Multi-Option Descending Clock Auction

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Abstract

A descending clock auction (DCA) is a mechanism for buying items from multiple sellers. The auctioneer starts by offering bidders high prices and gradually decreases the prices while there is competition. The literature has focused on the vanilla case where each bidder has two options: to accept or reject the offered price. However, in many settings—such as the FCC's imminent incentive auction—each bidder may be able to sell one from a *set of options*. We present a multi-option DCA (MDCA) framework where at each round, the auctioneer offers each bidder different prices for different options, and a bidder may find multiple options still acceptable. A key component is the technique for deciding how to set prices during the MDCA. This is significantly more difficult in an MDCA than in a DCA. We develop a Markov chain model for representing the dynamics of each bidder's state (which options are still acceptable), as well as an optimization model and technique for finding prices to offer to the different bidders for the different options in each round—using the Markov chain. The optimization minimizes total payment while ensuring feasibility in a stochastic sense. We also introduce percentile-based approaches to decrementing prices. Experiments with real FCC incentive auction interference constraint data reveal that the optimization-based approach dramatically outperforms the simple percentile-based approach both under symmetric and asymmetric bidder valuation distributions—because it takes feasibility into account in pricing. Both techniques scale to the large.

Keywords: Descending Clock Auction, Incentive Auction, Spectrum Auction.

1 Introduction

A *descending clock auction (DCA)* is a mechanism for buying items from multiple potential sellers. A vanilla DCA works as follows, and has remarkably strong incentive properties [18]. Consider the following setting where the auctioneer wants to buy items. Each seller $i \in N$ has a specific type of item and decides to sell it or not depending on the offer price. The items from the sellers could be substitutable and complementary to the buyer. The auctioneer has a target number of items to buy, T, and there is a feasibility function $F : 2^N \to \{0, 1\}$ that specifies, for each subset of potential sellers, S, whether the items from S can fulfill the target T or not, that is, $F(S) = 1$ if the combined items fulfill the target. A simple example of this is the case where the sellers have identical items and the auctioneer wants to buy a target number, T, of them. In that case, the feasibility function is simply $F(S) = 1$ if $|S| \geq T$, and $F(S) = 0$ otherwise. In real-world applications—such as the FCC spectrum reverse auctions discussed below—the feasibility function can be highly complex. Often it cannot be given in closed form, but rather is stated through constraints as an optimization problem.

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In the vanilla DCA, the auctioneer sends offer prices to the sellers and checks whether they accept those prices. Bidders who accept the offers are called *active*. If the combined items from the active bidders fulfill the target, then the auctioneer reduces the prices further in the next round and repeats the process. If at some point the items from the active bidders do not fulfill the target, then the auctioneer goes back to the last step and conducts a last-round adjustment to offer higher prices to some declined bidders so that feasibility is obtained.

The DCA framework is agnostic to how offered prices are decremented across rounds. Doing that well is a key problem for which no solutions had been published until recently. Nguyen and Sandholm [19] presented techniques for this.

In their *percentile-based* family of techniques, the approach is to set the offer price at some fixed percentile of the (buyer's model of the) distribution of that bidder's valuation. For example, the prices could be set so that each bidder has the same probability of accepting her offer. The choice of the percentile would depend on what the auctioneer aims for on the trajectory of the sizes of sets of active bidders through the rounds. For example, the trajectory could be set so that the expected number of rejections in each round is distributed evenly throughout the auction. Another example of a trajectory is to set a fixed percentage of rejection in each round, that is, the expected number of rejections would be proportional to the size of the remaining set of active bidders.

Those methods have several drawbacks. First, having a fixed percentile means there is no way to distinguish bidders with greater influence on the feasibility function; hence the final payment will likely be unnecessarily high due to the probabilistic inclusion of high-priced bidders. More importantly, those methods do not have any special treatment for the degree of interaction among the items in the feasibility function.

Nguyen and Sandholm [19] also presented an optimization model for setting the prices in the vanilla D-CA. The model is designed to minimize the expected final payment while ensuring feasibility in a stochastic sense. It is flexible in that it can incorporate bidder-specific characteristics with respect to feasibility.

That paper—and, to our knowledge, all other papers on incentive auctions and on the DCA to date [18, 21]—consider the setting where bidders have only two options, that is, either to sell or not.

In contrast, in many settings, each seller may be able to sell *one from a set of options* to the auctioneer. The DCA can be generalized to this setting by offering each bidder a separate price for each of her options in each round. However, the problem of decreasing prices appropriately during the DCA is drastically more intricate in this *multi-option DCA (MDCA)* setting.

We present an MDCA framework and price-decrementing techniques for it. The model captures a broad set of applications, including the imminent flagship application of DCAs, the FCC incentive auctions—where an MDCA will be needed and used.

1.1 Incentive auctions

The FCC has been selling radio spectrum licenses via auctions since 1994 [2, 15]—in recent years via combinatorial auctions [3, 4]. However, there is not enough spectrum left to sell for the new high-value spectrum uses that have arisen. The idea of *incentive auctions*, therefore, is to buy some of the existing licenses back from their current holders, which frees up spectrum, and then to sell spectrum to higher-value users. The idea of such incentive auctions was introduced in the 2010 National Broadband Plan [6]. It is motivated by the fact that the demand and the value of over-the-air broadcast television has been declining while the demand for mobile broadband and wireless services has increased dramatically in recent years. Given the limited spectrum resources, incentive auctions were introduced as a voluntary, market-based means of repurposing spectrum. This is done by creating a market that exchanges the usage rights among the two groups of users: (a) existing TV broadcasters and (b) wireless broadband networks. Three key players in this market are existing spectrum owners, spectrum buyers, and the FCC, which acts as the intermediator.

An incentive auction consists of three stages [11] (see also a whitepaper about design choices by Hazlett et al. [12]):

1. *Reverse auction*: some spectrum currently used by TV broadcasters is bought back.[1, 2]

2. *Repacking*: remaining broadcasters are reallocated to a smaller spectrum band.

3. *Forward auction*: freed spectrums is sold via a (combinatorial) auction for use in wireless broadband networks.[3]

In the reverse auction, we need to find a set of stations to be reallocated to lower-band channels and a separate set of stations to be bought off the air, in order to achieve the following goals: (a) meet some target on the number of contiguous channels freed on the higher spectrum band and (b) minimize total payment by the FCC. The FCC is required to respect the broadcasters' carry-right, which means, in the context of a DCA, that a station that rejects the offer still has the right to stay on the air, but possibly on a lower spectrum band. This repacking stage needs to ensure that all the stations that rejected their offers can be feasibly repacked into the allocated band without violating the engineering constraints, that is, interference-free population coverage, as we detail later in the paper.

There are $n = 2177$ stations and $m = 49$ channels (ranging from channels 2 to 51, with channel 37 not available). The channels are divided into two bands: the very-high frequency (VHF) band (54-216 MHz) and the ultra-high frequency (UHF) band (614-698 MHz) [6]. These correspond to VHF channels 2-13 and UHF channels 14-51. The VHF band is further divided into two bands: lower VHF (LVH) with channels 2-6 and upper VHF (UVH) with channels 7-13.

The aim of the reverse auction is to clear a number of channels in the high-frequency band, say channels 33-51. This means all stations that are currently in this band need to either go off-air or be reallocated to lower-frequency channels 2-32. Stations in channels 2-32 could alternatively go off-air or be reallocated to different channels. It is these options that beget the need for a *multi-option* DCA, as we detail in the next section.

At each round of the DCA, the repacking problem needs to be solved in order to check whether the remaining stations can be feasibly reassigned to the targeted lower-band channels. Other groups have recently also tackled the repacking part (e.g., Leyton-Brown [14]). There are a large number—2.9×10^6—of engineering interference constraints requiring pairs of stations not to be allocated in the same or adjacent channels. Also, some stations are restricted to being allocatable to only a subset of the channels. The FCC has published all these engineering constraints on the FCC web site [9], and we use these real constraints in our experiments, as detailed later.

The FCC announced in June 2014 that a multi-option DCA will be used for the reverse auction [8], but left open the important question of how prices will be decremented across rounds. Our paper is, to

[1]The FCC has decided to use some form of DCA for the reverse auction instead of a VCG mechanism because 1) the VCG winner determination problem with the interference constraints is prohibitively complex [17] (and would have to be solved $|N| + 1$ times to obtain VCG prices also), and 2) a small approximation error in solving can lead to significant over-payment [16].

[2]Combinatorial reverse auctions are used extensively for sourcing goods and services in industry and government (e.g., [25, 20]). Typically, pay-your-winning-bids (i.e., first-price) pricing is used. Usually, 1-to-3 rounds of bidding are used, with feedback to bidders between rounds. Also, continuous variants have been used (when the number of items in the auction is small), where tentative winner determination is conducted each time a bid is submitted or revised [25]. Feedback and bidder strategies in combinatorial auctions have also been studied in laboratory experiments [1]. Combinatorial reverse auctions have been proposed for sourcing carrier-of-last-resort responsibility for universal service [13]. In contrast to combinatorial reverse auctions, in the DCA, prices are non-combinatorial: only individual items are priced.

[3]Once the reverse auction phase is completed and the remaining stations are repacked, the FCC announces the cleared spectrum that is now available for purchase. Buyers then submit bids on bundles of licenses. The FCC solves a winner determination problem to decide which bids to accept. This does not involve the engineering constraints, so it resembles a standard combinatorial auction. Existing algorithms (e.g., [23, 24, 25, 26, 5, 22]) can be used for this. (The FCC may iterate between the reverse and forward auctions to try to (approximately) equilibrate supply and demand before actual purchases and sales are made [10].)

our knowledge, the first paper on pricing techniques for multi-option DCAs. We present general techniques, both ones based on percentiles and ones based on optimization. We present experiments using real interference constraints from the imminent FCC incentive auction.[4]

The problem of designing the price-decrementing methodology is a key design element in the DCA, and the FCC has postponed the incentive auction multiple times because that has not yet been figured out (e.g., wireless.fcc.gov/ incentiveauctions/learn-program/rule-option/reverse-auction.html). The FCC was planning to run the first incentive auction in 2014. Then it was postponed to mid-2015 [27], and most recently to 2016 [10].

2 Multi-option descending clock auction (MDCA)

We present our MDCA in the domain of the FCC incentive auction, where the number of options per bidder is at most three (plus the option of rejecting all three options), but the techniques can be directly extended to any number of options per bidder.

In the FCC setting, bidders (stations) that are currently in the UHF band, have four choices[5]: go off-air, go down to LVH (lower VHF), go down to UVH (upper VHF), or reject all of these options. Stations currently in the UVH band, have three choices: go off-air, go down to LVH (lower VHF), or reject both of these options. Stations currently in the LVH band have two choices: go off-air or reject. When a station rejects all options, it will be (re)allocated to a channel in its original band (without any payment).

We denote the set of options $D = \{\text{OFF, LVH, UVH}\}$ with indices $k \in \{1, 2, 3\}$, respectively. D_i is the set of options that station i can choose. $D_i = \{1, 2, 3\}$ for station in UHF, $D_i = \{1, 2\}$ for stations in UVH, and $D_i = \{1\}$ for stations in LVH. We denote by v_{ik} be the valuation of station i for option k, that is, the price at which the station is indifferent between accepting the option and rejecting all options.

For each station $i \in \mathcal{N}$, let $\mathcal{A}_i^{(r)} \in \{0, 1\}^3$ be the binary vector that indicates whether, at round r, station i is still active for the three participation options. That is, $\mathcal{A}_{i1}^{(r)} = 1$ if station i is still active to go off-air and $\mathcal{A}_{i1}^{(r)} = 0$ otherwise. Similarly, $\mathcal{A}_{i2}^{(r)} = 1$ ($\mathcal{A}_{i3}^{(r)} = 1$) if station i is still active to be downgraded to LVH (UVH). (No upgrading to higher bands is allowed so $\mathcal{A}_{i2}^{(r)} = 1$ is possible only if station i is currently in the UHF band and $\mathcal{A}_{i1}^{(r)} = 1$ is possible only if the station is currently in either the UHF or the ULV band.) Figure 1 shows the bidders' options and the corresponding possible allocations. The first row shows which band the stations currently belong to while the first column shows 8 possible scenario for $\mathcal{A}_i^{(r)}$. The second row includes 4 possible outcomes of the final allocation, i.e., OFF (off-air), LVH, UVH, or UHF (reject all offers and stay in UHF). For each of the 8 rows, cells that are marked with a cross X are possible allocations. Because upgrading is not allowed, some cells are colored in gray. Empty cells are not applicable due to the corresponding choices of $\mathcal{A}_i^{(r)}$.

