15–399: Recitation - Week 4

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1 Set $\mathcal{B}$

A *set* is a set ;-)

2 Partially ordered set $(\mathcal{B}, \leq)$

A *partially ordered set* is a set with a partial order $\leq$.

- $\leq$ satisfies reflexivity, transitivity, and antisymmetry.

3 Lattice $(\mathcal{B}, \leq, \wedge, \vee, \top, \bot)$

A *lattice* is a partially ordered set with $\wedge, \vee, \top, \bot$.

- If $z \leq x$ and $z \leq y$, then $z \leq x \wedge y$.
  Therefore, to prove $z \leq x \wedge y$, it suffices to prove $z \leq x$ and $z \leq y$.
  Draw a diagram showing the relation between z, $x \wedge y$, x, and y, and it will give you an easy way to memorize this property.
  Let's call this property $\mathit{Rule}_{\wedge}$.

- If $x \leq z$ and $y \leq z$, then $x \vee y \leq z$.
  Therefore, to prove $x \vee y \leq z$, it suffices to prove $x \leq z$ and $y \leq z$.
  Draw a diagram showing the relation between z, $x \vee y$, x, and y, and it will give you an easy way to memorize this property.
  Let's call this property $\mathit{Rule}_{\vee}$.

4 Heyting algebra $(\mathcal{B}, \leq, \wedge, \vee, \top, \bot, \rightarrow)$

A *Heyting algebra* is a lattice with an exponential operator $\rightarrow$.

- $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.
  This is essentially the so-called standard dirty trick.
  Let's call this property $\mathit{Rule}_{\rightarrow}$.

  $y \rightarrow z$ is the least expensive ticket that you have to purchase at y in order to move to z (if this sentence makes any sense to you).
  For example, y and $\bot$ are also valid tickets that lead you to z, but they are too expensive!

$\top$ is an insanely inexpensive ticket, but unfortunately it doesn't contribute to your trip! Thus you have to spend at least as much as $y \to z$.

- A Heyting algebra satisfies the distributivity property:
  - $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
    A proof was given in class (and is analogous to the proof shown below).
  - $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
    Let's prove this equality. Our weapon of choice is $Rule_\wedge$, $Rule_\vee$, and $Rule_\to$.
    We need to prove the following two relations:
    1. $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.
       Try to prove it yourself!
       This is relatively easy, because both $Rule_\wedge$ and $Rule_\vee$ are applicable.
       By $Rule_\wedge$, we need to prove $x \vee (y \wedge z) \leq (x \vee y)$ and $x \vee (y \wedge z) \leq (x \vee z)$.
       Consider the first case. Note that only $Rule_\vee$ is applicable.
       By $Rule_\vee$, we need to prove $x \leq (x \vee y)$ and $(y \wedge z) \leq (x \vee y)$.
       They follow from $x \leq (x \vee y)$ and $(y \wedge z) \leq y \leq (x \vee y)$.
       Now consider the second case. Note that only $Rule_\vee$ is applicable.
       By $Rule_\vee$, we need to prove $x \leq (x \vee z)$ and $(y \wedge z) \leq (x \vee z)$.
       They follow from $x \leq (x \vee z)$ and $(y \wedge z) \leq z \leq (x \vee z)$.
    2. $(x \vee y) \wedge (x \vee z) \leq x \vee (y \wedge z)$.
       Try to prove it yourself!
       This is a bit tough, but there is nothing special in here – all you need to do is to apply $Rule_\to$, $Rule_\wedge$, $Rule_\vee$ when applicable and make progress.
       Note that neither $Rule_\wedge$ nor $Rule_\vee$ is applicable. So the only option left is $Rule_\to$!
       By $Rule_\to$, we need to prove $(x \vee y) \leq (x \vee z) \to x \vee (y \wedge z)$.
       Now $Rule_\vee$ is applicable (it is stupid to apply $Rule_\to$ again).
       By $Rule_\vee$, we need to prove $x \leq (x \vee z) \to x \vee (y \wedge z)$ and $y \leq (x \vee z) \to x \vee (y \wedge z)$.
       In both cases, only $Rule_\to$ is applicable.
       By $Rule_\to$, we need to prove $x \wedge (x \vee z) \leq x \vee (y \wedge z)$ and $y \wedge (x \vee z) \leq x \vee (y \wedge z)$.
       The first case follows from $x \wedge (x \vee z) \leq x \leq x \vee (y \wedge z)$.
       The second case requires further work.
       Note that neither $Rule_\wedge$ nor $Rule_\vee$ is applicable to the second case.
       So, again, the only option left is $Rule_\to$!
       By the rule $Rule_\to$, we need to prove $(x \vee z) \leq y \to x \vee (y \wedge z)$.
       Here you move y to the right side, not $x \vee z$!
       Now $Rule_\vee$ is appliable (it is stupid to apply $Rule_\to$ again).
       By $Rule_\vee$, we need to prove $x \leq y \to x \vee (y \wedge z)$ and $z \leq y \to x \vee (y \wedge z)$.
       By $Rule_\to$, we need to prove $x \wedge y \leq x \vee (y \wedge z)$ and $z \wedge y \leq x \vee (y \wedge z)$.
       They follow from $x \wedge y \leq x \leq x \vee (y \wedge z)$ and $z \wedge y \leq x \vee (z \wedge y)$.
- A Heyting algebra satisfies $x \wedge \neg x = \bot$ where $\neg x$ is defined as $x \to \bot$.
- A Heyting algebra *does not* satisfy $x \vee \neg x = \top$ (like constructive logic does not prove $A \vee \neg A$).

5 Boolean algebra $(\mathcal{B}, \leq, \wedge, \vee, \top, \bot, \rightarrow)$

A *Boolean algebra* is a special case of Heyting algebra where the exponential operator $\rightarrow$ satisfies $\neg\neg x = x$ (called the stability condition).

– A Heyting algebra *does* satisfy $x \vee \neg x = \top$ (like classical logic proves $A \vee \neg A$).