is interested in a single bundle a_i, and $a_i \cap a_j = \emptyset$, then $f(v) = (a_1, \ldots, a_n)$. We also allow some players to have a zero value for all bundles, in which case they are allocated the empty bundle.*

For example, the greedy approximation algorithm of [18] is Unanimity-respecting (even its extension to the non single minded case).

Lemma 1 *Any truthful CA or MUA that is a c-approximation of the welfare, and satisfies IIA and unanimity-respecting, must be affine maximizer, with the γ_a constants being equal to zero, and $(1/c) \leq (\omega_i/\omega_j) \leq c$ for all players i, j.*

proof: First notice that, by the unanimity-respecting property, it follows that f has full range, since every possible allocation is obtained when the players are unanimous for it. Also notice that, since f is a c-approximation, it must be player decisive.

We now show that V^* from the proof of the main theorem now becomes $V^* = \{v \in V \mid v_i(a) > 0 \forall \text{i and } a \in A \setminus 0_i\}$. This will immediately follow from the claim that, for any $a \in A$: $V^a = \{v(a) | v_i(a) > 0 \text{ for all i s.t. } a \notin 0_i\}$, where the V^a's are defined in the proof of the main theorem. For $a \in S$ ($S = \{a \in A | a_1 \notin 0_1\}$), then by definition $x \in V^a$ iff there exists $v \in V$ s.t. $f(v) = a$ and $v(a) = x$. Therefore, take v_i so that player i is interested in a_i with $v_i(a_i) = x_i$ (and if $a_i \in 0_i$ then i has a value of zero for all bundles), and so $f(v) = a$, since f is unanimity-respecting. For the c^i alternatives, $y \in V^{c^i}$ iff there exists some $a \in S$ and $x \in V^a$ such that $y@c^i > x@a$. Take some allocation a s.t. $a_1, a_i \neq \emptyset$. Thus $a \in S$. For any $\epsilon > 0$, let $x = (\epsilon, \ldots, \epsilon)$. As shown before, $x \in V^a$. Let v be some type in which all players are interested in the single bundle a_i with a value of ϵ, and player i, in addition, has a value of $2nc\epsilon$ for c^i. Since f is a c-approximation, it follows that $f(v) = c^i$, and so $y@c^i > x@a$ (where we choose ϵ so that $y_i = 2nc\epsilon$). For the other alternatives a, $x \in V^a$ iff there exists $y \in V^{c^i}$ s.t. $x@a > y@c^i$ (for all i s.t. $a \notin 0_i$). For this, start with a type v in which all players are unanimous for a with value x, except player i who has value $x_i - \epsilon$. Thus $f(v) = a$. Now, if we raise all non-zero coordinates of i by ϵ, then by S-MON f will still choose a, and so $x@a > y@c^i$, where $y \in V^{c^i}$, as needed.

Therefore, by our main theorem, f is affine maximizer for any $v \in V^*$. We now show that the γ_a constants are all zero. Assume w.l.o.g that $\gamma_a \geq 0$ for any $a \in A$. Let b an alternative with $\gamma_b = 0$. Suppose all players are unanimous with value δ for b. Then $f(v) = b$. Suppose that $b_j \neq \emptyset$. Then, if j raises all his non-zero values by ϵ, b is still chosen. Since $v(b) \in V^b$ and $v^{c^j} \in V^{c^j}$, then by claim 32, $\gamma_{c^j} \leq \gamma_b + \sum_i \omega_i \delta$. Since this is true for any $\delta > 0$ it follows that $\gamma_{c^j} = 0$. Now suppose by contradiction that $\gamma_a > \gamma_{c^j}$ for some alternative a. Consider the type where j values all the goods for a value of $c\epsilon$, and all players value a by some ϵ' s.t. $\sum_i \omega_i \epsilon' < \epsilon$. Then, for small enough ϵ, ϵ', j will not receive all the goods since $\gamma_a > \gamma_{c^j}$. But this contradicts the approximation guarantee.

To verify that for any i, j, $(\omega_i/\omega_j) \le c$, suppose i, j are interested only in the bundle that contains all the goods, for a value of 1 and $c+\epsilon$, respectively. From the approximation ratio it follows that j wins, and therefore $\omega_i \cdot 1 \le \omega_j(c+\epsilon)$.

We are left to show that f is an affine maximizer for any $v \in V$. Suppose by contradiction that there exists a type v s.t. $f(v) = a$ but $\sum_i \omega_i v_i(a) < \sum_i \omega_i v_i(b)$ for some alternative b. We first verify that $v(a) \in V^a$ by adding ϵ to all non-zero coordinates of any player i with $a_i \ne \emptyset$ (by S-MON the result remains a). By claim 32, it follows that for every c^i, $\omega_i v_i(c^i) \le \sum_i \omega_i v_i(a)$ (as shown above, if $v_i(c^i) > 0$ then $v(c^i) \in V^{c^i}$). We turn this inequality to be strict by choosing two players i, j with $a_i, a_j \ne \emptyset$ and raise all their non-zero values by ϵ. [19] We now move to some $u \in V^*$ in which the measure of a is still smaller than that of b: For every player i, increase all non-zero coordinates by ϵ, and c^i's value by 2ϵ. Let u_i denote this new type of i. By S-MON, $f(v_{-i}, u_i)$ is either a or c^i, and since $\omega_i u_i(c^i) + \gamma_{c^i} < \sum_i \omega_i v_i(a) + \gamma_a$ (we choose a small enough ϵ) we have that $f(v_{-i}, u_i) = a$. By induction, $f(u) = a$. But now $u \in V^*$, and we still have $\sum_i \omega_i u_i(a) + \gamma_a < \sum_i \omega_i u_i(b) + \gamma_b$, thus a contradiction. ∎

For two players, where all the goods are always allocated, we can drop even the remaining qualifiers:

Lemma 2 *Any truthful $(2-\epsilon)$-approximation CA or MUA for two players (that always allocates all the goods) must be affine maximizer, with full range, the γ_a constants being equal to zero, and $0.5 \le (\omega_i/\omega_j) \le 2$.*

proof: We observe the following:

1. Any c-approximation algorithm must satisfy player decisiveness: Fix any player i and v_{-i}. If $v_i(\Omega) = (c+1)\max_{j\ne i} v_j(\Omega)$ then $f(v_i, v_{-i})$ must allocate all goods to i in order to c-approximate the optimal welfare. (In fact this is true for any number of players).

2. Any $(2-\epsilon)$-approximation that always allocates all the goods must have a full range, even in its interior: Fix any allocation $a = (a_1, a_2)$ where $a_2 = \Omega \setminus a_1$. If player i wants a_i with some value x (for $i = 1, 2$) then a has welfare of $2x$ and any other allocation has a value at most x (if player i is allocated a bundle that contains a_i then the bundle of player j is partial to a_j). This type is on the boundary of V, but we can easily shift it to the interior by choosing a small enough δ (w.r.t. ϵ and x) and "space" the values for other bundles with δ jumps: a bundle $X \not\supseteq S_i$ has value $l\delta$, where $l = |X|$, and a bundle $X \supset S_i$ has value $x + l\delta$. We need to choose δ so that $2x/(x + L\delta) > 2 - \epsilon$ (where L is the number of goods).

