Theoretical Cryptography	Manuel Blum and Steven Rudich
HW3	Due: 4/19/06

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Ryan's Factorization Homework. This homework will familiarize you with some simple randomized factoring methods, culminating in an algorithm that runs in $O(N^{1/4} \cdot poly(\log N))$ steps. Note this is a quadratic speedup over the obvious algorithm that tries all possible factors. Each problem builds upon the previous one, so it's probably best to do them in order. **Start Early!**

For simplicity, let $N = pq$ where p and q are prime in the following problems.

1. **Take a Wild Guess.** The first method requires roughly $\sqrt{N}$ GCD computations in the worst case, so in that sense it is no better than trial division. But (we hope) it is still interesting to think about.

   (a) Prove that a random $r \in \mathbb{Z}_N^+$ has probability at least $1-(p+q-1)/N$ of being relatively prime to N.

   (b) Let e be the base of the natural logarithm.
   Prove that for all integers $x > 1$, $(1-1/x)^{x-1} \geq 1/e \geq (1-1/x)^x$.
   (Note: of course it suffices to prove the inequality for all real $x > 1$, if that's easier.)

   (c) Prove that there is a fixed constant $c > 0$ such that the following algorithm factors N in $O(poly(\log N) \cdot N/(p+q))$ time, with probability at least c:

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   Repeat until a factor is found:
       Choose r ∈ Z_N^+ uniformly at random.
       If GCD(r, N) ≠ 1 then return r as a non-trivial factor of N.
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2. **Applying the Birthday Paradox.** The second method is *still* not asymptotically better than trial division, but it will bring us a little closer to a method that *is* better.

   (a) The birthday paradox says that if you have 23 "randomly chosen" people in a room, then the probability that two in the room have the same birthday is actually quite high, at least 1/2. (This assumes that each person has a birthday selected uniformly at random over all 365 days in the year.)
   Prove that there is a constant $c > 0$ such that, for $r_1, \ldots, r_{\sqrt{n}}$ chosen from $\mathbb{Z}_n^+$ uniformly at random, the probability that there exists $i \neq j$ such that $r_i = r_j$ is at least c.

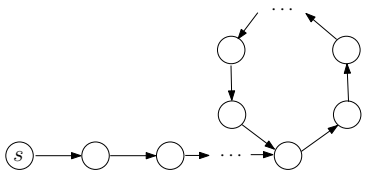
   (b) Let $x_1, \ldots, x_{N^{1/4}}$ be uniformly randomly chosen integers from $[1, N]$.
   Prove: The probability that there exists $i \neq j$ such that $|x_i - x_j|$ has p as a factor is at least some universal constant c.
   Hint: Use modular arithmetic.

   (c) Propose another simple randomized algorithm for factoring N that runs in roughly $N^{1/2}$ steps, and say intuitively why it works. (Your algorithm and analysis should rely crucially on 2b.)

3. **Saving Some Time.** Our final algorithm will run in roughly $N^{1/4}$ steps. To achieve it, we need to take an unexpected turn.

   (a) Let $G = (V, E)$ be a directed graph whose nodes have outdegree at most 1. Let s be a node in G. You have two node-pointers p_1 and p_2; both pointers initially point to node s. The only access to G available is that you are allowed to move one of the two pointers to the successor of that pointer's current node (if the successor exists– if there is no successor then such a move would return an error).

   Under this kind of graph access, give an algorithm to determine if s has a path to a node in a cycle. That is, you want to detect the following pattern:



   Your algorithm should run in linear time ($O(|V|+|E|)$) with *constant* additional workspace. That is, all that one really keeps track of are the node-pointers, with perhaps $O(1)$ additional information.

   (b) Suppose you are given access to a function $f : \mathbb{Z}_N^+ \to \mathbb{Z}_N^+$ that was chosen uniformly at random over all such functions (*i.e.* a random oracle).
   Prove that, if we choose $x_1 \in \mathbb{Z}_N^+$ at random, and set $x_k = f(x_{k-1})$ for $k = 2, \ldots, N^{1/4}$, then there is a non-zero constant probability (over the choice of f and x_1) that $i \neq j$ exist such that $x_i \equiv x_j \mod p$.

   (c) Suppose p is given. Let f, x_1, and x_k's be chosen as in the previous problem. Give an $O(N^{1/4} \cdot \mathit{poly}(\log N))$ time algorithm that *finds* x_i and x_j such that $i \neq j$ and $x_i \equiv x_j \mod p$, with probability at least some $c > 0$.
   Hint: Define a graph G where each vertex is a congruence class modulo p. (So, the vertices are $\{0, \ldots, p-1\}$.) Put directed edges $(u, f(u))$ in G. Pick a random vertex ($x_1 \mod p$) in G...

   (d) You may assume access to a random oracle in this problem. Propose an algorithm that factors N in $O(N^{1/4} \cdot \mathit{poly}(\log N))$ steps, with probability at least some $c > 0$.

   (Note in this case, p is NOT given!)

   For extra credit, prove the correctness of your algorithm.