Recitation 3:  
Gödel’s System T  
15-312: Foundations of Programming Languages  
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1 Syntax

We now define and explore a language called **System T**. System T extends E with function types and replaces E’s primitive arithmetic operations with a more general operation on the natural numbers: **primitive recursion**. The syntax of System T is given by the following grammar:

\[
\begin{align*}
\text{Typ} & \quad \tau ::= \text{nat} & \text{number} \\
& \quad \tau_1 \rightarrow \tau_2 & \text{function} \\
\text{Exp} & \quad e ::= x & \text{variable} \\
& \quad z & \text{zero} \\
& \quad s(e) & \text{successor} \\
& \quad \text{rec}\{e_0; y.e_1\}(e) & \text{recursion} \\
& \quad \lambda (x : \tau) e & \text{abstraction} \\
& \quad e_1(e_2) & \text{application}
\end{align*}
\]

Surprisingly, despite the loss of the arithmetic operations, T is capable of expressing every numeric computation in E and much more.

2 Abstraction and Application

Abstraction and application behave much as we would intuitively expect. An abstraction (function) binds a variable of type \(\tau\) in \(e_1\), and an application substitutes an expression \(e_2 : \tau\) for that bound variable. Abstractions are first-class expressions: they have a type and can be passed to and returned from other abstractions. Because of this, System T is a language with higher-order functions.

The statics and dynamics for abstraction and application are given below.

2.1 Statics

\[
\begin{align*}
\Gamma, x : \tau_1 & \vdash e_2 : \tau_2 & \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 & \Gamma \vdash e_2 : \tau_1 \\
\Gamma & \vdash \lambda (x : \tau_1) e_2 : \tau_1 \rightarrow \tau_2 & \Gamma \vdash e_1(e_2) : \tau_2
\end{align*}
\]
2.2 Dynamics

These dynamics rules are for the *eager* form of System T. All arguments are evaluated before being substituted into the body of a function. For a lazy dynamics, the \( e_2 \mapsto e'_2 \) rule would be left out, along with the requirement on the last rule that \( e_2 \) be a value. Note the first rule, which states that functions are values.\(^1\)

\[
\frac{\lambda (x : \tau) e \text{ val}}{
  e_1 \mapsto e'_1 \\
  e_1(e_2) \mapsto e'_1(e_2) \\
  e_1 \text{ val} \quad e_2 \mapsto e'_2 \\
  e_1(e_2) \mapsto e_1(e'_2) \\
  e_2 \text{ val} \\
  (\lambda (x : \tau) e)(e_2) \mapsto [e_2/x]e
}
\]

3 Natural Numbers

In System T, the natural numbers are defined as either zero, or the successor of a natural number. In addition to this definition, we also now have a single operation that works on naturals: recursion. The statics and dynamics of \texttt{nats} is given below, while recursion is discussed in the next section.

3.1 Statics

\[
\frac{\Gamma \vdash z : \text{nat}}{
  \frac{\Gamma \vdash e : \text{nat}}{
    \frac{}{\Gamma \vdash s(e) : \text{nat}}}
}
\]

3.2 Dynamics

For a lazy form of System T, the requirement \( e \text{ val} \) would be removed.

\[
\frac{z \text{ val}}{e \text{ val}} \\
\frac{e \text{ val}}{s(e) \text{ val}}
\]

4 Recursion

Now let’s consider the recursion operation for System T:

\[
\text{rec}\{e_0; y.e_1\}(e)
\]

This operation cases on the value of \( e \) (either \( z \) or \( s(e') \)). If \( e \) is \( z \) then the expression evaluates to \( e_0 \), the base case. If \( e \) is \( s(e') \) for some natural number \( e' \), then it recurs on \( e' \), binding the result of the recursion to \( y \) for use in \( e_1 \).