3. For the case of a $(2-\epsilon)$-approximation CA (or MUA) for two players (that always allocates all the goods), V^* from the main theorem's proof equals V, as follows. Notice that definition of V^a's for this case become: $V^{c^1} = V|_{c^1}$, $V^a = \{v(a) | f(v) = a\}$ (for any $a \ne c^i$), and $V^{c^2} = \{y | \exists a, x \in V^a \text{ s.t. } y@c^2 > x@a\}$. For any $a \in A$ s.t. $a_i \ne \emptyset$ for $i = 1, 2$, for any $x > 0$ we have seen in the last section that there exists $v \in V$ such that $v_1(a) = v_2(a) = x$ and $f(v) = a$. Thus $(x, x) \in V^a$. For any $y \ge (x, x)$ it follows from the closure under positive translation, that $y \in V^a$. Since this is true for any $x > 0$ it follows that $V^a = \mathcal{R}^2_+$. For the alternative c^2, and any $y@c^i$, since $x = (y_2/4, y_2/4) \in V^a$ for some $a \in A$, it follows that $y@c^i > x@a$ (since this is a $(2-\epsilon)$-approximation), and so $y \in V^{c^i}$. Thus $V^{c^i} = \mathcal{R}^2_+$, and so $V^* = \overset{\circ}{V}$.

[19] The possibility that only one player, say j, receives a non-empty bundle in a is handled by performing the move from v_j to u_j last. Then, neither a nor c^j can be chosen since both measures are strictly smaller than b's measure – and now $v(b) \in V^b$.

Since all the goods are always allocated, we conclude, by theorem 4, that f is affine maximizer in its interior.

We now claim that the γ_a constants are all equal to zero: Otherwise, suppose there are two allocations a, b s.t. $\gamma_b > \gamma_a$. Then, if we choose $x = (\gamma_b - \gamma_a)/4$ to be the x of observation 2 (as the value of a_i), we get that a will not be chosen, contradicting the fact that a must be chosen in order to be a $(2-\epsilon)$-approximation (as shown there). Therefore, by theorem 4 again, f is affine maximizer.

We are left to show that $\omega_i \leq 2\omega_j$. Otherwise suppose $\omega_i > 2\omega_j$, and consider the case where player i is interested only in Ω, for a value of 1, and player j is also interested only in Ω, for a value of 2. Then, f will allocate Ω to i, contradicting the $(2-\epsilon)$-approximation. ∎

C.3 Polynomial-time algorithms

Our characterization views the number of goods, k, and number of players, n, as fixed constants. This implies that, if k and n are given to the auction as parameters, then the constants ω_i, γ_a, and the range A, may all depend on k and n. We denote these by the superscript k, i.e. $\omega_i^k, \gamma_a^k, A^k$ (usually it is clear from the context what is n). Notice that, if these constants are large (w.r.t. n and k), then this may limit the range of the auction in a way that will enable it to become polynomial (e.g. if ω_i is much larger than the input size and the other constants, this depicts that player i will always receive all goods). This motivates the following definition:

Definition 19 *An affine maximizer CA or MUA has* **polynomially bounded constants** *if there exists a constant c such that $(\omega_i^k/\omega_j^k), \gamma_a^k \leq 2^{(n \cdot \log k)^c}$ for all number of goods k, players $i, j \in \{1, \ldots, n\}$ and $a \in A^k$.*

Note that $\omega_i^k/\omega_j^k, \omega_j^k/\omega_i^k, \gamma_a^k$ are real numbers with possibly infinitely many digits. The only consideration about these numbers is that they are not too small or too large.

When considering an auction algorithm from a computational point of view, we need to define exactly the input format, i.e. the *bidding language* [23]. In fact, for claiming that affine maximization is as computationally hard as exact maximization, we only need the bidding language to have the following two elementary properties:

Definition 20 *A bidding language L is* **elementary** *if,*

1. *For any bid $b \in L$ that implicitly represents some valuation v, there exists a polynomial time procedure to construct a bid $b' \in L$ that represents the valuation $\alpha \cdot v$, i.e. multiplying all values of all bundle by some constant $\alpha > 0$.*

2. *There exists a valid bid in which all bundles except Ω are valued as 0, and Ω is valued as α, for any $\alpha \geq 0$.*

For example, OR bids and XOR bids (see details below) are elementary: the first property is satisfied by just going over all the bid's blocks and multiplying their value by α.

Using these two definitions, we can now state formally that affine maximizers CAs and MUAs are as hard to compute as exact welfare maximizers:

Lemma 3 *Any affine maximizer CA or MUA with an elementary bid language, with polynomially bounded constants, and with the additive constants being equal to zero, is as computationally hard as the exact welfare maximization problem (with the same bidding language and the same range A).*

proof: Denote by AM the affine maximizer CA or MUA, and by EM the exact welfare maximizer. To prove the claim, we need to show a reduction from EM to AM. Before showing this, we need a method to calculate a close enough bound on the constants ω_i. We assume w.l.o.g that $\omega_1 = 1$ (any affine maximizer with constants $\{\omega_1, \ldots, \omega_n\}$ is also an affine maximizer with constants $\{\omega_1/\omega_1, \ldots, \omega_n/\omega_1\}$). Suppose the input bid is of size l (i.e. it contains l bits), let $M = 2^l$ be an upper bound on the value of any bundle, and $1/R = 1/2^l$ be a lower bound on the precision of the bundle values, i.e. if $v_i(X) > v_i(Y)$ then $v_i(X) \geq v_i(Y) + (1/R)$.

Claim 33 *There is a polynomial time procedure that computes $\tilde{\omega}_i$ such that $1 \leq (\omega_i/\tilde{\omega}_i) \leq 1 + 1/(2nMR)$.*

proof: We describe a simple iterative procedure: maintaining an interval I that contains ω_i, while reducing its size half until it is sufficiently small. We use a bid $b(\alpha_1, \alpha_i)$, which represents n players, where players 1 and i are interested only in the bundle Ω for a value of α_1, α_i, respectively, and the other players have a value of zero for all bundles.