In the beginning, we can assume that all the bidders are active for all their available options. That is, in the first round $r = 0$, $\mathcal{A}_i^{(r)} = \iota(D_i)$ which is the indicator vector in $\{0, 1\}^3$ with non-zero indices in D_i. Stations in UHF will be offered three prices. Stations in UVH and LVH will be offered two prices and one price, respectively. We denote by $p_{ik},\ k \in \{1, 2, 3\}$, the offer price to station i for option k.

In each DCA round, the auctioneer offers each bidder prices for all the options for which the bidder is still active. The bidder then evaluates them and decides which of those options are still acceptable. As long as a bidder is still active for an option, the bidder enters the next round where the same process will

[4]Very recently, in December 2014, in parallel with our work, the FCC put up for comment an MDCA design that includes a price adjustment heuristic [10]. It is much more rigid than what we propose. Also, it does not take feasibility into account to nearly the same extent as our pricing technique does. To our knowledge, no theory or experiments have been published so far to analyze the design choices. In Appendix A.1 we discuss how our techniques can be of benefit even within the confines of that proposal.

[5]The option of stations sharing channels is not included as an option in the auction—by the FCC or by us—as it can be viewed as one station going off-air and the other bidding in the DCA.

	LVH Stations				UVH Stations				UHF Stations			
$A^{(r)}_i$	OFF	LVH	UVH	UHF	OFF	LVH	UVH	UHF	OFF	LVH	UVH	UHF
0,0,0		X					X					X
0,1,0						X	X			X		X
0,0,1											X	X
0,1,1										X	X	X
1,0,0	X	X			X		X		X			X
1,1,0					X	X	X		X	X		X
1,0,1									X		X	X
1,1,1									X	X	X	X

Figure 1: Bidders' options and allocation possibilities.

be repeated for the remaining active options. If a bidder becomes inactive for all options, that station needs to be allocated to its current band without payment.

ALGORITHM 1: A Multi-Option DCA Framework

Input: A set of stations $\mathcal{N} = \{1, \dots, n\}$, an auctioneer with a feasibility function $F : \prod_{i \in \mathcal{N}} \mathcal{A}_i \to \{0, 1\}$. A target number of rounds allowed m. Initial valuation function estimates $v_{ik},\ k \in D_i$.

Output: For each station owner i, a feasible final set of active options $\mathcal{A}_i$, i.e. $F(\mathcal{A}) = 1$, the corresponding offer price vector $\boldsymbol{p}$ for achieving $\mathcal{A}$, and the final assignment of stations to their options to minimize the expected payment.

1. Set the initial prices $\boldsymbol{p}$ at the reserves. Set $r = 0$ and let $\mathcal{A}_i^{(r)} = \iota(D_i)$;

2.**for** *round* $r = 1...m$ **do**

  2.1. Find a vector of prices $\boldsymbol{p}$ to offer the bidders on their active options;

  2.2. Update the sets $\mathcal{A}^{(r)}$ as follows: $\mathcal{A}_{ik}^{(r)} = 1$ only if option k is still available for station i, i.e. $\mathcal{A}_{ik}^{(r-1)} = 1$, and if bidder i accepts price p_{ik}, i.e. $p_{ik} \geq v_{ik}$;

  if *bidders' options according to* $\mathcal{A}^{(r)}$ *still leads to feasible packing, i.e.* $F(\mathcal{A}^{(r)}) = 1$, **then**

    2.2.1. Update the distributions of the bidders' valuations;

  else

    2.2.2. Reset the sets $\mathcal{A}^{(r)}$ to those in the previous round and enter Step 3;

  end

end

3. Final round adjustment to find the winners and their prices;

There are two missing pieces in Algorithm 1, which are to find the offer prices in step 2.1 and to check the repacking feasibility within step 2.2. We will now discuss these pieces, starting from the latter.

2.1 Repacking feasibility problem

Let $\mathcal{S}$ be a set of stations that needs to be repacked into a set of channels $\mathcal{C}$. We use i and j as indices for stations and k as indices for channels. Let $\mathcal{C}_i \subset \mathcal{C},\ i \in \mathcal{S}$, be the set of feasible channels for station i. Let $\mathcal{I}_c$ be the list of triplets (i, j, k) such that stations i and j cannot be assigned to the same channel k. Let $\mathcal{I}_a$ be the list of triplets (i, j, k) such that stations (i, j) cannot be assigned to channels k and $k + 1$, respectively. Data for $\mathcal{C}_i$, $\mathcal{I}_c$ and $\mathcal{I}_a$ are available from the domain file and the interference-paired file on the FCC web site [9], which we use in our experiments.

Let $F : \prod_{i \in \mathcal{N}} \mathcal{A}_i \to \{0, 1\}$ be the feasibility function

$$F(\mathcal{A}) = \begin{cases} 1, & \text{if } \mathcal{P}(\mathcal{A}, \mathcal{C}) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}(\mathcal{A}, \mathcal{C})$ is the set of feasible assignments of stations in $\mathcal{A}$ to available channels $\mathcal{C}$:

$$\mathcal{P}(\mathcal{A}, \mathcal{C}) = \left\{ \boldsymbol{z} : \begin{array}{c} z_{ik} \in \{0,1\}, \forall i \in \mathcal{N} \text{ and } k \in C_i, \\ \sum_{k \in C_i} z_{ik} \geq 1 - \mathcal{A}_{i1}^{(r)}, \forall i \in \mathcal{N}, \\ \sum_{k \in LVH} z_{ik} \leq \mathcal{A}_{i2}^{(r)}, \forall i \notin LVH, \\ \sum_{k \in UVH} z_{ik} \leq \mathcal{A}_{i3}^{(r)}, \forall i \notin UVH, \\ z_{ik} + z_{jk} \leq 1, \forall (i,j,k) \in \mathcal{I}_c, \\ z_{ik} + z_{jk+1} \leq 1, \forall (i,j,k) \in \mathcal{I}_a, \end{array} \right\} \quad (1)$$

Here, z_{ik} is a binary variable that indicates whether station i is assigned to channel k. The second set of constraints requires that, if a station decided not to go off-air ($\mathcal{A}_{i1}^{(r)} = 0$), the station needs to be (re)allocated to reserve its carry-right. The third set of constraints requires that, for stations currently not in LVH and that do not accept to move to LVH, they will not be allocated to LVH. The fourth set of constraints enforces the analogous requirement for the UVH band. The last two sets of constraints ensure that the allocation avoids interference. Later in the paper we present an optimization technique for decrementing prices that incorporates this feasibility problem.

3 Setting offer prices

A key component of a DCA is how the prices offered to active bidders are decremented across rounds. The auctioneer needs to consider the tradeoff between minimizing payment to the accepted bidders and fulfillment of the target (repacking feasibility in the case of incentive auctions).

Furthermore, the pricing affects the number of rounds the auction takes. This begets another tradeoff. If the prices decrease too slowly, many rounds are required, and that may be undesirable from the perspective of minimizing logistical effort. If the prices decrease too quickly, many bidders reject and the auction ends without properly serving its price-discovery purpose.

How the prices are changed across rounds should depend on (a) the estimated value functions of the bidders, (b) the importance of the items for the target to be fulfilled (including interference in the case of incentive auctions), and (c) the desired number of rounds. We provide an optimization model that incorporates these considerations.

3.1 A stochastic program for optimizing offer prices

We first present a stochastic program for finding optimal offer prices. Although we will not attempt to solve this model due to its complexity, it provides us an understanding of the uncertainty involved. For each station i and for each option $k \in \{1,2,3\}$, suppose the valuations v_{ik} are drawn from some distribution on support $[l_{ik}, u_{ik}]$. Assume that the auctioneer knows these distributions but not the valuations.

Let $X_{ik}(p_{ik})$ be the Bernoulli random variable that indicates whether bidder i will accept the offer for option k at price p_{ik}. Let $\mathbf{X} = (X_{ik}), i \in \mathcal{N}, k \in \{1,2,3\}$ be one such matrix of realized outcomes. We denote by $Q(\boldsymbol{p})$ the probability distribution over $\mathbf{X}$. Given outcome $\mathbf{X}$, the auctioneer faces a problem of choosing which active options from each station to choose so as to minimize total payment while achieving a feasible assignment. This can be formulated as the integer program

$$f(\boldsymbol{p}, \mathbf{X}) = \min_{\boldsymbol{z}} \sum_{i \in \mathcal{N}} \sum_{k \in X_i} z_{ik} p_{i,o(k)} \text{ s.t. } \mathbf{z} \in \mathcal{P}(\mathbf{X}, \mathcal{C}); \text{ See (1) for a formulation of } \mathcal{P}(\cdot, \cdot). \quad (2)$$

Here $\boldsymbol{z}$ is the assignment and $o(k)$ is the band that channel k is in. If the current channel of station i and channel k belong to the same band, we can assume $o(k) = 0$ with $p_{i,0} = 0$. The set of constraints here is similar to those in repacking feasibility (1). The only difference is that we have replaced $\mathcal{A}^{(r)}$ with $\mathbf{X}$. Note that the repacking problem might be infeasible. In that case, we define $f(\boldsymbol{p}, \mathbf{X}) = +\infty$.

Given that $\mathbf{X}$ is a random matrix generated by Bernoulli random variables with success rates dependent on $\boldsymbol{p}$, $f(\boldsymbol{p}, \mathbf{X})$ is also a random variable. A natural approach in stochastic programming is to find the optimal offer price $\boldsymbol{p}$ such that the expected total payment is minimized:

$$\min_{\boldsymbol{p}} \quad E_{\mathbf{X}\sim Q(\boldsymbol{p})} \left[f(\boldsymbol{p}, X) \right]. \tag{3}$$

This is very difficult to solve, if not impossible, because even a functional evaluation of the term $f(\boldsymbol{p}, \mathbf{X})$ inside the expectation for a given $\boldsymbol{p}$ and a realization $\mathbf{X}$ involves solving a large-scale integer program that is very difficult to solve to optimality. (This problem is actually a generalized graph coloring problem and its IP formulation is of the same form as the final settlement problem described later in Section 4.) We will design a deterministic approximation of (3). One major difficulty in designing a deterministic program to approximate the stochastic program (3) lies in the complexity caused by the multiple possible states that a station can be in throughout the DCA. To develop a percentile-based model or a deterministic optimization model, we need to model the constraints in relation to the decision variable $\boldsymbol{p}$. This requires a mathematical representation for the (expected) number of stations that will finally end up in each band—which is non-trivial: these will be highly nonlinear functions of $\boldsymbol{p}$.

To elaborate, consider a station that is in the UHF band and has all three options still active. Suppose the auctioneer offers prices to the options in such a way that the acceptance probabilities are q_1, q_2, q_3. Now, what is the probability of the station ending up in the UHF band (i.e., rejecting all offers) after m rounds? What about the other bands? We propose using Markov chains to model the dynamic of the station's state throughout the DCA.

3.2 Modeling the dynamics of a station in the DCA using a Markov chain

We denote by q_{ik} the probability that station i finds price p_{ik} acceptable for option k. For example, if the valuation distribution is uniform, $q_{ik} = \frac{u_{ik} - p_{ik}}{u_{ik} - l_{ik}}$. For convenience, we regard $\boldsymbol{q}$ as the decision variables that the auctioneer has to set.

Consider a station that is currently in the UHF band. Suppose at the current round r, the auctioneer offers acceptance rates (q_1, q_2, q_3) to the three available options. What is the station's probability distribution over its states in the next round? How about after m rounds if the acceptance rates in each round are kept the same at (q_1, q_2, q_3)? To answer these questions, we need to understand the state evolution of each station. We will use Markov chains for this.

We define the state of a station to depend on which of the three options are still active. There are $2^3 = 8$ possible states formed by the power set of {OFF, LVH, UHF}:

$S_1 =$ {OFF, LVH, UVH}	$S_2 =$ {OFF, LVH}	$S_3 =$ {OFF, UVH}	$S_4 =$ {LVH, UVH}
$S_5 =$ {OFF}	$S_6 =$ {LVH}	$S_7 =$ {UVH}	$S_8 =$ {None}

Table 1: States of a station with respect to active options.

Theorem 1. *The state transition of the m-round DCA with fixed acceptance probabilities* (q_1, q_2, q_3) *is equivalent to a single-round DCA with acceptance probabilities* q_1^m, q_2^m, q_3^m.

We now present an exposition of this result, which also serves as the proof.