\(^1\)As they say in 15-150.
4.1 Statics

\[
\Gamma \vdash e : \text{nat} \quad \Gamma \vdash e_0 : \tau \quad \Gamma, y : \tau \vdash e_1 : \tau
\]
\[
\Gamma \vdash \text{rec}\{e_0; y.e_1\}(e) : \tau
\]

4.2 Dynamics

\[
e \mapsto e'
\]
\[
\text{rec}\{e_0; y.e_1\}(e) \mapsto \text{rec}\{e_0; y.e_1\}(e')
\]
\[
\text{rec}\{e_0; y.e_1\}(z) \mapsto e_0
\]
\[
s(e) \text{ val}
\]
\[
\text{rec}\{e_0; y.e_1\}(s(e)) \mapsto [\text{rec}\{e_0; y.e_1\}(e)/y]e_1
\]

4.3 Examples for Recursion

4.3.1 Doubling

Understanding the recursor can be tricky, so let’s go through an example. We’ll write a function that doubles a number using the recursor. To do this, let’s consider how we would implement doubling in Standard ML given the following datatype for natural numbers:

```ml
datatype nat = z | s of nat
```

We can double a number by doubling its predecessor and then taking the successor of that number twice:

```ml
fun double z = z
| double (s x) = s (s (double x))
```

Let’s rewrite this so that it matches the format of the recursor, with the predecessor of \( e \) bound to \( x \) and the result of the recursion bound to \( y \):

```ml
fun double e =
  case e of
  z => z
  | s x => let val y = double x in s (s y) end
```

This makes it easier to now implement this using the recursor:

\[
\lambda(e : \text{nat}) \text{rec}\{z; y.s(s(y))\}(e)
\]

Let’s see how our dynamics rule captures the computation exactly the way we want:

\[
\text{double } s(z) \mapsto \text{rec}\{z; y.s(s(y))\}(s(z)) \quad \text{Application}
\]
\[
\mapsto [\text{rec}\{z; y.s(s(y))\}(z)/y]s(s(y)) \quad \text{Since } s(z) \text{ val}
\]
\[
= s(s[\text{rec}\{z; y.s(s(y))\}(z)]) \quad \text{Substitution}
\]
\[
\mapsto s(s(z)) \quad \text{By evaluation rule of } s()
\]

As an exercise to make sure you understand the recursor, try to implement addition in the same manner.
5 Expressiveness of System T

5.1 Termination

Every well-typed expression $e$ in System T terminates, i.e. if $\cdot \vdash e : \tau$, then $e \rightarrow^* v$ where $v \text{ val}$. However, this seemingly harmless property makes our language not Turing-complete. The full proof is in PFPL; here we present the general idea.

We first define some relevant concepts:

In eager dynamics\(^2\), we say a function $f : \mathbb{N} \to \mathbb{N}$ is \textbf{definable} in T, iff there exists an expression $e : \text{nat} \rightarrow \text{nat}$ in T such that for all $n \in \mathbb{N}$, $e(n) \rightarrow^* f(n)$.\(^3\)

Two assumptions in this proof:

1. We can encode every expression in T as natural numbers. (One example: Gödel’s numbering) Fix some encoding, let $\lceil e \rceil$ be the encoded numeral of expression $e$ in T.
2. We curry products in the domain of mathematical functions. So a function $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ has its correspondence in T as $e_f : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$. Consider a function $\text{eval} : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined as $\text{eval}(\lceil e \rceil, m) = n$ iff $e(m) \rightarrow^* \pi$.

Claim: $\text{eval}$ is not definable in System T.

The proof is by the classic diagonalization argument. Assume for the sake of contradiction that it is definable, i.e. there exists $e_{\text{eval}} : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ such that $e_{\text{eval}}(\lceil e \rceil)(m) \rightarrow^* \text{eval}(\lceil e \rceil, m)$

Then consider an expression $e_\delta = \lambda(x : \text{nat}) s(e_{\text{eval}}(x)(x))$.

By termination property of T, we know $e_\delta(\lceil e_\delta \rceil) \rightarrow^* v$ for some natural number $v$.

Then by definition $e_\delta(\lceil e_\delta \rceil) \rightarrow^* s(e_{\text{eval}}(\lceil e_\delta \rceil)(\lceil e_\delta \rceil)) \rightarrow^* s(e_\delta(\lceil e_\delta \rceil)) \rightarrow^* v + 1$.