The procedure works as follows. Initially, find some α s.t. $AM(b(\alpha, 1)) = 1$ (i.e. the auction allocates all goods to player 1). This is done by starting with $\alpha = 1$ and doubling it until the desired allocation is achieved. Since ω_i is polynomially bounded, i.e. $\omega_i \leq 2^{(n\cdot\log k)^c}$, this requires at most $2(n \cdot \log k)^c$ steps. Since the auction choose the allocation with maximal weighted welfare, we have that $\omega_i \cdot 1 \leq \omega_1 \alpha = \alpha$. Then we find c_1 such that $AM(b(\alpha, c_1)) = i$, using the same doubling method. This again takes polynomial time in the number of players and the input size. Therefore we now have that $\omega_i \in [(\alpha/c_1), (\alpha/c_0)]$, where $c_0 = 1$. We now set $c^* = (c_1 + c_0)/2$. And test $AM(b(\alpha, c^*))$. If this equals 1 then we set $c_0 = c^*$, otherwise this equals i and we set $c_1 = c^*$. Thus we maintain $\omega_i \in [(\alpha/c_1), (\alpha/c_0)]$. We repeat this until $c_1 - c_0 \leq 1/(2nMR)$, and then determine $\tilde{\omega}_i = \alpha/c_1$. Therefore $1 \leq \omega_i/\tilde{\omega}_i \leq c_1/c_0$. Since $c_0 \geq 1$ it follows that $c_1/c_0 = (c_1 - c_0)/c_0 + 1 \leq 1 + 1/(2nMR)$. This binary search procedure takes $\log(\beta(2nMR))$, where β is the initial length of the interval. This is again polynomial in the number of players and the input size. ∎

We can now describe a reduction from EM to AM:

1. Given an input bid $b = (b_1, \ldots, b_n)$ for EM, first compute the bounds $\{\tilde{\omega}_i\}_i$ according to claim 33.

2. Create a bid $\tilde{b}$ such that $\tilde{b}_i$ represents the valuation $\tilde{v}_i = v_i/\tilde{\omega}_i$ (where v_i is the valuation that b_i represents) – there is an efficient method to compute $\tilde{b}$ from b since the bid language is elementary.

3. Return the allocation $AM(\tilde{b})$ (as the allocation that EM outputs).

The correctness of this reduction immediately follows from the following claim:

Claim 34 *For any two allocations $a, b \in A$, if $\sum_i \omega_i \tilde{v}_i(a) \geq \sum_i \omega_i \tilde{v}_i(b)$ then $\sum_i v_i(a) \geq \sum_i v_i(b)$.*

proof: We show that $\sum_i v_i(a) < \sum_i v_i(b)$ implies $\sum_i \omega_i \tilde{v}_i(a) < \sum_i \omega_i \tilde{v}_i(b)$. First note that $\sum_i \omega_i(\tilde{v}_i(b) - \tilde{v}_i(a)) = \sum_i (\omega_i/\tilde{\omega}_i)(v_i(b) - v_i(a)) = \sum_i (v_i(b) - v_i(a)) + \sum_i ((\omega_i/\tilde{\omega}_i) - 1)(v_i(b) - v_i(a))$. Since $\sum_i v_i(b) > \sum_i v_i(a)$ then $\sum_i (v_i(b) - v_i(a)) \geq 1/R$. Since $0 \leq ((\omega_i/\tilde{\omega}_i) - 1) \leq 1/(2nMR)$ and $(v_i(b) - v_i(a)) \geq -M$, it follows that, for every i, $((\omega_i/\tilde{\omega}_i) - 1)(v_i(b) - v_i(a)) \geq (-M)(1/(2nMR)) = -(1/(2nR))$. Therefore: $\sum_i ((\omega_i/\tilde{\omega}_i) - 1)(v_i(b) - v_i(a)) \geq n(-1/(2nR)) = -1/(2R)$. So we can conclude that $\sum_i \omega_i(\tilde{v}_i(b) - \tilde{v}_i(a)) \geq 1/R - 1/(2R) > 0$. ∎

This concludes the proof of the lemma. ∎

The above Lemma holds for any bid language with which the welfare maximization problem is hard. We next consider two such specific bid languages. If the bid language forces the input to be long, e.g. the value of all possible bundles must be specified, then clearly we can construct an affine maximizer that will take linear time in the size of this input. Therefore, we need to allow short inputs. The Single Minded bidding language was considered in [18], where it was also showed that exact welfare maximizing CA with it is NP-hard. The OR bids language was considered in [23].

Definition 21 (Single Minded Bids) *A single minded bid of player i has the form $(q^i,\ v^i)$, which implies the following valuation: for MUA, any quantity not smaller than q^i has a value v^i, and, for CA, any bundle that contains the bundle q^i has value v^i. All other bundles have value 0.*

Definition 22 (OR Bids) *Player i's valuation is represented by OR bids if it is a collection of pairs $(q_1^i,\ v_1^i),(q_2^i,\ v_2^i),\dots,(q_l^i,\ v_l^i)$, where each v_j^i is the value of i for the bundle q_j^i – for MUA q_j^i specifies just the number of items in the bundle, where in CA it identifies uniquely some bundle. From this representation, it is implicit that the value of any bundle X is: $v_i(X) = \max\ \{\ \sum_{j\in I} v_j^i \mid I \subseteq \{1,\dots,l\}$ s.t. $\cup_{j\in I}\, q_j^i \subseteq X$ and for all $j,j' \in I,\ q_j^i \cap q_{j'}^i = \emptyset\}$* [20].

Claim 35 *Any welfare maximizing CA or MUA for n players (where n is not fixed), with full range, is NP-hard, even with single minded bids. If the number of players is fixed, then the above holds with OR bids as the bidding language.*

proof: We give the proof in the next subsection. ∎

To integrate our main characterization with this computational hardness, we need a bidding language that will be rich enough to express all possible valuations, since the characterization does not assume any limitations on the possible valuation of the players. Notice that single minded bids and OR bids are not rich enough (OR bids can express only super-additive valuations).

Definition 23 *A bidding language L* **generalizes** *the bidding language L' if,*

1. *L contains all valid bids of L'.*
2. *L can express all possible player valuations.*

For example, XOR bids generalize single minded bids. And, OR bids with dummy items, and XOR of ORs, both generalize OR bids.

We can now integrate the above claims with our characterization of truthful welfare approximations (as described in section C.2):

Theorem 5 *Any Unanimity-respecting truthful polynomial-time combinatorial (or multi-unit) auction, with a bidding language that generalizes single minded bids, and that satisfies IIA, cannot obtain $poly(n,k)$ welfare approximation (unless $P = NP$).*

proof: By Lemma 1, any truthful CA or MUA that satisfies Unanimity-respecting and IIA, and is a $poly(n,k)$ welfare approximation is an affine maximizer with polynomially bounded constants and the additive constants are zero. By Lemma 3, the affine maximization problem is as computationally hard as the exact maximization problem, and by claim 35, this problem is NP-hard. Therefore the auction cannot be polynomial (unless $P = NP$). ∎

[20]In MUA, X and the q_j^i are number of goods, and so the condition becomes $\sum_{j\in I} q_j^i \le X$.