Figure 2 (a) shows the Markov chain for a station currently in the UHF band. For example, the transition probability to go from S_1 to S_1 is $q_1 q_2 q_3$ since this occurs when all three offers are acceptable. We can derive the transition probabilities between the remaining states analogously.

We denote by Γ_{UHF} the corresponding transition matrix

$$\begin{bmatrix} q_1q_2q_3 & q_1q_2(1-q_3) & q_1(1-q_2)q_3 & (1-q_1)q_2q_3 & q_1(1-q_2)(1-q_3) & (1-q_1)q_2(1-q_3) & (1-q_1)(1-q_2)q_3 & (1-q_1)(1-q_2)(1-q_3) \\ 0 & q_1q_2 & 0 & 0 & q_1(1-q_2) & (1-q_1)q_2 & 0 & (1-q_1)(1-q_2) \\ 0 & 0 & q_1q_3 & 0 & q_1(1-q_3) & 0 & (1-q_1)q_3 & (1-q_1)(1-q_3) \\ 0 & 0 & 0 & q_2q_3 & 0 & q_2(1-q_3) & (1-q_2)q_3 & (1-q_2)(1-q_3) \\ 0 & 0 & 0 & 0 & q_1 & 0 & 0 & (1-q_1) \\ 0 & 0 & 0 & 0 & 0 & q_2 & 0 & (1-q_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & (1-q_3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

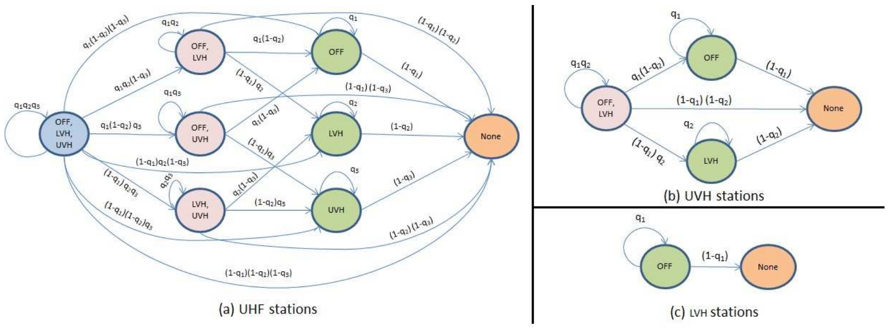

Figure 2: Markov chain for a station currently in UHF (a), UVH (b), and LVH (c).

We denote by $I^{(r)}_{UHF,i} \in \mathbb{R}^8$ the probability vector of station i being in state $S_1, \ldots, S_8$. At the beginning of the auction, $I^{(0)}_{UHF,i} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. By the end of round m,

$$I^{(m)}_{UHF,i} = \Gamma_{UHF} I^{(m-1)}_{UHF,i} = \cdots = \Gamma^m_{UHF} I^{(0)}_{UHF,i}$$

The formulation for E_{UHF}, E_{UVH} and E_{LVH} involves Γ^m_{UHF} which is a polynomial of q_1, q_2, q_3 and of order $3m$. We need a fast way to evaluate this. It turns out that the eigenvalues of Γ_{UHF} have special forms of $\prod_{j \in S} q_j$ for each of the 2^3 subset $S \subset \{1, 2, 3\}$. Let Λ denote the diagonal matrix with these eigenvalues on the diagonal and let V be the matrix containing the corresponding eigenvectors on its columns:

$$\Lambda = diag\left(\begin{bmatrix} 1 \\ q_1 \\ q_2 \\ q_3 \\ q_1q_2 \\ q_1q_3 \\ q_2q_3 \\ q_1q_2q_3 \end{bmatrix}\right) \quad \text{and} \quad V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We observe that the eigenvectors do not depend on q_i and the eigenvalues have a nice form as a function of q_i. Thus we can utilize the single value decomposition for Γ_{UHF} as $\Gamma_{UHF} = V\Lambda V^{-1}$. From that, we have $\Gamma^m_{UHF} = V\Lambda^m V^{-1}$. By defining $\kappa_i = q_i^m$,

$$\Gamma^m_{UHF} = \begin{bmatrix} \kappa_1\kappa_2\kappa_3 & \kappa_1\kappa_2(1-\kappa_3) & \kappa_1(1-\kappa_2)\kappa_3 & (1-\kappa_1)\kappa_2\kappa_3 & \kappa_1(1-\kappa_2)(1-\kappa_3) & (1-\kappa_1)\kappa_2(1-\kappa_3) & (1-\kappa_1)(1-\kappa_2)\kappa_3 & (1-\kappa_1)(1-\kappa_2)(1-\kappa_3) \\ 0 & \kappa_1\kappa_2 & 0 & 0 & \kappa_1(1-\kappa_2) & (1-\kappa_1)\kappa_2 & 0 & (1-\kappa_1)(1-\kappa_2) \\ 0 & 0 & \kappa_1\kappa_3 & 0 & \kappa_1(1-\kappa_3) & 0 & (1-\kappa_1)\kappa_3 & (1-\kappa_1)(1-\kappa_3) \\ 0 & 0 & 0 & \kappa_2\kappa_3 & 0 & \kappa_2(1-\kappa_3) & (1-\kappa_2)\kappa_3 & (1-\kappa_2)(1-\kappa_3) \\ 0 & 0 & 0 & 0 & \kappa_1 & 0 & 0 & (1-\kappa_1) \\ 0 & 0 & 0 & 0 & 0 & \kappa_2 & 0 & (1-\kappa_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_3 & (1-\kappa_3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

At any round r, each station i currently in the UHF band has 8 possible states. Let $\Phi_i^r \in \{0, 1\}^8$ be the indicator vector that denotes the current state: if the station is in state S_1, the first binary indicator equals 1, and so on. The probabilities of that station to be in each of the 8 states at round $(r + m)$ are $\Phi_i^r \Gamma^m_{UHF}$, which is a vector of dimension 8 where a component at position $j = 1, \ldots, 8$ represents the probability of being in S_j.

For a station currently in UVH, Figure 2 (b) shows the Markov chain. The states are S_2, S_5, S_6 and S_8. The probability of remaining in S_2 is q_1q_2 because this occurs when the station is willing to accept both the offer to go off-air (with probability q_1) and the offer to be downgraded to LVH (with probability q_2). The probability to go from S_2 to S_5 is $q_1(1 - q_2)$ because this occurs when the station is willing to accept the offer to go off-air and declines the offer to be downgraded to LVH. Similarly, we derive the transition probabilities between other states as shown in the figure. The transition matrix for S_2, S_5, S_6 and S_8 using this ordering is

$$\Gamma_{UVH} = \begin{bmatrix} q_1 q_2 & q_1(1-q_2) & (1-q_1)q_2 & (1-q_1)(1-q_2) \\ 0 & q_1 & 0 & (1-q_1) \\ 0 & 0 & q_2 & (1-q_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is exactly the same as in the transition matrix for Γ_{UHF} between states S_2, S_5, S_6 and S_8. Using the same singular value decomposition technique, we obtain

$$\Gamma^m_{UVH} = \begin{bmatrix} \kappa_1 \kappa_2 & \kappa_1(1-\kappa_2) & (1-\kappa_1)\kappa_2 & (1-\kappa_1)(1-\kappa_2) \\ 0 & \kappa_1 & 0 & (1-\kappa_1) \\ 0 & 0 & \kappa_2 & (1-\kappa_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a station currently in LVH, Figure 2 (c) shows the Markov chain. The state can be $S_5 = \{\text{OFF}\}$, which means the station is still active to go off-air, or $S_8 = \{\text{None}\}$, which means the station is no longer participating. The transition probabilities are shown in the figure. Here, the transition probability to go from S_5 to S_5 is q_1 while that to go from S_5 to S_8 is $(1-q_1)$. The transition matrix is

$$\Gamma_{LVH} = \begin{bmatrix} q_1 & (1-q_1) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma^m_{LVH} = \begin{bmatrix} \kappa_1 & (1-\kappa_1) \\ 0 & 1 \end{bmatrix}$$

During the auction, each UVH station will be in state S_2, S_5, S_6, or S_8. We denote by $I^{(r)}_{UVH,i} \in \mathbb{R}^4$ the corresponding probability vector. At the beginning of the auction, $I^{(0)}_{UVH,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. By the end of round m, $I^{(m)}_{UVH,i} = \Gamma_{UVH} I^{(m-1)}_{UVH,i} = \cdots = \Gamma^m_{UVH} I^{(0)}_{UVH,i}$.

Similarly, each LVH station will be in state S_5 or S_8. We denote by $I^{(r)}_{LVH,i} \in \mathbb{R}^2$ the probability vector. At the beginning of the auction, $I^{(0)}_{LVH,i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By the end of round m, $I^{(m)}_{LVH,i} = \Gamma_{LVH} I^{(m-1)}_{LVH,i} = \cdots = \Gamma^m_{LVH} I^{(0)}_{LVH,i}$. This completes the proof of the theorem.

The significance of Theorem 1 is that, instead of having to keep track of the transitions through m rounds, we can apply the change of variables and view it as a single-round DCA. This helps simplify the expression of the state probabilities, and will help tame the complexity of the nonlinear model as we will show in Section 3.4.

To proceed further, one needs to model how the auctioneer will deal with stations with more than one active option remaining. Which one of those options will the auctioneer offer to the stations? In principle one could treat this as a combinatorial optimization problem with the objective of payment minimization, but that has the same structure as a winner determination problem and is often prohibitively difficult to solve.[6] Instead, we assume that the auctioneer will choose among the bidder's available options with equal probabilities.[7] For example, if a station is in state S_1, that is, it offers to be off-air, downgraded to LVH, or downgraded to UVH, then there is one third probability that it will be taken off-air, one third probability that it will be moved to LVH, and one third probability that it will be moved to UVH.[8]

So, if the auctioneer fixes the percentiles for acceptance to be q_1, q_2, and q_3 for the three options OFF, LVH and UVH, respectively, then by round m,

- the number of stations in state S_1 is $N_{UHF} I^{(m)}_{UHF,i}(1)$,

[6] Solving this WDP is challenging because it involves thousands of binary variables and millions of interference-avoidance constraints. The FCC attempted to solve it when they considered using a seal-bid auction framework for the reverse auction. Solving an instance of this WDP with a state-of-the-art optimization package takes weeks without finding the optimal solution [17].

[7] It is possible to use unequal probabilities as well.

[8] In December 2014 the FCC put out for comment the idea of introducing a 'preferred option' into their DCA; each station that still has multiple active options must state which option is preferred [10]. In Appendix A.1 we discuss how our framework can be extended to capture that and other aspects under consideration by the FCC.

- the number of stations in state S_2 is $N_{UHF} I^{(m)}_{UHF,i}(2) + N_{UVH} I^{(m)}_{UVH,i}(1)$, i.e., it can include stations that came from UHF or UVH,
- the number of stations in state S_3 is $N_{UHF} I^{(m)}_{UHF,i}(3)$,
- the number of stations in state S_4 is $N_{UHF} I^{(m)}_{UHF,i}(4)$,
- the number of stations in state S_5 is $N_{UHF} I^{(m)}_{UHF,i}(5) + N_{UVH} I^{(m)}_{UVH,i}(2) + N_{LVH} I^{(m)}_{LVH,i}(1)$,
- the number of stations in state S_6 is $N_{UHF} I^{(m)}_{UHF,i}(6) + N_{UVH} I^{(m)}_{UVH,i}(3)$,
- the number of stations in state S_7 is $N_{UHF} I^{(m)}_{UHF,i}(7)$, and
- the number of stations in state S_8 is $N_{UHF} I^{(m)}_{UHF,i}(8) + N_{UVH} I^{(m)}_{UVH,i}(4) + N_{LVH} I^{(m)}_{LVH,i}(2)$.

We define the weight matrix $W_{UHF} \in \mathbb{R}^{8\times 4}$ where each row corresponds to a state among 8 possible states, S_1 to S_8, and the four columns correspond to the probabilities of having the final assignment be UHF, UVH, LVH, or OFF, respectively:

$$W_{UHF} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In this example, the first row states that a station that ended up in state S_1 will have $\frac{1}{3}$ chance to be assigned to each of UVH, LVH, and OFF. The 8th row states that a station that ended up in state S_8, that is, rejected all three options, will be assigned to UHF.