Contradiction! $\text{eval}$ is not definable in System T. However, $\text{eval}$ is clearly computable.\(^4\)

Notice the argument above applies to System T, as well as any other language that is total.

5.2 Extra Reading: Ackermann

System T is notable for its only explicit recursion operator being primitive recursion. However, its higher-order functions means that it is capable of computing non-primitive-recursive functions, like the well-known Ackermann function $A(m, n)$, defined as follows:

$$A(0, n) = n + 1$$
$$A(m + 1, 0) = A(m, 1)$$
$$A(m + 1, n + 1) = A(m, A(m + 1, n))$$

Ackermann is not primitive recursive since with a given recursive call, it is possible for $n$ to increase. This is incompatible with the recursor construct, which requires its argument be deconstructed at every step. However, consider currying $A(m, n)$:

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\(^2\)this definition is not so great under lazy dynamics, why and how do we fix that?

\(^3\)\(\pi\) is a shorthand for numeral values in System T. So $\overline{2} = s(s(z))$

\(^4\)If you are not convinced, you will get to implement an interpreter for System T in Assignment 2.
\[ A(0)(n) = s(n) \]
\[ A(s(m))(0) = A(m)(1) \]
\[ A(s(m))(s(n)) = A(m)(A(s(m))(n)) \]

If we treat \( A(s(m)) \) as the function in question, we observe that whenever it is called recursively, its argument \( n \) decreases in value. We arrive at an insight: \( A(s(m)) \) is a primitive recursive function in as of itself, and we should try writing it as a recursor.

However, there is one hiccup in computing \( A(s(m)) \): the intermediate value we are collecting is not a number, but a function which applies \( A(m) \) every step. Fortunately, System \( T \) allows us to write this. Consider the definitions:

\[
\begin{align*}
\text{id} & : \text{nat} \to \text{nat} \\
\text{id} & \triangleq \lambda (x : \text{nat}) x \\
\text{comp} & : (\text{nat} \to \text{nat}) \to (\text{nat} \to \text{nat}) \to \text{nat} \to \text{nat} \\
\text{comp} & \triangleq \lambda (f : \text{nat} \to \text{nat}) \lambda (g : \text{nat} \to \text{nat}) \lambda (x : \text{nat}) f(g(x)) \\
\text{iter} & : (\text{nat} \to \text{nat}) \to \text{nat} \to \text{nat} \to \text{nat} \\
\text{iter} & \triangleq \lambda (f : \text{nat} \to \text{nat}) \lambda (n : \text{nat}) \text{rec}\{\text{id}; y.\text{comp}(f)(y)\}(n)
\end{align*}
\]

What does \( \text{iter} \) do? Given a function \( f \) and a number \( n \), it computes the \( n \)-th iterate of \( f \), \( f^n \). That’s exactly what we need!

Rearranging, we have:

\[
\begin{align*}
A(0)(n) &= s(n) \\
A(s(m))(n) &= \text{iter}(A(m))(n)(A(m)(1))
\end{align*}
\]

Now we can move up one level to express \( A \) as a recursor, and write the Ackermann function in \( T \) (using a \text{succ} function that just takes the successor of a \text{nat}):

\[
\begin{align*}
\text{succ} & : \text{nat} \to \text{nat} \\
\text{succ} & \triangleq \lambda (n : \text{nat}) s(n) \\
\text{ack} & : \text{nat} \to \text{nat} \to \text{nat} \\
\text{ack} & \triangleq \lambda (m : \text{nat}) \text{rec}\{\text{succ}; y.\lambda (n : \text{nat}) \text{iter}(y)(n)(y(s(z)))(m)\}(m)
\end{align*}
\]

This is a constructive proof that despite not being primitive recursive, Ackermann is higher-order primitive recursive. System \( T \) allows us to compute a large set of functions like Ackermann, though all expressions in \( T \) provably terminate (cannot diverge). We showed in the previous section that this means System \( T \) is \textit{not} Turing-complete.