Theorem 6 *Any truthful polynomial-time multi-unit (or combinatorial) auction between two players, with a bidding language that generalizes OR bids, and that always allocates all goods, cannot obtain a welfare approximation better than* 2 *(unless $P = NP$).*

Remark: In contrast, for MUA without the truthfulness requirement there exists a FPAS [26]. Also notice that a truthful 2-approximation can be easily obtained by allocating all goods to the player with the highest value for them.

proof: Essentially the same as for the previous theorem, replacing Lemma 1 with Lemma 2. ∎

C.4 The hardness of welfare maximization

In this section we prove that CA or MUA that is an exact welfare maximizer (with the appropriate bid language) is NP-hard. For two players, we prove this for any affine maximizer, even with additive constants not equal to zero (this claim is stronger then proving NP-hardness for exact welfare maximization and using Lemma 3, since Lemma 3 requires the additive constants to be zero). For n players, we prove this for exact welfare maximizers.

Lemma 4 *An affine maximizer CA or MUA for two players, with OR bids as the input, that has polynomially bounded constants and full range* [21]*, is an NP-complete problem.*

proof: We show this in two parts. First, we show how to calculate polynomial bounds on the constants ω_i and γ_a (we omit the superscript k when it is clear from the context), in polynomial time. We then use these bounds to describe a reduction of exact-subset-sum to MUA, and of independent-set to CA.

We also assume w.l.o.g that $\omega_1 = 1$ and $\gamma_a \geq 0$ for all $a \in A$ (since f is also an affine maximizer with all the constants multiplied by $1/\omega_1$, and with all the γ_a constants increased by the same value). Denote by c the constant implied from the polynomially bounded constants definition.

By an abuse of notation, we denote by k the alternative that allocates all goods to player 1, by $k-1$ the alternative that allocates $k-1$ goods to player 1 and 1 good to player 2 (for CA, there are several such alternatives - we define below exactly to which one we refer), and by 0 the alternative that allocates all goods to player 2. We need three bounds on the constants, according to the following three claims:

Claim 36 *There exists a polynomial time procedure to calculate a bound $\bar{\gamma} > \max\{(\gamma_a - \gamma_k),\ (\gamma_a - \gamma_{k-1})\}$, for all $a \in A$.*

proof: We first show how to find a bound on $\gamma_a - \gamma_k$. Assume that player 1 has the single OR bid $(k : 2)$, and 2 has a single OR bid $(1 : 1)$. We double 1's price (for k) l times, until k is chosen. We denote this as $f(k : 2^l \mid 1 : 1) = k$. Since the auction is affine maximizer with $\omega_1 = 1$, we have: $2^l + \gamma_k \geq \omega_2 + \gamma_a$ for every $a \in A$, therefore $2^l \geq \gamma_a - \gamma_k$, so we can take the bound to be 2^l. To verify that l is polynomial, notice that $2^{l-1} \leq \omega_2 + \gamma_a$ (where $f(k : 2^{l-1} \mid 1 : 1) = a \neq k$), and so $l \leq 4(\log k)^c$, i.e. the number of bits and iterations l is linear in $(\log k)^c$.

To bound $\gamma_a - \gamma_{k-1}$, we use a similar procedure: we iteratively find the minimal r s.t. $f(k-1 : 2^r \mid 1 : 2^r) = k-1$. Notice first that such an r exists: if $2^r > \gamma_a - \gamma_{k-1}$ this implies that $2^r + \omega_2 2^r + \gamma_{k-1} > \omega_2 2^r + \gamma_a$ and so any $a \neq k, k-1$ cannot be chosen, and if $\omega_2 2^r > \gamma_a - \gamma_{k-1}$ then this

[21] In fact it is enough to assume that the range contains the following three allocations: allocating all goods to player 1, allocating $k-1$ goods to player 1, and one goods to player 2, and allocating all goods to player 2.

implies that $2^r+\omega_2 2^r+\gamma_{k-1} > 2^r+\gamma_k$, and so k cannot be chosen. Since $f(k-1:2^r \mid 1:2^r)=k-1$ it follows that $2^r+\omega_2 2^r+\gamma_{k-1} > \gamma_a$, and so $\gamma_a-\gamma_{k-1} \le 2^r(\bar{\omega}+1)$, where $\bar{\omega}$ is the upper bound on ω_2 that is calculated in the next claim. To verify that r is polynomial, notice that either $2^{r-1} \le \gamma_a-\gamma_{k-1}$ or $\omega_2 2^r \le \gamma_a-\gamma_{k-1}$, and hence r is linear in $(\log k)^c$. ∎

Claim 37 *There exists a polynomial time procedure to calculate a bound $\bar{\omega} \ge \omega_2$ in polynomial time.*

proof: We start with $f(k:1 \mid k:2)$, and double 2's bid until $f(k:1 \mid k:2^{r_2})=0$, that is 2 wins all the goods. Thus, $\omega_2 2^{r_2}+\gamma_0 \ge 1+\gamma_k$, and so $2^{r_2}\omega_2 \ge \gamma_k-\gamma_0$.

We continue with $f(k:2 \mid k:1+2^{r_2})$ and double 1's bid until $f(k:2^{r_1} \mid k:1+2^{r_2})=k$. Now, $2^{r_1}+\gamma_k \ge \omega_2(1+2^{r_2})+\gamma_0$. In particular $\omega_2(1+2^{r_2}) \le 2^{r_1}+\gamma_k-\gamma_0 \le 2^{r_1}+2^{r_2}\omega_2$. We conclude that $\omega_2 \le 2^{r_1}=\bar{\omega}$.

To verify that r_1 and r_2 are polynomial, notice first that $\omega_2 2^{r_2-1}-\gamma_0 \le 1+\gamma_k$, and therefore r_2 is linear in $(\log k)^c$. Similarly, r_1 is polynomial since $2^{r_1-1}+\gamma_k \le \omega_2(1+2^{r_2})+\gamma_0$, in fact r_1 is $O(\log k)^{2c}$. ∎

Claim 38 *There exists a polynomial time procedure to calculate a bound $\underline{\omega} \le \omega_2$ in polynomial time.*

proof: We start by iteratively finding $\hat{\beta}$ s.t. $f(k:1 \mid k:\hat{\beta})=0$, and then m s.t. $f(k:m \mid k:\hat{\beta})=k$. Therefore $\gamma_a \le \omega_2\hat{\beta}+\gamma_0 \le m+\gamma_k$ (for all $a \in A$), i.e. $0 \le \omega_2\hat{\beta}-(\gamma_k-\gamma_0) \le m$. Define $\beta=\hat{\beta}+1$. It follows that $f(k:1 \mid k:\beta)=0$. We note that finding m takes $O(\log k)^{2c}$ time (as detailed in the proof of claim 37).

