Computational Higher Type Theory

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Happy Birthday Gordon!

Thanks

Joint work with Carlo Angiuli (CMU) and Todd Wilson (CSUF).

Thanks to Dan Licata for many discussions.

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Two Kinds of Type Theory

Two traditions in type theory, both embodied by Martin-Löf:

- Formal, or axiomatic, as in ITT and HoTT.
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What is the computational content of HoTT?

Semantic Type Theory

Martin-Löf; Constable, et al

Meaning explanations define types and elements semantically:

- Computational: as programs with deterministic dynamics.
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Inverts conceptual order compared to formal type theory:

- Type theory as a theory of truth.
- Proof theory accesses the truth.

Computational Meaning Explanation

Martin-Löf: Constr. Math. and Comp. Prog.

Start with computation on closed expressions (types and terms):

- Transition: $M \longmapsto M'$, one step of execution.
- Termination: M val is canonical/complete.

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Extend to open forms by functionality aka extensionality:

- Types: $a_1{:}A_1, \ldots, a_n{:}A_n \gg A \doteq B$ type $[\Psi]$.
- Terms: $a_1{:}A_1, \ldots, a_n{:}A_n \gg M \doteq N \in A$ $[\Psi]$.

Cubical Type Theory

BCH; CCHM; LH; LB

Bezem, et al.: model of (a version of) HoTT valued in constructive cubical sets.

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Formal cubical type theories are under active development (Coquand, et al.; this work.)

Cubical Programming Language

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Syntax is organized cubically:

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- Lines represent identifications of elements and types.

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- Finite set of dimension variables $x, y, z, \ldots$.

Substitutions $\psi : \Psi' \to \Psi$ send $x \in \Psi$ to $\psi(x) = 0/1/x' \in \Psi'$.

Cubical Programming Language

Substitutions define the aspects of a cube E:

- Faces: $E\langle 0/x\rangle$, $E\langle 1/x\rangle$.
- Diagonals: $E\langle x', x'/x, y\rangle$.
- Symmetries: $E\langle y, x/x, y\rangle$.
- Degeneracy: silent/implicit.

$$\begin{tikzcd}
E\langle 0/x\rangle\langle 0/y\rangle \arrow[r, "E\langle 0/y\rangle"] \arrow[d, "E\langle 0/x\rangle"'] \arrow[dr, phantom, "E"] & E\langle 1/x\rangle\langle 0/y\rangle \arrow[d, "E\langle 1/x\rangle"] \\
E\langle 0/x\rangle\langle 1/y\rangle \arrow[r, "E\langle 1/y\rangle"'] & E\langle 1/x\rangle\langle 1/y\rangle
\end{tikzcd}$$

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Unconventional functional programming constructs:

- Circle: $\mathbb{S}^1, \mathsf{base}, \mathsf{loop}_x, \mathbb{S}^1\text{-}\mathsf{elim}_{a.A}(M; M_{\mathsf{b}}, x.M_{\mathsf{l}})$.
- Negation: not_x, a type line, and glueing, $\mathsf{notel}_x(M)$.
- Kan operations: coe, hcom.

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The Kan operations are computational content of the Kan condition (cf, LB14, CCHM16).

Kan Operations

Coercion along a type line: $\mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M)$.

- Heterogeneous along line $x.A$.
- Evaluates A to effect coercion from $A\langle r/x\rangle$ to $A\langle r'/x\rangle$.

Composition: $\mathsf{hcom}_{A}^{\overrightarrow{r_i}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^{\varepsilon}})$.

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- The tubes $\overrightarrow{y.N_i^{\varepsilon}}$ with extent $\overrightarrow{r_i}$ in dimension $\overrightarrow{y_i}$.

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- The tubes $\overrightarrow{y.N_i^{\varepsilon}}$ with extent $\overrightarrow{r_i}$ in dimension $\overrightarrow{y_i}$.
- Evaluates A to define composite, which may or may not be the hcom itself.

Two-Dimensional Compositions

$$
\begin{tikzcd}
{} \arrow[r, "x"] \arrow[d, "y"'] & {} \\
{} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge]
\cdot \arrow[r, "M"] \arrow[r, dashed, bend right, "\mathsf{hcom}^{x}_{A}(0 \rightsquigarrow 0, M; y.N^0, y.N^1)"'] \arrow[d, "N^0"'] & \cdot \arrow[d, "N^1"] \\
N^0\langle 1/y \rangle & N^1\langle 1/y \rangle
\end{tikzcd}
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N^0\langle 1/y \rangle \arrow[r, dashed, "\mathsf{hcom}^x_A(0 \rightsquigarrow {\color{red}1}, M; y.N^0, y.N^1)"'] & N^1\langle 1/y \rangle
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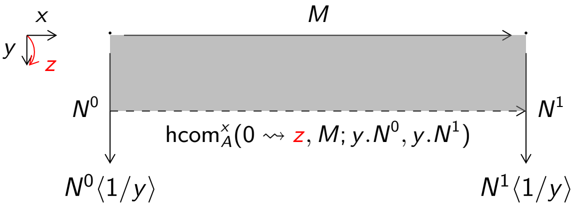
Two-Dimensional Compositions



Cubical Meaning Explanation

Explanation proceeds in stages:

- Define the canonical types and their elements at each dimension Ψ.
- Define pre-types to be cubical, ie with coherent aspects.
- Define types to be Kan pre-types.