Similarly, the weight matrices for UVH and LVH stations are

$$W_{UVH} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W_{LVH} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then the expected number of stations that end up in the UHF band is

$$\begin{aligned} E_{UHF} &= \sum_{i \in UHF} \Phi_i^r \Gamma^m_{UHF} W_{UHF}(1) \\ &= N_{UHF}(S_1)(1-\kappa_1)(1-\kappa_2)(1-\kappa_3) + \\ &\quad N_{UHF}(S_2)(1-\kappa_1)(1-\kappa_2) + N_{UHF}(S_3)(1-\kappa_1)(1-\kappa_3) + N_{UHF}(S_4)(1-\kappa_2)(1-\kappa_3) + \\ &\quad N_{UHF}(S_5)(1-\kappa_1) + N_{UHF}(S_6)(1-\kappa_2) + N_{UHF}(S_7)(1-\kappa_3) + N_{UHF}(S_8), \end{aligned}$$

where $W_{UHF}(j)$ denotes the jth column of W_{UHF}, and $N_{UHF}(S_j)$ is the number of UHF stations currently in state S_j. The expected number of stations that end up in UVH is

$$\begin{aligned} E_{UVH} &= \sum_{i \in UHF} \Phi_i^r \Gamma^m_{UHF} W_{UHF}(2) + \sum_{i \in UVH} \Phi_i^r \Gamma^m_{UVH} W_{UVH}(1) \\ &= N_{UHF}(S_1)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + N_{UHF}(S_3)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3) + \\ &\quad N_{UHF}(S_4)(\kappa_3 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_7)(\kappa_3) + N_{UVH}(S_2)(1-\kappa_1)(1-\kappa_2) + \\ &\quad N_{UVH}(S_5)(1-\kappa_1) + N_{UVH}(S_6)(1-\kappa_2) + N_{UVH}(S_8) \end{aligned}$$

Finally, the expected number of stations that end up in the LVH band is

$$\begin{aligned}
E_{LVH} &= \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(3) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(2) + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m W_{LVH}(1) \\
&= N_{UHF}(S_1)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + N_{UHF}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + \\
&\quad N_{UHF}(S_4)(\kappa_2 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_6)(\kappa_2) + N_{UVH}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + \\
&\quad N_{UVH}(S_6)\kappa_2 + N_{LVH}(S_5)(1-\kappa_1) + N_{LVH}(S_8)
\end{aligned}$$

3.3 Percentile-based approach for price setting in each round

A very simple way to adjust prices across DCA rounds is to decrease each price by a given percentage—starting from the upper bound of the support of the station's valuation distribution support for each of the options. We present experiments with that approach later on in the paper.

However, one can do better even within the percentile-based family by using the Markov chain approach presented above. Suppose the auctioneer offers prices to the options so that the acceptance probabilities of the three options are q_1, q_2 and q_3. How should these probabilities be set so that, after m rounds, the expected numbers of stations allocated to the bands match given targets? The auctioneer can solve for (q_1, q_2, q_3) so that the expected number of stations allocated to each band equals its target. For this, we need to find these expectations. This involves finding the probability of the station ending in each band after m rounds. We provide detailed calculation of these expectations in Appendix A.2.

3.4 Optimization model for price setting in each round

We consider the case where the auctioneer wants to offer different acceptance rates to different stations. This will provide more flexibility and hence intuitively should lead to lower aggregate payment by the auctioneer. Instead of aiming to find the same (q_1, q_2, q_3) for all stations, we aim to find the optimal (q_{i1}, q_{i2}, q_{i3}) for each station $i \in \mathcal{N}$.

For notational convenience we denote $d_{ik} = (u_{ik} - l_{ik})$. At round r, the offer price to station i and option k is $p_{ik}^{(1)} = l_{ik} + q_{ik} d_{ik}$.[9] If this option is still available at round $(r+1)$, then the upper bound u_{il} is updated to $p_{ik}^{(1)}$ and the new offer price $p_{ik}^{(2)} = l_{ik} + q_{ik}(p_{ik}^{(1)} - l_{ik}) = l_{ik} + q_{ik}^2 d_{ik}$. Similarly, if option k is still active for station i at round $(r+m)$, then the offer price at that round will be $p_{ik}^{(m)} = l_{ik} + q_{ik}^m d_{ik} = l_{ik} + \kappa_{ik} d_{ik}$.

The payment vector $C_{UHF,i} \in \mathbb{R}^8$ for station i in UHF that corresponds to the 8 possible states is

$$C_{UHF,i} = \begin{bmatrix} \frac{1}{3}(p_{i1}^{(m)} + p_{i2}^{(m)} + p_{i3}^{(m)}) & \frac{1}{2}(p_{i1}^{(m)} + p_{i2}^{(m)}) & \frac{1}{2}(p_{i1}^{(m)} + p_{i3}^{(m)}) & \frac{1}{2}(p_{i2}^{(m)} + p_{i3}^{(m)}) & p_{i1}^{(m)} & p_{i2}^{(m)} & p_{i3}^{(m)} & 0 \end{bmatrix}$$

Similarly, the payment vector $C_{UVH,i} \in \mathbb{R}^4$ for station i in the UVH band that corresponding to 4 possible states is $C_{UVH,i} = \begin{bmatrix} \frac{1}{2}(p_{i1}^{(m)} + p_{i2}^{(m)}) & p_{i1}^{(m)} & p_{i2}^{(m)} & 0 \end{bmatrix}$. The payment vector $C_{LVH,i} \in \mathbb{R}^2$ for station i in the LVH band that corresponding to 2 possible states is $C_{LVH,i} = \begin{bmatrix} p_{i1}^{(m)} & 0 \end{bmatrix}$.

The expected payment at round $(r+m)$ is

$$E[c(\boldsymbol{p})] = \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m C_{UHF,i} + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m C_{UVH,i} + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m C_{LVH,i}.$$

[9] The derivations in this section assume that the valuation distributions are uniform. The techniques can be generalized to the nonuniform case, but the final optimization problem can end up having a higher-order polynomial objective and higher-order polynomial constraints than the optimization problem we derive in this section. Also, note that one can use the uniformity assumption in the price setting even if the assumption does not actually hold.

Our OPT-SCHED model for minimizing expected payment while ensuring that the expected number of accepted bidders in each band equals its target is

$$\begin{aligned}
\min_{\boldsymbol{q}} \quad & \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m C_{UHF,i} + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m C_{UVH,i} + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m C_{LVH,i}, \\
\text{s.t.} \quad & \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(1) \le C_{UHF} \\
& \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(2) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(1) \le C_{UVH} \\
& \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(3) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(2) + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m W_{LVH}(1) \le C_{LVH}
\end{aligned} \tag{4}$$

This is a polynomial optimization problem where the objective function is of order $4m$ and the constraint involve polynomials of order $3m$. This is very challenging to solve. We can use the transformation of decision variables from q_{ik} to κ_{ik}. This will lead to a new polynomial optimization problem with the objective function having degree 4 and the constraints having degree 3, which is much more manageable. That fully expanded form is given in Appendix A.3. This problem has $3n$ continuous variables. Solving this problem directly is still not easy due to nonlinearity and nonconvexity. However, the problem has a separable objective and separable constraints. Hence we can apply Lagrangian relaxation. By introducing notation g, h, and u, we can rewrite that problem (i.e., Problem (7)) as

$$\begin{aligned}
\min_{\boldsymbol{\kappa}} \quad & \textstyle\sum_{i\in\mathcal{N}} f_i(\boldsymbol{\kappa_i}) \\
\text{s.t.} \quad & \sum_{i\in\mathcal{N}} g_i(\boldsymbol{\kappa_i}) \le C_{UHF} \\
& \sum_{i\in\mathcal{N}} h_i(\boldsymbol{\kappa_i}) \le C_{UVH} \\
& \sum_{i\in\mathcal{N}} u_i(\boldsymbol{\kappa_i}) \le C_{LVH}
\end{aligned} \tag{5}$$

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be the Lagrangian multipliers for the three constraints in Model (5). The Lagrangian dual function is

$$\begin{aligned}
\mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\kappa}) &= \sum_{i\in\mathcal{N}} f_i(\boldsymbol{\kappa_i}) + \lambda_1 \left(C_{UHF} - \sum_{i\in\mathcal{N}} g_i(\boldsymbol{\kappa_i}) \right) + \lambda_2 \left(C_{UVH} - \sum_{i\in\mathcal{N}} h_i(\boldsymbol{\kappa_i}) \right) + \lambda_3 \left(C_{LVH} - \sum_{i\in\mathcal{N}} u_i(\boldsymbol{\kappa_i}) \right) \\
&= \lambda_1 C_{UHF} + \lambda_2 C_{UVH} + \lambda_3 C_{LVH} + \sum_{i\in\mathcal{N}} \left(f_i(\boldsymbol{\kappa_i}) \lambda_1 g_i(\boldsymbol{\kappa_i}) + \lambda_2 h_i(\boldsymbol{\kappa_i}) + \lambda_3 u_i(\boldsymbol{\kappa_i}) \right)
\end{aligned}$$

The Lagrangian dual problem can be derived as

$$\max_{\lambda\ge 0} \quad \left\{ \lambda_1 C_{UHF} + \lambda_2 C_{UVH} + \lambda_3 C_{LVH} + \sum_{i\in\mathcal{N}} \min_{0\le\boldsymbol{\kappa_i}\le 1} \lambda_1 g_i(\boldsymbol{\kappa_i}) + \lambda_2 h_i(\boldsymbol{\kappa_i}) + \lambda_3 u_i(\boldsymbol{\kappa_i}) \right\}$$

For each fixed set of Lagrangian multipliers, the Lagrangian dual problem includes n separable nonlinear problems, each with three decision variables lying in the box $[0, 1]^3$ and with an objective that is a 4th-order polynomial that can be solved efficiently. (In the experiments, we used the *Knitro* nonlinear optimization solver to do this.) As the "outer loop" around this subproblem, we apply a conjugate gradient method to solve for the Lagrangian multipliers λ.

4 Finding the best assignment in the final round settlement

After the last round of the auction—i.e., the round where the packing ceases to be feasible—the auctioneer has to decide an outcome for each bidder so as to minimize total payment. For each bidder, the outcome has to be selected from among the options that the bidder's bids indicate are still acceptable to her. In case none of the three options are acceptable, the bidder has to stay in its current band, but can be reallocated to a different channel within the band.

We use the notation from Section 2.1 since the problem of finding the best assignment shares the set of constraints with the feasibility problem. That is, z_{ik} is a binary variable that indicates whether station i is assigned to channel k. We have $\sum_{k\in C_i} z_{ik} \leq 1$ since each station will be assigned to at most one channel. If $\sum_{k\in C_i} z_{ik} = 0$, then the station goes off-air and the auctioneer needs to pay the offer price $p_{i,OFF}$. Otherwise, the station needs to be assigned to one of the three bands. Let $C_{ib} \in C_i$ be the list of feasible channels for station i in band $b \in \{LVH, UVH, UHF\}$. If $\sum_{k\in C_{ib}} z_{ik} = 1$, the auctioneer will pay $p_{i,b}$ to the station to be allocated into band b if that differs from the station's current band. Thus, for an UHF station i, the payment is $p_{i,LVH} \sum_{k\in C_{i,LVH}} z_{ik} + p_{i,UVH} \sum_{k\in C_{i,UVH}} z_{ik} + p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik})$. The payment for a UVH station is $p_{i,LVH} \sum_{k\in C_{i,LVH}} z_{ik} + p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik})$. The total payment for an LVH stations is $p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik})$. Summing the payments to individual stations provides us with the objective function that the FCC wants to minimize. So, the overall final-round winner determination problem is

$$\begin{aligned}
\min_{z} \quad & \sum_{i\in\mathcal{N}} p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik}) + \sum_{i\in UVH,UHF} p_{i,LVH}(1 - \sum_{k\in C_{i,LVH}} z_{ik}) + \sum_{i\in UHF} p_{i,UVH}(1 - \sum_{k\in C_{i,UVH}} z_{ik}) \\
\text{s.t.} \quad & \mathbf{z} \in \mathcal{P}(\mathcal{A}, \mathcal{C}); \qquad \text{See (1) for a formulation of } \mathcal{P}(\mathcal{A}, \mathcal{C}). \qquad (6)
\end{aligned}$$

This is a mixed integer linear program with similar structure to the WDP in the VCG. It has 616,907 binary variables and 2.9×10^6 constraints, which make it very difficult to solve. The FCC has attempted to solve it, but according to Milgrom and Segal [17], solving an instance of it with a state-of-the-art integer programming package takes weeks without finding an optimal solution.