Consider the interval $I=[\omega_2\hat{\beta}-(\gamma_k-\gamma_0)\ ,\ \omega_2\beta-(\gamma_k-\gamma_0)]$. The length of I is ω_2 but we do not have exactly its two ends. We shall find 2 distinct points in I, then the distance between these 2 points is a lower bound for ω_2. However we have an interval $[1,m]$ that contains I. From this we can find a point $\alpha \in I$, using binary search as follows: Set $l_0=1$, $l_1=m$. Iteratively, let $\alpha=(l_0+l_1)/2$. (notice that $f(k:\alpha \mid k:\hat{\beta})$ may only be either 0 or k, and the same for β instead of $\hat{\beta}$, since $f(k:1 \mid k:\hat{\beta})=0$). Test if $f(k:\alpha \mid k:\hat{\beta})=0$: If so, $\alpha \le \omega_2\hat{\beta}-(\gamma_k-\gamma_0)$. Therefore set $l_0=\alpha$ and start another iteration. Otherwise, test if $f(k:\alpha \mid k:\beta)=k$: If so, $\alpha \ge \omega_2\beta-(\gamma_k-\gamma_0)$. Therefore set $l_1=\alpha$ and start another iteration. Otherwise we have that $\alpha \in I$. Let L be the number of iterations performed. Thus, after $L-1$ iterations we still have that $I \subseteq [l_0,l_1]$. Therefore $m/2^{L-1}=l_1-l_0 \ge |I|=\omega_2$, and so the procedure will iterate at most $O(\log k)^{3c}$ times.

To find a second point in I we find ϵ s.t. either $(\alpha+\epsilon) \in I$ or $(\alpha-\epsilon) \in I$. We start with $\epsilon=1$, and check if either $(\alpha+\epsilon) \in I$ or $(\alpha-\epsilon) \in I$, using the test described above. If not, we decrease ϵ by half and continue. When we stop, we will have $\epsilon > \omega_2/4$ - the distance between α to one of I's ends must be at least $|I|/2=\omega_2/2$. Thus, since $\omega_2 > 1/(\log k)^c$ (this follows since $\omega_1/\omega_2 < (\log k)^c$), we conclude that finding ϵ took polynomial time as well (we assume that to represents a polynomial proper fractional number we separately store its denominator and numerator as integers occupying together polynomial number of bits). Now, we have an interval of size ϵ that is contained in I. Therefore $\omega_2=|I| \ge \epsilon$. On the other hand, we have $\epsilon \ge \omega_2/4 \ge 1/4(\log k)^c$, so we can take $\underline{\omega}=\epsilon$.
∎

Reducing Exact Subset Sum to MUA: In order to show that an affine maximizer MUA (with OR bids, denoted below as AMOR) is NP complete we show that it is harder from the following NP-complete problem:

The Problem **Exact Subset Sum** *denoted below as* **Exact***:*

1. *Input: a finite collection of positive integers* $S, r_1, r_2, \ldots, r_d$.
2. *Output: "yes" if there is a sub-collection* $I \subseteq \{1, \ldots, d\}$ *of* r_i*'s that amounts to* S*, that is* $\Sigma_{i \in I} r_i = S$*, and "no" otherwise.*

The reduction: *Given an input* $J = (S, r_1, r_2, \ldots, r_d)$ *for EXACT, the reduction constructs the following input* $\tau(J)$ *for AMOR:*

1. $k = S$.
2. *compute the bounds* $\bar{\gamma}, \bar{\omega}$*, and* $\underline{\omega}$ *(notice that these are computed for this specific* k*).*
3. *The OR bids for player* 1 *are:* $(r_1,\ c_1 r_1), \ldots, (r_d,\ c_1 r_d)$*, where* $c_1 = \frac{2 \cdot \bar{\gamma} \cdot \bar{\omega}}{\underline{\omega}}$
4. *The OR bids for player* 2 *is:* $(1,\ c_2)$*, where* $c_2 = \frac{\bar{\gamma}}{\underline{\omega}}$.

And then, answer "yes" if and only if all goods are allocated to player 1.

c_1 can be viewed as the average price of one item for player 1 $c_1 > \omega_2 c_2$ implies that the donation of player 2 to the welfare is always smaller in case both compete the same item. Intuitively, the prices of both players are factored by $\bar{\gamma}$ and so the γ_a's never affect the chosen allocation.

Claim 39 *If Exact*(J) *is "yes" then AMOR*$(\tau(J))$ *allocates all the* S *items to player* 1.

proof: If Exact(J) is "yes" then there is $I \subseteq \{1, \ldots, d\}$ such that $\Sigma_{i \in I} r_i = S$. The weighted welfare of allocating all items to player 1 is then $c_1 \cdot S + \gamma_S$. We show that in this case any other allocation achieves a sub optimal weighted welfare. The following is an upper bound for the weighted welfare achieved whenever at least one item is allocated to player 2: $c_1(S-1) + \omega_2 c_2 + \gamma_{x_0}$, where $\gamma_{x_0} \geq \gamma_a$ for all alternatives $a \neq k$. We argue that $c_1 \cdot S + \gamma_S$ is greater than this upper bound and hence AMOR$(\tau(J))$ would allocate all the items to player 1.

$c_1 \cdot S + \gamma_S > c_1(S-1) + \omega_2 c_2 + \gamma_{x_0}$ if and only if $c_1 > \omega_2 c_2 + \gamma_{x_0} - \gamma_S$. Now, $c_1 = \frac{2 \cdot \bar{\gamma} \cdot \bar{\omega}}{\underline{\omega}} \geq \frac{2 \cdot \bar{\gamma} \cdot \omega_2}{\underline{\omega}} = 2\omega_2 c_2$. Thus, it is suffice show that $2\omega_2 c_2 > \omega_2 c_2 + \gamma_{x_0} - \gamma_S$. This is true since $\omega_2 c_2 = \omega_2 \frac{\bar{\gamma}}{\underline{\omega}} \geq \bar{\gamma} > \gamma_{x_0} - \gamma_S$. ∎

Claim 40 *If Exact*(J) *is "no" then AMOR*$(\tau(J))$ *allocates at least one item to player* 2.

proof: Assume by contradiction that AMOR$(\tau(J))$ allocates all the items to player 1. We argue that the allocation of $S-1$ items to player 1 and one item to player 2 has a higher welfare. That is, we argue that $v_1(S) + \gamma_S < v_1(S-1) + \omega_2 v_2(1) + \gamma_{S-1}$. Note that in this case $v_1(S) = v_1(S-1)$, since otherwise this implies that there exists $I \subseteq \{1, \ldots, d\}$ s.t. $\Sigma_{i \in I} r_i = S$. Thus it is suffice to show that $\gamma_S < \omega_2 c_2 + \gamma_{S-1}$, or equivalently $\gamma_S - \gamma_{S-1} < \omega_2 c_2$. But, $\gamma_S - \gamma_{S-1} < \bar{\gamma} \leq \omega_2 c_2$. ∎

This completes the proof w.r.t. MUA.

We now show a reduction from Independent Set to an affine maximizer CA for two players. This is in the spirit of [18], but for two players, and where the CA obtains the weighted optimum, and not simply the optimum.

The Problem **Max. Independent Set***:*

1. *Input: An undirected graph* $G = (V, E)$.

2. *Output: The size of the maximal independent set*[22] *of* G.