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The main criteria for a higher type system:

- All aspects of a type or element must be types or elements.
- Taking aspects must commute with evaluation.
- Equal types must have the same element equality.
- Equal types must be equally Kan.

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- $A \sim^{\Psi} B$ iff $A \longmapsto^* A_0$ and $B \longmapsto^* B_0$ and $A_0 \approx^{\Psi} B_0$.
- $M \sim^{\Psi}_{A} N$ iff $M \longmapsto^* M_0$, $N \longmapsto^* N_0$, $A \longmapsto^* A_0$, and $M_0 \approx^{\Psi}_{A_0} N_0$.

Pre-Types: Coherent Aspects

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- Let $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$.
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- Require:

$$
\begin{tikzcd}
A \arrow[r, Rightarrow, "\psi_1"] \arrow[d, Rightarrow, "\psi_1\psi_2"'] & A_1 \arrow[d, Rightarrow, "\psi_2"] \\
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Similarly for exact equality of types and of elements: substitute-then-evaluate is functorial.

Pre-Types and Types

A pretype $[\Psi]$ is cubical: its values have coherent aspects:

- If $\psi : \Psi' \to \Psi$ and $M \approx^{\Psi'}_{A\psi} N$, then $M \doteq N \in A\psi\ [\Psi']$.

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A type is a Kan pre-type:

- Supports coercion and composition.
- Certain equational requirements are met.

Kan Conditions for Coercion

For any $\psi : (\Psi', x) \to \Psi$, if

$$M \in A\psi\langle r/x \rangle \ [\Psi'],$$

then

$$\mathsf{coe}_{x.A\psi}^{r \rightsquigarrow r'}(M) \in A\psi\langle r'/x \rangle \ [\Psi'].$$

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Constraints limit applicable substitutions; conditions can be vacuous.

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The following "rules" are theorems, not definitions.

Boolean Dynamics

$$\frac{}{\mathsf{bool}\ \mathsf{val}} \qquad \frac{\overrightarrow{r_i} = x_1, \ldots, x_{i-1}, \varepsilon, r_{i+1}, \ldots, r_n}{\mathsf{hcom}^{\overrightarrow{r_i}}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N^\varepsilon_i}) \longmapsto N^\varepsilon_i \langle r'/y \rangle}$$

$$\frac{r = r'}{\mathsf{hcom}^{x_1,\ldots,x_n}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N^\varepsilon_i}) \longmapsto M} \qquad \frac{}{\mathsf{true}\ \mathsf{val}} \qquad \frac{}{\mathsf{false}\ \mathsf{val}}$$

$$\frac{r \neq r'}{\mathsf{hcom}^{x_1,\ldots,x_n}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N^\varepsilon_i})\ \mathsf{val}}$$

Boolean Dynamics

$$\frac{M \longmapsto M'}{\mathsf{if}_{a.A}(M; T, F) \longmapsto \mathsf{if}_{a.A}(M'; T, F)} \qquad \frac{}{\mathsf{if}_{a.A}(\mathsf{true}; T, F) \longmapsto T}$$

$$\frac{}{\mathsf{if}_{a.A}(\mathsf{false}; T, F) \longmapsto F}$$

$$\frac{r \neq r' \quad H = \mathsf{hcom}^{x_1,\ldots,x_n}_{\mathsf{bool}}(r \rightsquigarrow z, M; \overrightarrow{y.N_i^\varepsilon})}{\begin{array}{c}\mathsf{if}_{a.A}(\mathsf{hcom}^{x_1,\ldots,x_n}_{\mathsf{bool}}(r \rightsquigarrow r', M; \overrightarrow{y.N_i^\varepsilon}); T, F) \\ \longmapsto \\ \mathsf{com}^{x_1,\ldots,x_n}_{z.A[H/a]}(r \rightsquigarrow r', \mathsf{if}_{a.A}(M; T, F); \overrightarrow{y.\mathsf{if}_{a.A}(N_i^\varepsilon; T, F)})\end{array}}$$

$$\frac{}{\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathsf{bool}}(M) \longmapsto M}$$

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Generally, values of positive type include compositions in higher dimensions.

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$$
\begin{tikzcd}[row sep=small, column sep=small]
{} \arrow[r, "x"] \arrow[d, "y"'] & {} \\
{}
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "\mathsf{bool}"] \arrow[d, mapsto, "\mathsf{not}(-)"'] & \cdot \arrow[d, mapsto, "\mathsf{id}"] \\
\mathsf{bool} \arrow[r, dashed, "\mathsf{not}_x"'] & \mathsf{bool}
\end{tikzcd}
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$x \rightarrow$, $y \downarrow$

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Cf. CCHM gluing of equivalences to a line of types.

Computational Higher Type Theory

Canonicity Theorem: closed points of bool evaluate to true or false.

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Huber has since proved canonicity for the CCHM theory, relative to weak head reduction.

Whither Proof Theory?

Validates expected formal rules where applicable.

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May be seen as abstract cubical extensional realizability interpretation.

- Abstract = open-ended: Church's Law not accepted.
- Extensional = full universal properties for usual type constructors.
- Realizability for certain formal theories, but that's not the point.

Ongoing and Future Work

Full account of univalence.

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- Glueing for composition of types not applicable (diagonals).
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- Line of types $\mathsf{ua}_x(E)$ for each equivalence E.
- Ensures equivalence between identifications and equivalences.

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Implementation in Sterling's RedPRL (`redprl.org`).

- NuPRL-like refinement rules.
- Extended with names for cubical programs.

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