We present a custom mathematical programming technique to find a near-optimal solution. Although the problem has a very large number of decision variables and a huge number of constraints, it has nice underlying structure. Stations can be viewed as nodes and interference constraints as edges. The problem becomes a generalized graph coloring problem with the following additional restrictions. First, each station has a set of feasible channels that it can be allocated to. Second, the adjacency restrictions should be represented by 'dotted edges' in the graph to indicate the certain pairs of stations cannot be allocated to adjacent channels. Observe that the constraints are of knapsack form and many of them can be combined by looking for all the maximal cliques as presented by Nguyen and Sandholm [19]. This reduces the number of constraints and produces a stronger LP relaxation. In addition, with the underlying graph, we apply a decomposition technique to divide the problem into smaller manageable subproblems through Lagrangian relaxation. Although the algorithm may not produce an optimal solution even if we let it run a long time, our experiments show that within a run-time limit of 10 minutes we usually obtain solutions that are 85-90% of optimal. We find this performance acceptable for the purpose of comparing the percentile-based approach and the optimization-based approach. (Without the decomposition, it took CPLEX 10-30 minutes to *load* the problem, not to talk about solving it.) The methods for decomposing the problem and for solving the Lagrangian relaxation are described in Appendix A.4.

5 Experiments

We implemented our optimization-based price-decrementing technique and conducted experiments using real FCC data. We compared the performance against the simple natural percentile-based method (described in the beginning of Section 3.3).

Since no incentive auctions have yet been conducted, we have to use generated data on the bounds of the bidders' valuations. The bounds for the first experiment (symmetric bidders) are generated using a uniform distribution where the upper and lower bound for the off-air option for bidder i are set to $u_{i1} = (1 + \delta)m_i$ and $l_{i1} = (1 - \delta)m_i$ and where m_i is a uniform random variable in $[0, 1]$. Here, $\delta = 0.2$ is a measure of how good the auctioneer's estimate of the bidders' valuations is. For each station, the upper and lower support bounds for the LVH option and the UVH option are set to 66.7% and 33.3% of the bounds for the off-air option, respectively. These percentages are consistent with the FCC estimates FCC [10]. We then draw random sample bid values from these ranges, that is, $\xi_{ik} \sim U[l_{ik}, u_{ik}]$ for each bidder $i = 1, \dots, n$ and for each option $k = \{1, 2, 3\}$. We draw M = 10 valuation vectors. Each vector corresponds to a DCA instance. The setting for the second experiment (asymmetric bidders) is similar except that the mean value m_i is set proportional to the population that station i serves [7]. In each experiment, the number of rounds allowed is 50.

We tried a number of possible acceptance probabilities for the percentile-based method. Early experiments showed that acceptance probabilities below 0.97 per round often make the DCA auctions end prematurely while acceptance probabilities above 0.995 have little effect on the price discovery. Thus we conducted detailed experiments with acceptance probabilities in $\{0.97, 0.975, 0.98, 0.985, 0.99, 0.995, 1\}$.

We also studied the role of final round settlement. Sections 5.1 and 5.2 assume that once encountering infeasibility, the DCA stops. We then report the final payments that the auctioneer has to pay to all active options of the second-to-last (i.e., last feasible) round. Section 5.3 studies the case with final round settlement.

5.1 Incentive auctions with symmetric valuation distributions

Figures 3 shows that OPT-SCHED outperforms the percentile-based approach on all instances—with 27% reduction in payment compared to the best choice of the per-round acceptance probability in the simple percentile-based approach. The optimization-based approach is able to reject high-value bids taking into account feasibility considerations.

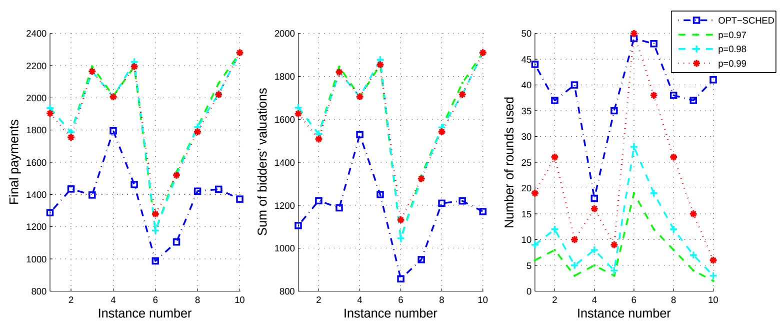

Figure 3: Final payments and active bidders in the symmetric valuations setting.

Figure 4-a illustrates how the auction proceeds across rounds. OPT-SCHED succeeds in proceeding through more rounds before reaching infeasibility by being more intelligent about taking feasibility into account in the pricing. It has a better way of rejecting high-priced bids and to balance payments against feasibility constraints.

5.2 Incentive auctions with asymmetric valuation distributions

In the next experiment, we set the mean value m_i proportional to the population that station i serves. Figure 5 shows the performance over $M = 10$ generated auction instances. OPT-SCHED yields lower final payment than the simple percentile-based approach for all choices of the fixed acceptance probability. It results in 25% lower payment on average.

Figure 6 shows that, again, OPT-SCHED succeeds in proceeding through more rounds before reaching infeasibility by more intelligently taking feasibility into account in the pricing.

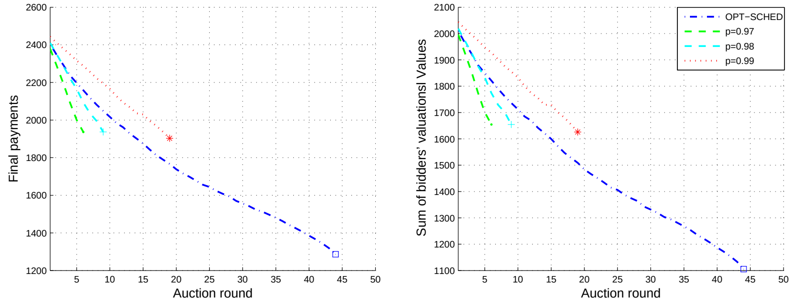

Figure 4: Auctio uations setting.

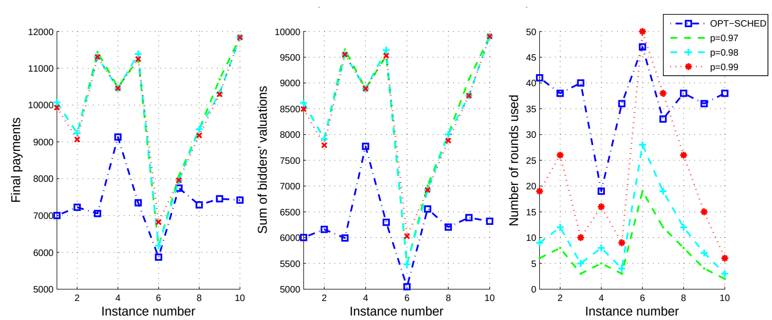

Figure 5: Final payments and active bidders in the asymmetric valuations setting.

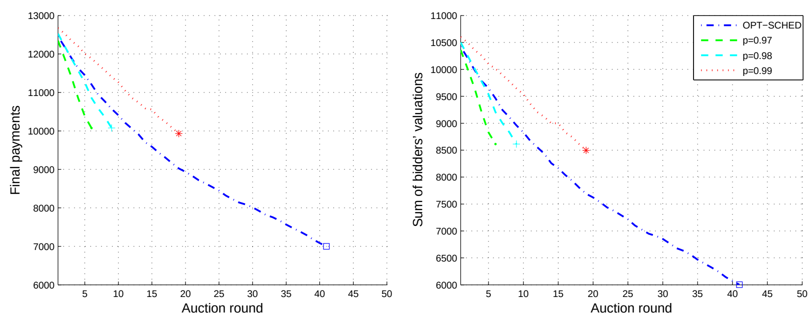

Figure 6: Auction trajectory on Instance 1 (a typical instance) in the asymmetric valuations setting.

5.3 Effect of final round settlement

The results presented above in Sections 5.1 and 5.2 are measured based on the sum (or equivalently, the average) of the payment to the active options of all bidders just before reaching repacking infeasibility. Instead, the auctioneer can choose the best set of offers from the bidders' active options in order to minimize total payment while ensuring repacking feasibility by solving the final round settlement model from Section 4. Figures 7 and 8 report results with final round settlement. (We do not show the results of the percentile-based approach for some instances because on those instances the payment and sum of bidders' valuations are much larger than on the rest of the instances, and thus their inclusion would make it difficult to visualize the rest of the results. These instances that are particularly bad for the percentile-based approach include Instance 6 with $p = 0.99$, Instance 7 with $p = 0.97$, and Instance 9 with $p = 0.97$.)

OPT-SCHED outperforms the percentile-based approach for all choices of acceptance probabilities in

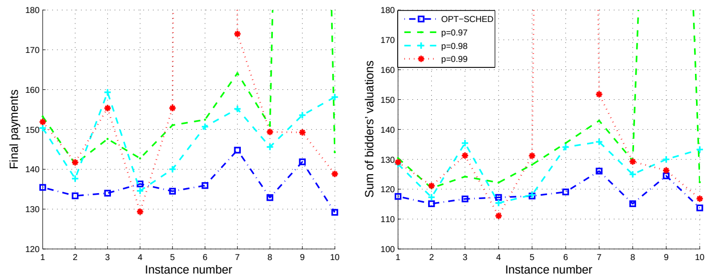

Figure 7: Results with final round settlement in the symmetric valuations setting.

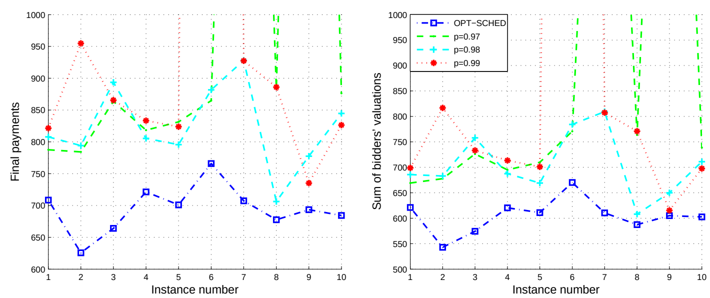

Figure 8: Results with final round settlement in the asymmetric valuations setting.

the asymmetric setting. In the symmetric setting, it almost always outperforms (except on Instance 4 with acceptance probability 0.99). The average payment by OPT-SCHED is 695 while that of the percentile-base approach is 1182, 823, and 1450 for the per-round acceptance probabilities 0.97, 0.98, and 0.99, respectively. In summary, OPT-SCHED dramatically outperforms the percentile-based approach.

5.3.1 Repacking solution

We report the OPT-SCHED repacking solution for Instance 1. The pattern on the other nine instances was similar.

In the asymmetric setting, among the 1647 UHF stations, 1278 were repacked to a UHF channel (channels 14-31), 57 were repacked to a UVH channel (7-13), 65 were repacked to an LVH channel (2-6), and 247 went off-air. Among the 428 UVH stations, 346 were repacked to a UVH channel, 21 were repacked to an LVH channel, and 61 went off-air. Among the 55 LVH stations, 49 were repacked to an LVH channel and 6 went off-air.

In the symmetric setting, among the 1647 UHF stations, 1277 were repacked to a UHF channel, 54 were repacked to a UVH channel, 56 were repacked to an LVH channel, and 260 went off-air. Among the 428 UVH stations, 341 were repacked to a UVH channel, 20 were repacked to an LVH channel, and 67 went off-air. Among the 55 LVH stations, 53 were repacked to a LVH channel and 2 went off-air.

6 Conclusions

We presented a multi-option DCA framework in which each bidder may be able to sell one from a set of options to the auctioneer. We developed a Markov chain model for representing the dynamics of each bidder's state in the auction, as well as an optimization model and technique for finding prices to offer to the different bidders for the different options in each round—using the Markov chain. The optimization minimizes total payment while ensuring feasibility in a stochastic sense. We also introduced percentile-based approaches to decrementing prices. Experiments with real FCC incentive auction interference constraint data revealed that the optimization-based approach dramatically outperforms the simple percentile-based approach both under symmetric and asymmetric bidder valuation distributions—because it takes feasibility into account in pricing. Both of our pricing techniques scale to the large.

Bibliography

[1] Adomavicius, G., Curley, S. P., Gupta, A., and Sanyal, P. 2012. Effect of information feedback on bidder behavior in continuous combinatorial auctions. *Mgmt. Sci. 58,* 4, 811–830.

[2] Cramton, P. 1997. The FCC spectrum auctions: an early assessment. *Journal of Economics and Management Strategy 6,* 3, 431–495. Special issue on market design and spectrum auctions.