The reduction: *Given a graph* G*, choose some node* $u_0 \in V$*, and define the graph* $G_{-u_0} = G \setminus \{u_0\}$ [23]*. Let* x, x_{-u_0} *be the size of the max. independent set of* G, G_{-u_0} *respectively. It is either the case that* $x = x_{-u_0}$ *or that* $x = x_{-u_0}+1$*. We first determine which case is it, using the following procedure. We then compute recursively* x_{-u_0} *and, by that, determine* x.

1. *Construct a set of items s.t. each edge becomes an item. Define the specific bundles (for any* $u \in V$*):* $B_u = \{(u,u') \in E | u' \in V\}$ *(i.e. all the edges of* u*).*
2. *Compute the bounds* $\bar{\gamma}, \bar{\omega}$*, and* $\underline{\omega}$ *for this problem instance. Here, the allocation termed* $k-1$ *is the allocation where player* 1 *receives* $E \setminus B_{u_0}$*, and player* 2 *receives* B_{u_0}.
3. *Define* $c_1 = (2\bar{\omega}\bar{\gamma})/\underline{\omega}$*, and* $c_2 = \bar{\gamma}/\underline{\omega}$.
4. *Construct The OR bids for player* 1*:* $(B_u, c_1)_{u \neq u_0}$ *– i.e.* 1 *values any bundle* B_u *(except* B_{u_0}*) by* c_1*. And the OR bids for player* 2*:* (B_{u_0}, c_2) *(i.e.* 2 *only wants the bundle* B_{u_0}*).*
5. *Execute the CA. If player* 2 *receives all the items in* B_{u_0} *then* $x = x_{-u_0} + 1$*, otherwise* $x = x_{-u_0}$.

Before proving the correctness of the reduction, it is useful to notice that the value of player 1 for the entire set of goods is $v_1(E) = x_{-u_0} \cdot c_1$, since there are x_{-u_0} (but no more) disjoint bundles that player 1 is interested in, and each has a value of c_1.

Claim 41 *If* $x = x_{-u_0}$ *then player* 2 *will not receive* B_{u_0}.

proof: If $x = x_{-u_0}$ then every max. IS for G_{-u_0} contains a neighbor of u_0 (otherwise, we can take this set, add u_0, and get an IS for G with size $x_{-u_0}+1$). Thus, $v_1(E \setminus B_{u_0}) \leq (x_{-u_0}-1)c_1 < xc_1 = v_1(E)$. Therefore, if 2 receives all the items of B_{u_0}, then the maximal weighted welfare that can be achieved is $v_1(E \setminus B_{u_0}) + \omega_2 v_2(B_{u_0}) + \gamma_a \leq (x-1)c_1 + \omega_2 c_2 + \gamma_a$ (for some γ_a). Allocating all items to 1 will result in a weighted welfare of $v_1(E) + \gamma_k = xc_1 + \gamma_k$. We claim that the latter term is strictly larger, and by that the claim is proved. But this follows since $c_1 - \omega_2 c_2 > \bar{\gamma}$. ∎

Claim 42 *If* $x = x_{-u_0} + 1$ *then player* 2 *will receive all the items of* B_{u_0}.

proof: Since $x = x_{-u_0}+1$, every max. IS for G contains u_0 (if we had a max. IS that does not contain u_0, it will be also a max. IS for G_{-u_0}, contradicting $x_{-u_0} < x$). Let S be any max. IS for G. Since S contains u_0 it does not contain any of its neighbors. Thus it contains $x-1$ nodes s.t. none of them has an edge in B_{u_0}. Therefore, the set of goods $E \setminus B_{u_o}$ contains $x-1$ disjoint bundles that player 1 values, and so $v_1(E \setminus B_{u_o}) \geq (x-1)c_1$. Suppose by contradiction that player 2 does not receive B_{u_0}. Then the maximal weighted welfare is at most $v_1(E)+\gamma_a = x_{u0}c_1+\gamma_a = (x-1)c_1+\gamma_a$. But the following allocation has a larger weighted welfare: $v_1(E \setminus B_{u_o}) + \omega_2 v_2(B_{u_o}) + \gamma_{k-1} \geq (x-1)c_1 + \omega_2 c_2 + \gamma_{k-1}$ (this follows since $\omega_2 c_2 > \gamma_a - \gamma_{k-1}$), a contradiction. ∎

The correctness of the reduction now follows from these two claims: if player 2 receives B_{u_0} then it cannot be the case that $x = x_{u_0}$, and therefore $x = x_{u_0}+1$. And if player 2 does not receive B_{u_0} then it cannot be the case that $x = x_{u_0}+1$, and therefore $x = x_{u_0}$. ∎

[22] a set of vertices $I \subseteq V$ s.t. for any $u, v \in I$, $(u,v) \notin E$

[23] I.e. $V_{-u_0} = V \setminus \{u_0\}$ and $E_{-u_0} = \{(u,u') \in E | u, u' \neq u_0\}$

Lemma 5 *A exact welfare maximizer MUA or CA for n players, with single minded bids as the input and a full range is an NP-hard problem.*

proof: For CA this was proved by [18]. We prove this for MUA.

Reducing Exact subset sum to MUA: *Given an input $J = (S, r_1, r_2, \ldots, r_d)$ for EXACT, the reduction constructs the following input $\tau(J)$ for Multi Unit Auction with Single Minded Bids and n players (denoted as MU-SMB):*

1. $k = S$, $n = d + 1$.
2. *The Single minded bids for players $i = 1, \ldots, d$ are: $(r_i,\ 2 \cdot r_i)$, i.e. every player desires r_i items for a value of $2 \cdot r_i$.*
3. *The Single minded bid for player $d+1$ is: $(1,\ 1)$.*

And then, answer "yes" if and only if none of the items is allocated to player $d+1$.

Claim 43 *If Exact(J) is "yes" then MU-SMB$(\tau(J))$ allocates none of the items to player $d+1$.*

proof: If Exact(J) is "yes" then there is $I \subseteq \{1, \ldots, d\}$ such that $\Sigma_{i \in I} r_i = S$. Thus allocating r_i items to players $i = 1, \ldots, d$ has total welfare of $2S$. Allocating one item to player $d+1$ means that at least one player from $i = 1, \ldots, d$ will now be allocated a quantity less than r_i, and thus his value will be zero. Since this player has a value of at least 2 for r_i items, we have that the total welfare when allocating player $d+1$ a non-empty bundle is at most $2S - 2 + 1 < 2S$. Therefore MU-SMB will not allocate any item to player $d+1$. ∎

Claim 44 *If Exact(J) is "no" then MU-SMB$(\tau(J))$ allocates at least one item to player $d+1$.*

proof: Let I be the set of players that received a non empty bundle. Suppose by contradiction that $d+1 \notin I$. Since Exact(J) is "no" then $\Sigma_{i \in I} r_i < S$. Therefore there exists an item who is not allocated, or is allocated to someone that is indifferent to not having it. Delivering this item to player $d+1$ will increase the welfare by 1, a contradiction. ∎

D Unrestricted Domains

In this section we give an alternative proof to Roberts' Theorem, using the notions of our main theorem. Notice that an unrestricted domain is (trivially) an order based domain with conflicting preferences. Therefore, our main theorem applies for it (with its qualifiers). However, if we want to remove the qualifiers, we can prove requirements one to three in a different (and in fact easier) way. We choose $V^* = V$, and show that for any alternative $c \in A$, $\vec{1}@c$ is a transitive reference for $A \setminus \{c\}$, and that any alternative $a \in A$ is a calibrator for any player i. For this we assume only that f satisfies S-MON:

Proposition 8 *If V is unrestricted and f is strongly monotone, then Requirements 1 to 3 are satisfied.*

proof: We show this using several claims: the first one summarizes some nice properties of an unrestricted domain:

Claim 45 *Suppose V is unrestricted and f is strongly monotone. Then:*

1. *$x@a > y@b$ implies $(x + \delta \cdot e_i)@a > (y + \delta \cdot e_i)@b$ for any i and $\delta > 0$.*

2. *For any $x@a$ there exists $v \in V$ such that $f(v) = a$ and $v(a) = x$.*

3. *Any $x@a$ and $y@b$ are comparable.*

proof: (1) Take $v \in V$ such that $f(v) = a, v(a) = x$, and $v(b) = y$. By S-MON, $f(v_i + \delta \cdot \vec{1}, v_{-i}) = a$ [24], and the claim follows.

(2) Fix some $u \in V$ such that $f(u) = a$. Let $\delta_i = x_i - u_i(a)$, and $v_i = u_i + \delta_i \cdot \vec{1}$. By S-MON, $f(v_i, u_{-i}) = a$, and thus also $f(v) = a$. Since $v(a) = x$, the claim follows.

(3) Fix any $u \in V$ such that $u(a) = x$ and $f(u) = a$. Let us construct some $v \in V$ as follows: $v(a) = x, v(b) = y$, and for any $c \neq a, b$: $v(c) = u(c)$. Since $x@a > u(c)@c$ then by claim 16, it follows that $f(v) \neq c$ (otherwise $x@a < u(c)@c$). Therefore $f(v) \in \{a, b\}$, and the claim follows. ∎

Requirement 1 follows from the part 1 of the claim.

Claim 46 *For any $c \in A$, $\vec{1}@c$ is a transitive reference for $A \setminus \{c\}$. Therefore, requirement 2 is satisfied.*

proof:

Measurability: Fix any $v \in V$ such that $f(v) = a$ and $v(a) = x$. Therefore $x@a > y@c$ (where $y = v(c)$). By claims 16 and 45, $x@a > (\alpha \cdot \vec{1})@c$ for any α such that $\alpha \cdot \vec{1} \leq y$. For the other direction, fix any $u \in V$ such that $f(u) = c$. By claim 45 we can assume w.l.o.g that $u(a) = x$. Therefore, for any β such that $\beta \cdot \vec{1} \geq u(c)$ it follows that $x@a < (\beta \cdot \vec{1})@c$.

Transitivity: Assume $x@a < (\alpha \cdot \vec{1})@c$ and $\neg(y@b < (\alpha \cdot \vec{1})@c)$, but, by contradiction, $x@a > y@b$. Choose $v \in V$ "for" $x@a > y@b$. By claim 16 it must be the case that $v(c) < (\alpha \cdot \vec{1})$. By S-MON, $f(v|^{c=\alpha \cdot \vec{1}} \in \{a, c\}$. But if this equals a it contradicts $x@a < (\alpha \cdot \vec{1})@c$, and if this equals c it contradicts $\neg(y@b < (\alpha \cdot \vec{1})@c)$.

R-monotonicity: follows immediately from claim 16.

L-monotonicity: Suppose by contradiction that $(x + \delta \cdot e_i)@a < (\alpha \cdot z)@c$ but $\neg(x@a < (\alpha \cdot z)@c)$. Therefore $x@a > (\alpha \cdot z)@c$, contradicting to claim 16.

∎

Since a transitive reference cannot measure itself, we need some "rich enough" structure of transitive references:

Definition 24 (A zero player) *Fix any reference $z@c$. Player i is a zero player w.r.t. M_c if for any $a \in M_c$ and any $x \in V^a$: $x@a < (\alpha \cdot z)@c \;\Rightarrow\; (x + \delta \cdot e_i)@a < (\alpha + \epsilon)z@c$, for any $\epsilon, \delta > 0$.*

Claim 47 *For any $c \in A$ and any non-zero player i (w.r.t. $A \setminus \{c\}$) there exists a calibrator for i. Therefore, requirement 3 is satisfied.*

proof: Since i is a non-zero player, there exists $b \in A \setminus \{c\}, y@b, \alpha \cdot z@c$, and $\epsilon, \delta > 0$ such that the first calibrator requirement holds. For the second requirement, first notice that $0_i = \emptyset$, and $S_c = A \setminus \{c\}$. For any $y@b$ choose some a and $x@a$ such that $y@b > x@a$. Similarly, for $\tilde{y}@b$ choose some $\tilde{x}@a$ such that $\tilde{y}@b > \tilde{x}@a$. Then, for $x' = min(x, \tilde{x})$ it follows that $x'@a$ calibrates both $y@b$ and $\tilde{y}@b$. The third and fourth requirements are also immediate. ∎

This concludes the proof of the proposition. ∎

[24] By definition, $\vec{1} = (1, \ldots, 1)$.

Requirement 4: Connected references. f has (a set of) "connected references" if there exists some $R \subseteq A$ such that, for every $c \in R$, $z@c$ is a transitive reference for M_c (for some $z \in \mathcal{R}^n,\ z \geq 0$), and,

1. For any $a, b \in A$ there exists $c \in R$ such that $a, b \in M_c$.
2. For any $c, d \in R$ and any player i there exists $a \in A \setminus 0_i$ such that $a, c \in M_d$, and $a, d \in M_c$.
3. For any $c \in R$ and any player i, either i is a zero player w.r.t. M_c, or there exists a calibrator $c^i \in M_c$ for i (the calibrators c^i, c^j are not necessarily distinct).

Proposition 9 *If V is unrestricted, f is strongly monotone, and $|A| \geq 3$, then f has connected references.*

proof: Fix any three alternatives as R, the set of connected references. For any $c \in R$, it follows from claim 46 that $\vec{1}@c$ is a transitive reference for $A \setminus \{c\}$. The first two requirements from R immediately follow from this, and the third requirement follows from claim 47. ∎

Lemma 6 *Suppose f has connected references. Then there exist constants $\omega_1, \ldots, \omega_n$ and $\{\gamma_a\}_{a \in A}$ such that:*

$$f(v) \in argmax_{a \in A} \{ \sum_{i=1}^{n} \omega_i \cdot v_i(a) + \gamma_a \}$$

for all $v \in V^*$.