[3] Cramton, P., Shoham, Y., and Steinberg, R. 2006. *Combinatorial Auctions.* MIT Press.

[4] Day, R. and Milgrom, P. 2008. Core-selecting package auctions. *International Journal of Game Theory 36,* 3, 393–407.

[5] Day, R. W. and Raghavan, S. 2007. Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science 53,* 9, 1389–1406.

[6] FCC. 2012. Notice of proposed rulemaking. Tech. rep., FCC. http://hraunfoss.fcc.gov/edocs_public/attachmatch/FCC-12-118A1.pdf.

[7] FCC. 2013. Fcc baseline data and maps. Tech. rep., FCC. http://data.fcc.gov/download/incentive-auctions/OET-69/Baseline_Data_and_Maps_2013July.pdf.

[8] FCC. 2014a. Broadcast incentive auction 101. Tech. rep., FCC. http:// wireless.fcc.gov/ incentiveauctions/learn-program/Broadcast_Incentive_Auction_101_slides.pdf.

[9] FCC. 2014b. FCC repacking constraint files. http://data.fcc.gov/download/incentive-auctions/Constraint_Files/.

[10] FCC. 2014c. Notice of proposed rulemaking. Tech. rep., FCC. http-s://apps.fcc.gov/edocs_public/attachmatch/FCC-14-191A1.pdf.

[11] FCC. 2014d. The path to a successful incentive auction. FCC presentation 1/30/2014, wireless.fcc.gov/incentiveauctions/learn-program/Incentive_Auction_Jan_30_Present_9am.pdf.

[12] Hazlett, T., Porter, D., and Smith, V. 2012. "Incentive auctions" economic and strategic issues. Arlington Economics white paper 6/12/2012.

[13] Kelly, F. and Steinberg, R. 2000. A combinatorial auction with multiple winners for universal service. *Management Science 46,* 4, 586–596.

[14] Leyton-Brown, K. 2013. Investigating the viability of exact feasibility testing. Working Paper, University of British Columbia / Auctionomics, http://www.cs.ubc.ca/ ˜ kevinlb/talk.php?u=2013-Feasibility-Testing.pdf.

[15] Milgrom, P. 2004. *Putting Auction Theory to Work.* Cambridge University Press.

[16] Milgrom, P. and Segal, I. 2012. Heuristic auctions and U.S. spectrum repurposing. Presentation, Stanford University, www.stanford.edu/ ˜ isegal/fcc.ppt.

[17] Milgrom, P. and Segal, I. 2013. Deferred-acceptance heuristic auctions. Working Paper, Stanford University, http://www.milgrom.net/downloads/heuristic.pdf.

[18] Milgrom, P. and Segal, I. 2014. Deferred-acceptance auctions and radio spectrum reallocation. In *Proc. ACM Conference on Economics and Computation (EC).*

[19] Nguyen, T.-D. and Sandholm, T. 2014. Optimizing prices in descending clock auctions. In *Proc. ACM Conference on Economics and Computation (EC)*. 93–110.

[20] Olivares, M., Weintraub, G. Y., Epstein, R., and Yung, D. Combinatorial auctions for procurement: An empirical study of the chilean school meals auction. *Mgmt. Sci. 58,* 8.

[21] Paul Duetting, V. G. and Roughgarden, T. 2014. The performance of deferred-acceptance auctions. In *Proc. ACM Conference on Economics and Computation (EC)*.

[22] Rothkopf, M. H., Pekeč, A., and Harstad, R. M. 1998. Computationally manageable combinational auctions. *Management science 44,* 8, 1131–1147.

[23] Sandholm, T. 2002. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence 135*, 1–54.

[24] Sandholm, T. 2006. Optimal winner determination algorithms. In *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, Eds. MIT Press, 337–368. Chapter 14.

[25] Sandholm, T. 2013. Very-large-scale generalized combinatorial multi-attribute auctions: Lessons from conducting $60 billion of sourcing. In *Handbook of Market Design*, Z. Neeman, A. Roth, and N. Vulkan, Eds. Oxford University Press.

[26] Sandholm, T., Suri, S., Gilpin, A., and Levine, D. 2005. CABOB: A fast optimal algorithm for winner determination in combinatorial auctions. *Management Science 51,* 3, 374–390.

[27] Wheeler, T. 2013. The path to a successful incentive auction. FCC Chairman's post on the official FCC blog 12/6/2013.

APPENDIX

A.1 Incorporating a one-way hierarchy of options

Very recently (in December 2014), the FCC put up for comment a specific proposal for the DCA to be used for the reverse auction part of the imminent incentive auction, including a sketch of a price adjustment method [10]. It is much more rigid than what we propose in this paper. Also, the pricing heuristic does not take feasibility into account to nearly the same extent as our pricing technique does. To our knowledge, no theory or experiments have been published so far to analyze the design choices.

There is also another confining—but potentially interesting—aspect of that DCA design. The options are considered to form a hierarchy. A bidder has to declare a preferred option (which is the option that he might get) at each point in the auction. A bidder is allowed to move the declared preferred option only downward in the hierarchy. So, a bidder can go from off-air to a lower band to an even lower band and then to accepting no offer, but not in the other direction. Also, a bidder is allowed to only move downward in the hierarchy from the option that she holds before the auction begins. Our Markov modeling and optimization techniques can be adapted to that setting as well. In the rest of this appendix we describe how to do that.

We denote by $k = \{1, 2, 3\}$ the options that correspond to Off-air, LVH and UVH. We denote by p_{ik} the price the auction offers to station i for option k. For simplicity, here we assume the v_{ik} are uniformly distributed with support $[l_{ik}, u_{ik}]$.

At the beginning of each DCA round, the station receives offer prices for its active options and evaluates its surplus $(p_{ik} - v_{ik})$ for each option k. Whenever the station switches its preferred option, options higher in the hierarchy become permanently inactive.

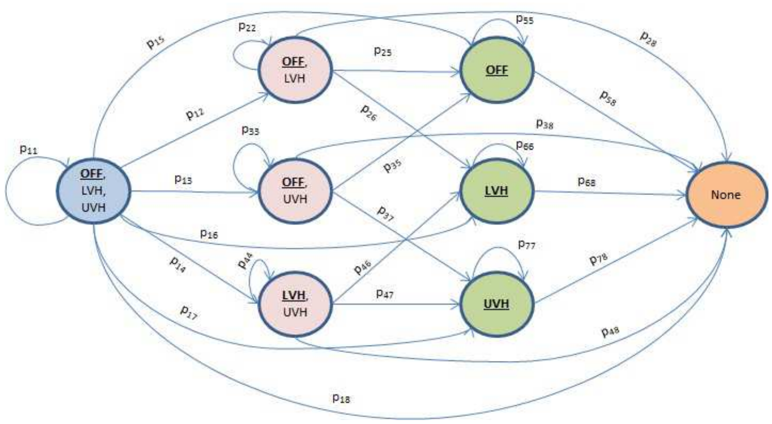

Figure 9: Markov chain on bidder status.

Figure 9 shows the states and transition probabilities for a station that is currently in the UHF band. Each node represents a state and the words inside describe the options that are still active. One of these active options is written in bold and underlined to highlight it as the preferred option. This preferred option is always the highest of the remaining active ones in the hierarchy. There are eight states which include:

- State S_1 with all three options still active and with off-air being the preferred option.
- State S_2 with options off-air and LVH still active and with off-air being the preferred option.
- State S_3 with options off-air and UVH still active and with off-air being the preferred option.

- State S_4 with options LVH and UVH still active and with LVH being the preferred option.
- State S_5 with the only option off-air still active which is also the preferred option.
- State S_6 with the only option LVH still active which is also the preferred option.
- State S_7 with the only option UVH still active which is also the preferred option.
- State S_8 with none of the options active.

We observe that the four states of S_1, S_2,S_3 and S_5 still have off-air as an active and the preferred option. The transition among these four states requires at least two conditions: (a) off-air is still active and (b) off-air is still the best option among those available. The remaining four states of S_4, S_6, S_7 and S_8 do not include the off-air option. This mean that transition from states (S_1, S_2, S_3, S_5) to the four states of (S_4, S_6, S_7, S_8) requires at least either (a) off-air no longer being active or (b) off-air still being acceptable but not as attractive as some other options and hence the station requested a switch. In the latter case, the switch would require delisting the off-air option and going down the hierarchy.

Due to the inclusion of the preferred option which could change throughout the auction, the transition among states is now different. We use the same notation of q_1, q_2, and q_3 being the acceptance probabilities to offer to the three options of off-air, LVH and UVH. We aim to derive the transition probabilities as a function of (q_1, q_2, q_3). Let us denote by p_{ij} the transition probability from state S_i to state S_j.

First let us consider the transition from state S_1 to S_2. This occurs under the following conditions: the UVH option is no longer attractive while the off-air and LVH options are still acceptable with off-air being the preferred option. This transition probability is calculated as

$$\begin{aligned} p_{12} &= \text{Prob}(p_{i1} - v_{i1} \geq p_{i2} - v_{i2} \geq 0 \geq p_{i3} - v_{i3}) \\ &= \text{Prob}(p_{i1} - v_{i1} \geq p_{i2} - v_{i2} \geq 0)\text{Prob}(p_{i3} - v_{i3} \leq 0) \end{aligned}$$

We observe that $(p_{ik} - v_{ik})$ is a uniform random variable with support $[p_{ik} - u_{ik}, p_{ik} - l_{ik}]$. Thus, $\text{Prob}(p_{i3} - v_{i3} \leq 0) = \frac{u_{i3}-p_{i3}}{u_{i3}-l_{i3}}$. In addition,

$$\begin{aligned} \text{Prob}(p_{i1} - v_{i1} \geq p_{i2} - v_{i2} \geq 0) &= \int_0^{min(p_{i1}-l_{i1}, p_{i2}-l_{i2})} \left[\int_x^{p_{i1}-l_{i1}} \frac{1}{d_1} dy \right] \frac{1}{d_2} dx \\ &= \begin{cases} \frac{1}{d_1 d_2} \int_0^{p_{i2}-l_{i2}} [p_{i1} - l_{i1} - x]\, dx, & \text{if } p_{i2} - l_{i2} \leq p_{i1} - l_{i1} \\ \frac{1}{d_1 d_2} \int_0^{p_{i1}-l_{i1}} [p_{i1} - l_{i1} - x]\, dx, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{d_1 d_2} \frac{(p_{i2}-l_{i2})(2(p_{i1}-l_{i1})-(p_{i2}-l_{i2}))}{2}, & \text{if } p_{i2} - l_{i2} \leq p_{i1} - l_{i1} \\ \frac{1}{d_1 d_2} \frac{(p_{i1}-l_{i1})^2}{2}, & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$p_{12} = \begin{cases} \frac{u_{i3}-p_{i3}}{u_{i3}-l_{i3}} \frac{1}{d_1 d_2} \frac{(p_{i2}-l_{i2})(2(p_{i1}-l_{i1})-(p_{i2}-l_{i2}))}{2}, & \text{if } p_{i2} - l_{i2} \leq p_{i1} - l_{i1} \\ \frac{u_{i3}-p_{i3}}{u_{i3}-l_{i3}} \frac{1}{d_1 d_2} \frac{(p_{i1}-l_{i1})^2}{2}, & \text{otherwise} \end{cases}$$

Similarly, we can derive

$$p_{13} = \begin{cases} \frac{p_{i2}-l_{i2}}{u_{i2}-l_{i2}} \frac{1}{d_1 d_3} \frac{(p_{i3}-l_{i3})(2(p_{i1}-l_{i1})-(p_{i3}-l_{i3}))}{2}, & \text{if } p_{i3} - l_{i3} \leq p_{i1} - l_{i1} \\ \frac{p_{i2}-l_{i2}}{u_{i2}-l_{i2}} \frac{1}{d_1 d_3} \frac{(p_{i1}-l_{i1})^2}{2}, & \text{otherwise} \end{cases}$$

The transition from state S_1 to S_4 occurs when off-air is not as attractive as LVH while both LVH and UVH are acceptable, that is,

$$\begin{aligned}
p_{14} &= \text{Prob}(p_{i1} - v_{i1} < p_{i2} - v_{i2} \text{ and } 0 \leq p_{i2} - v_{i2} \text{ and } 0 \leq p_{i3} - v_{i3}) \\
&= \text{Prob}(p_{i3} - v_{i3} \geq 0)\text{Prob}(p_{i1} - v_{i1} < p_{i2} - v_{i2} \text{ and } 0 \leq p_{i2} - v_{i2}) \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}}\text{Prob}(p_{i1} - v_{i1} < p_{i2} - v_{i2} \text{ and } 0 \leq p_{i2} - v_{i2}) \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \int_0^{p_{i2}-l_{i2}} \left[\int_{p_{i1}-u_{i1}}^{min(x, p_{i1}-l_{i1})} \frac{1}{d_1} dy \right] \frac{1}{d_2} dx \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \frac{1}{d_1 d_2} \int_0^{p_{i2}-l_{i2}} \left[min(x, p_{i1} - l_{i1}) - (p_{i1} - u_{i1}) \right] dx \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \frac{1}{d_1 d_2} \begin{cases} \frac{(p_{i2}-l_{i2})(2(u_{i1}-p_{i1})+(p_{i2}-l_{i2}))}{2}, & \text{if } p_{i2} - l_{i2} \leq p_{i1} - l_{i1} \\ d_1((p_{i2} - l_{i2}) - (p_{i1} - l_{i1})) + \frac{(p_{i1}-l_{i1})(2(u_{i1}-p_{i1})+(p_{i1}-l_{i1}))}{2}, & \text{otherwise} \end{cases}
\end{aligned}$$

Similarly, we can derive

$$\begin{aligned}
p_{15} &= \text{Prob}(p_{i1} - v_{i1} \geq 0 \text{ and } p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\
&= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

$$\begin{aligned}
p_{16} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \geq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{p_{i2} - l_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

$$\begin{aligned}
p_{17} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \geq 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

$$\begin{aligned}
p_{18} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

From state S_2:

$$\begin{aligned}
p_{25} &= \text{Prob}(p_{i1} - v_{i1} \geq 0 \text{ and } p_{i2} - v_{i2} \leq 0) \\
&= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}}
\end{aligned}$$

$$\begin{aligned}
p_{26} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \geq 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{p_{i2} - l_{i2}}{u_{i2} - l_{i2}}
\end{aligned}$$

$$\begin{aligned}
p_{28} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \leq 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}}
\end{aligned}$$

We can derive p_{22} in a similar way or we can simply use the formulation $p_{22} = 1 - p_{25} - p_{26} - p_{28}$.
From state S_3:

$$\begin{aligned} p_{35} &= \text{Prob}(p_{i1} - v_{i1} \geq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{37} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i3} - v_{i3} \geq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{38} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

We have $p_{33} = 1 - p_{35} - p_{37} - p_{38}$.

$$\begin{aligned} p_{46} &= \text{Prob}(p_{i2} - v_{i2} \geq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{47} &= \text{Prob}(p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \geq 0) \\ &= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{48} &= \text{Prob}(p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

We have $p_{44} = 1 - p_{46} - p_{47} - p_{48}$.
From state S_5:

$$\begin{aligned} p_{58} &= \text{Prob}(p_{i1} - v_{i1} \leq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \end{aligned}$$

$$\begin{aligned} p_{55} &= \text{Prob}(p_{i1} - v_{i1} \geq 0) \\ &= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \end{aligned}$$

From state S_6:

$$\begin{aligned} p_{68} &= \text{Prob}(p_{i2} - v_{i2} \leq 0) \\ &= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \end{aligned}$$

$$p_{66} = \text{Prob}(p_{i2} - v_{i2} \geq 0) = \frac{p_{i2} - l_{i2}}{u_{i2} - l_{i2}}$$

From state S_7:

$$p_{78} = \text{Prob}(p_{i3} - v_{i3} \leq 0) = \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}$$

$$p_{77} = \text{Prob}(p_{i3} - v_{i3} \geq 0) = \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}}$$

Figure 10 shows the state transition for stations that are currently in the LVH and UVH bands. For UVH stations, the Markov chain contains four states S_2, S_5, S_6 and S_8. The transition probabilities among these states are equal to those in the Markov chain for an UHF station. Similarly the Markov chain for an LVH station contains two states S_7 and S_8, also with the same transition probabilities

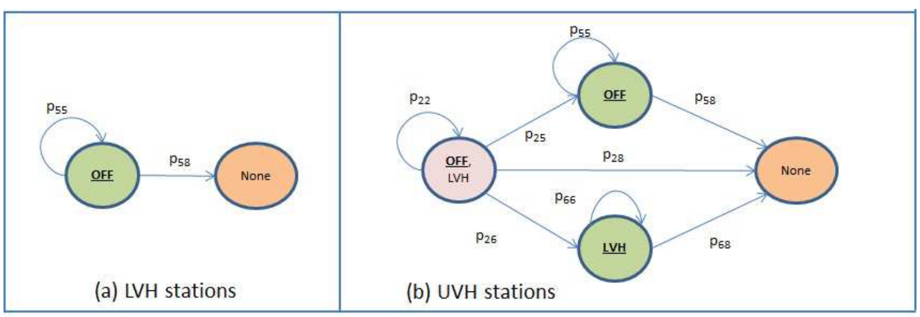

Figure 10: Markov chain on bidder status.

One disadvantage of this new setting with a preferred option and a hierarchy is that the resulting optimization model will be of more complicated form as a result of the new transition probabilities. We leave the computational approach for future research. However, a nice property of the new setting is that the auctioneer pays each station the price of the preferred option and hence we do not have to introduce the weight matrices presented in Section 3.2. In addition, since we know which band each station will be allocated to, the expected number of stations allocated to each band can be calculated with certainty.

A.2 Method for optimizing percentiles

Recall that for the purposes of this appendix, the auctioneer is trying to solve for (q_1, q_2, q_3) such that the expected number of stations allocated to each band (UHF, UVH, and LVH) equals its target. Denote these targets by $C_b,\ b \in \{\text{UHF, UVH,LVH}\}$.

We get three equations that correspond to three bands and three unknowns. The issue is that these expected values are polynomial of degree $3m$ in the unknown, and numerical methods for solving these might lead to approximation errors. We resolve this challenge by observing that the decision variables

(q_1, q_2, q_3) can be replaced by $(\kappa_1, \kappa_2, \kappa_3)$ and we arrive at the following system of three equations for three unknowns $(\kappa_1, \kappa_2, \kappa_3)$:

$$
\begin{aligned}
C_{UHF} &= N_{UHF}(S_1)(1-\kappa_1)(1-\kappa_2)(1-\kappa_3) + \\
&\quad N_{UHF}(S_2)(1-\kappa_1)(1-\kappa_2) + N_{UHF}(S_3)(1-\kappa_1)(1-\kappa_3) + N_{UHF}(S_4)(1-\kappa_2)(1-\kappa_3) + \\
&\quad N_{UHF}(S_5)(1-\kappa_1) + N_{UHF}(S_6)(1-\kappa_2) + N_{UHF}(S_7)(1-\kappa_3) + N_{UHF}(S_8) \\
C_{UVH} &= N_{UHF}(S_1)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + \\
&\quad N_{UHF}(S_3)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3) + N_{UHF}(S_4)(\kappa_3 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_7)(\kappa_3) + \\
&\quad N_{UVH}(S_2)(1-\kappa_1)(1-\kappa_2) + N_{UVH}(S_5)(1-\kappa_1) + N_{UVH}(S_6)(1-\kappa_2) + N_{UVH}(S_8) \\
C_{LVH} &= N_{UHF}(S_1)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + \\
&\quad N_{UHF}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + N_{UHF}(S_4)(\kappa_2 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_6)(\kappa_2) + \\
&\quad N_{UVH}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + N_{UVH}(S_6)\kappa_2 + N_{LVH}(S_5)(1-\kappa_1) + N_{LVH}(S_8).
\end{aligned}
$$

We observe that the right-hand sides of these equalities include polynomials of degree 3. Hence we can apply a numerical method—such as the Newton-Raphson's method—to solve for $(\kappa_1, \kappa_2, \kappa_3)$ easily. We then set $q_i = \kappa_i^{1/m}, \forall i \in \{1, 2, 3\}$.

In the case where the auctioneer sets the same acceptance rates for the three options, let $q = q_1 = q_2 = q_3$ and $r = q^m$. Then the expected numbers of stations in the bands are

$$
\begin{aligned}
E_{UHF} &= \sum_{i \in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(1) \\
&= N_{UHF}(S_1)(1-r)^3 + N_{UHF}(S_2, S_3, S_4)(1-r)^2 + N_{UHF}(S_5, S_6, S_7)(1-r) + N_{UHF}(S_8) \\
E_{UVH} &= \sum_{i \in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(2) + \sum_{i \in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(1) \\
&= N_{UHF}(S_1)(r - r^2 + \frac{1}{3}r^3) + N_{UHF}(S_3, S_4)(r - \frac{1}{2}r^2) + N_{UHF}(S_7)r + \\
&\quad N_{UVH}(S_2)(1-r)^2 + N_{UVH}(S_5, S_6)(1-r) + N_{UVH}(S_8) \\
E_{LVH} &= \sum_{i \in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(3) + \sum_{i \in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(2) + \sum_{i \in LVH} \Phi_i^r \Gamma_{LVH}^m W_{LVH}(1) \\
&= N_{UHF}(S_1)(r - r^2 + \frac{1}{3}r^3) + N_{UHF}(S_2, S_4)(r - \frac{1}{2}r^2) + N_{UHF}(S_6)r + \\
&\quad N_{UVH}(S_2)(3/2r - r^2) + N_{UVH}(S_6)r + N_{LVH}(S_5)(1-r) + N_{LVH}(S_8)
\end{aligned}
$$

We can then solve for r that minimizes the sum of square errors, i.e., $\sum_b (E_b - C_b)^2$.

A.3 Expanded form of the OPT-SCHED model

Let $UHF(S_j)$ (and similarly $UVH(S_j)$, $LVH(S_j)$) denote the list of UHF (UVH, LVH) stations that are currently in state S_j. The problem of minimizing the expected payment while ensuring the expected number of stations allocated to each band does not exceed the band capacity can be formulated as

$$
\begin{aligned}
\min_{\boldsymbol{\kappa}} \quad & f(\boldsymbol{\kappa}) \\
\text{s.t.} \quad & \sum_{i\in UHF(S_1)} (1-\kappa_{i1})(1-\kappa_{i2})(1-\kappa_{i3}) + \\
& \sum_{i\in UHF(S_2)} (1-\kappa_{i1})(1-\kappa_{i2}) + \sum_{i\in UHF(S_3)} (1-\kappa_{i1})(1-\kappa_{i3}) + \sum_{i\in UHF(S_4)} (1-\kappa_{i2})(1-\kappa_{i3}) + \\
& \sum_{i\in UHF(S_5)} (1-\kappa_{i1}) + \sum_{i\in UHF(S_6)} (1-\kappa_{i2}) + \sum_{i\in UHF(S_7)} (1-\kappa_{i3}) + \sum_{i\in UHF(S_8)} 1 \le C_{UHF}, \\
& \sum_{i\in UHF(S_1)} (\kappa_{i3} - \frac{1}{2}\kappa_{i1}\kappa_{i3} - \frac{1}{2}\kappa_{i2}\kappa_{i3} + \frac{1}{3}\kappa_{i1}\kappa_{i2}\kappa_{i3}) + \\
& \sum_{i\in UHF(S_3)} (\kappa_{i3} - \frac{1}{2}\kappa_{i1}\kappa_{i3}) + \sum_{i\in UHF(S_4)} (\kappa_{i3} - \frac{1}{2}\kappa_{i2}\kappa_{i3}) + \sum_{i\in UHF(S_7)} (\kappa_{i3}) + \qquad (7) \\
& \sum_{i\in UVH(S_2)} (1-\kappa_{i1})(1-\kappa_{i2}) + \sum_{i\in UVH(S_5)} (1-\kappa_{i1}) + \sum_{i\in UVH(S_6)} (1-\kappa_{i2}) + \sum_{i\in UVH(S_8)} 1 \le C_{UVH}, \\
& \sum_{i\in UHF(S_1)} (\kappa_{i2} - \frac{1}{2}\kappa_{i1}\kappa_{i2} - \frac{1}{2}\kappa_{i2}\kappa_{i3} + \frac{1}{3}\kappa_{i1}\kappa_{i2}\kappa_{i3}) + \\
& \sum_{i\in UHF(S_2)} (\kappa_{i2} - \frac{1}{2}\kappa_{i1}\kappa_{i2}) + \sum_{i\in UHF(S_4)} (\kappa_{i2} - \frac{1}{2}\kappa_{i2}\kappa_{i3}) + \sum_{i\in UHF(S_6)} (\kappa_{i2}) + \\
& \sum_{i\in UVH(S_2)} (\kappa_{i2} - \frac{1}{2}\kappa_{i1}\kappa_{i2}) + \sum_{i\in UVH(S_5)} \kappa_{i2} + \sum_{i\in LVH,\Phi_i=S_6} (1-\kappa_{i1}) + \sum_{i\in LVH,\Phi_i=S_8} 1 \le C_{LVH},
\end{aligned}
$$