We prove this Lemma in sub-section D.1 below. From all this we can immediately conclude:

Theorem 7 *If V is unrestricted, f is strongly monotone, and $|A| \geq 3$, then f is affine maximizer.*

Corollary 6 *Any truthful social choice function on an unrestricted domain, with at least three alternatives, must be affine maximizer.*

proof: By theorem 1, f satisfies W-MON. By theorem 2, since an unrestricted domain is an open set, there exists a function $\tilde{f}$ that satisfies S-MON, and that if $\tilde{f}$ is affine maximizer, then so is f. By theorem 7, $\tilde{f}$ must be affine maximizer, and the claim follows. ∎

D.1 Connected references

Since there are several references, we denote by $m_c(x@a)$ the measure of $x@a$ according to the reference $z@c$. Before proving the lemma, we need some useful claims:

Claim 48 *For any reference $c \in R$, $a \in M_c$ and $x \in V^a$,*

1. *If i is a zero player w.r.t. M_c, then $m_c((x + \delta \cdot e_i)@a) - m_c(x@a) = 0$, for any $\delta > 0$.*
2. *There exist $\omega_1^c, \ldots, \omega_n^c, \{\gamma_{a,c}\}_{a \in M_c}$ such that: $m_c(x@a) = \sum_{i=1}^n \omega_i^c \cdot x_i + \gamma_{a,c}$.*
3. *$\sum_{i=1}^n \omega_i^c \cdot z_i = 1$.*

proof: (1) From the definition of a zero player it follows that $m_c((x+\delta\cdot e_i)@a)\leq m_c(x@a)$. By claim 24, $m_c((x+\delta\cdot e_i)@a)\geq m_c(x@a)$, and the claim follows.

(2) Similarly to claim 32, since for every $c\in R$ and every player i, either there exists a calibrator for i or i is a zero player, it follows that $m_c((x+\delta\cdot e_i)@a)-m_c(x@a)=\omega_i^c\cdot\delta$, and the claim follows.

(3) Fix any $a\in S_c$, $x\in V^a$, and some $\beta>0$. By claim 25, $m_c((x+\beta\cdot z)@a)=m_c(x@a)+\beta$. Therefore $\sum_{i=1}^n\omega_i^c\cdot z_i\cdot\beta=\beta$, and the claim follows. ∎

Claim 49 *For any $c,d\in R$:*

1. *For any $a\in M_c\cap M_d$, and $x\in V^a$: $m_c(x@a)=m_d(x@a)-\gamma_{c,d}$.*
2. *For any i, $\omega_i^c=\omega_i^d=\omega_i$.*

proof: First suppose by contradiction that $m_c(x@a)<m_d(x@a)-\gamma_{c,d}$, and choose some α such that $x@a<(\alpha\cdot z)@c$, and $\alpha<m_d(x@a)-\gamma_{c,d}$. Since $c\in M_d$, and by claim 48, $m_d((\alpha\cdot z)@c)=\alpha+\gamma_{c,d}<m_d(x@a)$. By claim 23, this contradicts $x@a<(\alpha\cdot z)@c$. Similarly, suppose by contradiction that $m_c(x@a)>m_d(x@a)-\gamma_{c,d}$, and choose some α such that $x@a>(\alpha\cdot z)@c$, and $\alpha>m_d(x@a)-\gamma_{c,d}$. Thus $m_d((\alpha\cdot z)@c)=\alpha+\gamma_{c,d}>m_d(x@a)$, a contradiction.

The second claim follows from the first one by taking some $a\in M_c\cap M_d\setminus 0_i$, $x\in V^a$, and $\delta>0$, and therefore $\omega_i^c\cdot\delta=m_c((x+\delta\cdot e_i)@a)-m_c(x@a)=m_d((x+\delta\cdot e_i)@a)-m_d(x@a)=\omega_i^d\cdot\delta$. ∎

Claim 50 *For any $c,d\in R$, and any $a\in M_c\cap M_d$,*

1. *$\gamma_{c,d}=-\gamma_{d,c}$.*
2. *$\gamma_{a,c}=\gamma_{a,d}+\gamma_{d,c}$.*

proof: Fix any $x\in V^a$. By claim 49, $m_c(x@a)=m_d(x@a)-\gamma_{c,d}$, and also $m_d(x@a)=m_c(x@a)-\gamma_{d,c}$. Therefore $\gamma_{c,d}=-\gamma_{d,c}$. Since $m_c(x@a)=\sum_{i=1}^n\omega_i\cdot x_i+\gamma_{a,c}$, $m_d(x@a)=\sum_{i=1}^n\omega_i\cdot x_i+\gamma_{a,d}$, and $m_c(x@a)=m_d(x@a)-\gamma_{c,d}$, the second claim follows. ∎

Lemma 6 *Suppose f has connected references. Then there exist constants $\omega_1,\ldots,\omega_n$ and $\{\gamma_a\}_{a\in A}$ such that:*

$$f(v)\in argmax_{a\in A}\{\ \sum_{i=1}^n\omega_i\cdot v_i(a)+\gamma_a\ \}$$

for all $v\in V^*$.

proof: Fix any $c^*\in R$. For any $a\in A$ and $x\in V^a$, choose any $c\in R$ such that $a\in M_c$, and define $w(x@a)=m_c(x@a)-\gamma_{c^*,c}$ (where $\gamma_{c^*,c^*}=0$ by definition, also notice that for any $c\neq c^*$, $c^*\in M_c$ by the second property of R, and so $\gamma_{c^*,c}$ is well defined).

We first claim that for any $c\in R$, $a,b\in M_c$, $x\in V^a$, and $y\in V^b$, $w(x@a)-m_c(x@a)=w(y@b)-m_c(y@b)$. Let $d\in R$ (not necessarily distinct from c) be the reference that determined $w(x@a)$. Therefore: $w(x@a)=m_d(x@a)-\gamma_{c^*,d}=m_c(x@a)-\gamma_{d,c}-\gamma_{c^*,d}=m_c(x@a)-\gamma_{c^*,c}$ (where the second equality follows from claim 49, and the third equality follows from claim 50, since $c^*\in M_c\cap M_d$). Similarly, let $e\in R$ (not necessarily distinct from c,d) be the reference that determined $w(y@b)$. Therefore: $w(y@b)=m_e(y@b)-\gamma_{c^*,e}=m_c(y@b)-\gamma_{e,c}-\gamma_{c^*,e}=m_c(y@b)-\gamma_{c^*,c}$, and so $w(x@a)-m_c(x@a)=w(y@b)-m_c(y@b)$.

From this it follows that $f(v)\in\text{argmax}_{a\in A,x=v(a)}\{\ w(x@a)\}$ for any $v\in V^*$: by contradiction, let $v\in V^*$ be such that $f(v)=a$, but $w(x@a)<w(y@b)$ (for $x=v(a),y=v(b)$). Let $c\in R$ be such that $a,b\in M_c$. Since $w(x@a)-m_c(x@a)=w(y@b)-m_c(y@b)$, it follows that $m_c(x@a)<m_c(y@b)$, contradicting claim 27. From this the Lemma immediately follows. ∎