where

$$
\begin{aligned}
f(\boldsymbol{\kappa}) &= \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m C_{UHF,i} + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m C_{UVH,i} + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m C_{LVH,i}, \\
&= \sum_{i\in UHF(S_1)} \frac{1}{3}\kappa_{i1}\kappa_{i2}\kappa_{i3}(\sum_{k=1}^{3} l_{ik} + \kappa_{ik}d_{ik}) + \frac{1}{2}\kappa_{i1}\kappa_{i2}(1-\kappa_{i3})(\sum_{k=1,2} l_{ik} + \kappa_{ik}d_{ik}) + \\
&\quad \frac{1}{2}\kappa_{i1}(1-\kappa_{i2})\kappa_{i3}(\sum_{k=1,3} l_{ik} + \kappa_{ik}d_{ik}) + \frac{1}{2}(1-\kappa_{i1})\kappa_{i2}\kappa_{i3}(\sum_{k=2,3} l_{ik} + \kappa_{ik}d_{ik}) + \\
&\quad \kappa_{i1}(1-\kappa_{i2})(1-\kappa_{i3})(l_{i1} + \kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i2}(1-\kappa_{i3})(l_{i2} + \kappa_{i2}d_{i2}) + \\
&\quad (1-\kappa_{i1})(1-\kappa_{i2})\kappa_{i3}(l_{i3} + \kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UHF(S_2)} \kappa_{i1}\kappa_{i2}\frac{1}{2}(\sum_{k=1,2} l_{ik} + \kappa_{ik}d_{ik}) + \kappa_{i1}(1-\kappa_{i2})(l_{i1} + \kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i2}(l_{i2} + \kappa_{i2}d_{i2}) + \qquad (8) \\
&\quad \sum_{i\in UHF(S_3)} \kappa_{i1}\kappa_{i3}\frac{1}{2}(\sum_{k=1,3} l_{ik} + \kappa_{ik}d_{ik}) + \kappa_{i1}(1-\kappa_{i3})(l_{i1} + \kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i3}(l_{i3} + \kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UHF(S_4)} \kappa_{i2}\kappa_{i3}\frac{1}{2}(\sum_{k=2,3} l_{ik} + \kappa_{ik}d_{ik}) + \kappa_{i2}(1-\kappa_{i3})(l_{i2} + \kappa_{i2}d_{i2}) + (1-\kappa_{i2})\kappa_{i3}(l_{i3} + \kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UHF(S_5)} \kappa_{i1}(l_{i1} + \kappa_{i1}d_{i1}) + \sum_{i\in UHF(S_6)} \kappa_{i2}(l_{i2} + \kappa_{i2}d_{i2}) + \sum_{i\in UHF(S_7)} \kappa_{i3}(l_{i3} + \kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UVH(S_2)} \frac{1}{2}\kappa_{i1}\kappa_{i2}(\sum_{k=1,2} l_{ik} + \kappa_{ik}d_{ik}) + \kappa_{i1}(1-\kappa_{i2})(l_{i1} + \kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i2}(l_{i2} + \kappa_{i2}d_{i2}) + \\
&\quad \sum_{i\in UVH(S_5)} \kappa_{i1}(l_{i1} + \kappa_{i1}d_{i1}) + \sum_{i\in UVH(S_6)} \kappa_{i2}(l_{i1} + \kappa_{i2}d_{i2}) + \sum_{i\in LVH(S_2)} \kappa_{i1}(l_{i1} + \kappa_{i1}d_{i1})
\end{aligned}
$$

A.4 Network decomposition and Lagrangian relaxation

An interesting observation about the network of stations is that it contains many subnetworks, each corresponding to a physical region, where the interference constraints form cliques. These small subnetworks are linked together but these links are sparse, that is, parts of the network share only a few (or no) nearby stations and hence share few binding constraints from an optimization perspective. For example, consider the reordering of the network so that an interference matrix can be viewed in sub-figure (b) of Figure (11). If we divide the network into two groups, group A with stations 1-381 in the ordered list and group B with the remaining stations, then there is only one pair of stations from A and B that cannot co-share a channel. Without this binding constraint, the problem is separable and can be solved by solving two smaller problems. With the presence of binding constraints, a Lagrangian dual problem can be formulated and solved by using a conjugate gradient method. The method often works well if the number of binding constraints is small (e.g., in the above case with only one constraint)

Figure 11-a shows the adjacency matrix of $G(N, E)$. A dot at row i and column j in Figure 11-a appears if there is potential interference between stations i and j. Figure 11-b show the same interference matrix after we reordered the stations in a way that the non-zeros appear mostly on the diagonals.

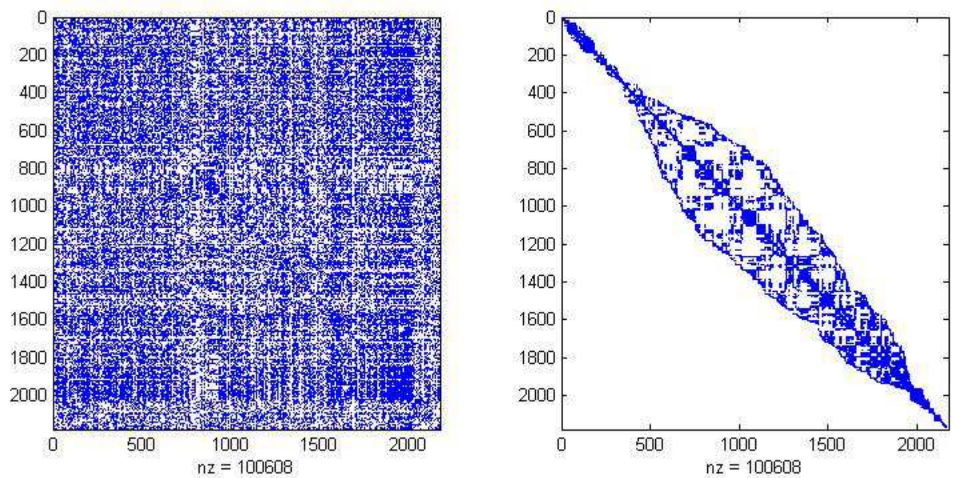

Figure 11: Interference matrix on the original ranking of stations from 1-2177 (left) and after reordering (right).

We need to divide the original network into $G > 2$ sub-networks. Having larger G implies smaller subproblems and hence there is a better chance that the integer program solver (CPLEX in our experiments) can handle the case. However, having larger G often leads to more binding constraints to be relaxed and hence can beget a larger optimality gap. Thus, we need a good way to divide the network that balances between the sizes of the subproblems and the number of constraints relaxed. Finding a division of the network into two parts with the lowest number of binding constraints is equivalent to the Mincut-Maxflow problem and can be solve efficiently. However, the optimal divisions often contains unbalanced subgraphs with a very large subproblem and the decomposition does not help. Therefore, we need to solve a network partitioning problem to divide the network into G subnetworks with similar sizes.

Suppose we divide the network of stations into G subgroups of stations $\mathcal{S}_1, \ldots, \mathcal{S}_G$. Let us define $\boldsymbol{z}_g$ to be the vector of decision variables that involve only stations in group $\mathcal{S}_g$. The objective function is linear and hence is separable to decision variables $\boldsymbol{z}_g$. The constraint set will contain two groups: (a) constraints that involves only decision variables in one of the subgroups and (b) constraints that involves decision variables from two subgroups. The constraints in (b) corresponds to edges that link stations in different groups. The idea of the Lagrangian relaxation technique is to formulate a Lagrangian dual problem where constraints in (b) are relaxed and are pushed into the objective function. The constraint set is now separable in $\boldsymbol{z}_g$ and the problem is separable for each fixed set of Lagrangian multipliers. In what follows we describe the algorithm in detail.

Let us define

$$\begin{aligned}\mathcal{I}_{cg} &= \{(i,j,k) \in \mathcal{I}_c : i \in \mathcal{S}_g, \}, \\ \mathcal{I}_{ag} &= \{(i,j,k) \in \mathcal{I}_a : i \in \mathcal{S}_g, \},\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{cx} &= \{(i,j,k) \in \mathcal{I}_c : \exists g_1, g_2 \in 1,\ldots,G, g_1 \neq g_2, i \in \mathcal{S}_{g_1}, j \in \mathcal{S}_{g_2}, \}, \\ \mathcal{I}_{ax} &= \{(i,j,k) \in \mathcal{I}_a : \exists g_1, g_2 \in 1,\ldots,G, g_1 \neq g_2, i \in \mathcal{S}_{g_1}, j \in \mathcal{S}_{g_2}, \},\end{aligned}$$

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ be the Lagrangian multipliers for co-channel constraints involving set $\mathcal{C}_{cx}$ and for adjacent-channel constraints involving set $\mathcal{C}_{ax}$, respectively. The Lagrangian dual function is

$$\begin{aligned}\mathcal{L}(\boldsymbol{z},\boldsymbol{\lambda},\boldsymbol{\gamma}) &= \sum_{i=1}^{n}(1-\sum_{k\in\mathcal{C}_i} z_{ik})b_i + \\ &\quad \sum_{(i,j,k)\in\mathcal{I}_{cx}} (1-z_{ik}-z_{jk})\lambda_{ijk} + \sum_{(i,j,k)\in\mathcal{I}_{cx}} (1-z_{ik}-z_{jk+1})\gamma_{ijk} \\ &= \sum_{i=1}^{n}|C_i|b_i + \sum_{(i,j,k)\in\mathcal{I}_{cx}} \lambda_{ijk} + \sum_{(i,j,k)\in\mathcal{I}_{ax}} \gamma_{ijk} - \\ &\quad \sum_{i=1}^{n}\sum_{k\in\mathcal{C}_i} z_{ik}\left\{b_i + \sum_{j:(i,j,k)\in\mathcal{I}_{cx}} \lambda_{ijk} + \sum_{j:(i,j,k)\in\mathcal{I}_{ax}} \gamma_{ijk} + \sum_{j:(j,i,k-1)\in\mathcal{I}_{ax}} \gamma_{ijk}\right\} \\ &= \sum_{g=1}^{G} \mathcal{L}_g(\boldsymbol{z}_g,\boldsymbol{\lambda},\boldsymbol{\gamma})\end{aligned}$$

Let us define

$$\mathcal{F}_g(\mathcal{C}) = \left\{\boldsymbol{z}_g : \begin{array}{l} z_{ik} \in \{0,1\}, \forall i \in \mathcal{S}_g \text{ and } k \in C_i,\ \sum_{k\in C_i} z_{ik} = 1, \forall i \in \mathcal{S}_g, \\ z_{ik} + z_{jk} \leq 1, \forall (i,j,k) \in \mathcal{I}_{cg},\ z_{ik} + z_{jk+1} \leq 1, \forall (i,j,k) \in \mathcal{I}_{ag} \end{array}\right\}$$

as the set of constraints that involve stations in group $\mathcal{S}_g$, $g = 1,\ldots,G$, only. The constraint set of the relaxed problem is $\{\boldsymbol{z}_g \in \mathcal{F}_g(\mathcal{C}), \forall g = 1,\ldots,G\}$ which is separable in $\boldsymbol{z}_g$. The Lagrangian dual problem is therefore equivalent to

$$\max_{\boldsymbol{\lambda},\boldsymbol{\gamma}} \min_{\boldsymbol{z}\in\mathcal{F}(\mathcal{C})} \mathcal{L}(\boldsymbol{z},\boldsymbol{\lambda},\boldsymbol{\gamma}) = \max_{\boldsymbol{\lambda},\boldsymbol{\gamma}} \left[\sum_{g\in 1,\ldots,G} \left\{\min_{\boldsymbol{z}_g\in\mathcal{F}_g(\mathcal{C})} \mathcal{L}_g(\boldsymbol{z}_g,\boldsymbol{\lambda},\boldsymbol{\gamma})\right\}\right]$$

For each fixed set of Lagrangian multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$, the Lagrangian dual problem can be solved by solving G subproblems. If the sizes of the subgroups are reasonable such that the subproblems can be solved efficiently, we can apply a sub-gradient method to find the optimal set of the Lagrangian multipliers. The final solution provides us an upper bound to the original problem.