15-399 Supplementary Notes: An Algebraic View of Logic

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Heyting and Boolean Algebras

The standard semantics for classical logic states that each proposition stands for, or *denotes*, a truth value, either $\top$ or $\bot$, according to the usual *truth tables* for each logical connective. According to this semantics, the categorical judgement $P\ \mathit{true}$ means that P denotes $\top$, and the hypothetical judgement $P_1\ \mathit{true}, \ldots, P_n\ \mathit{true} \vdash P\ \mathit{true}$ means that P denotes $\top$ whenever each P_i denotes $\top$.

The hypothetical judgement induces an ordering relation among propositions defined by taking $P \leq Q$ to hold iff $P\ \mathit{true} \vdash Q\ \mathit{true}$. (There is no loss of generality in restricting attention to a binary relation, because $P_1\ \mathit{true}, \ldots, P_n\ \mathit{true} \vdash P\ \mathit{true}$ holds iff $P_1 \wedge \cdots \wedge P_n\ \mathit{true} \vdash P\ \mathit{true}$.[1]) This relation is a *pre-order*.[2] This extends to a *partial order*[3] on equivalence classes of propositions under mutual entailment. Specifically, define

$$[P] = \{\, Q \mid P\ \mathit{true} \vdash Q\ \mathit{true} \text{ and } Q\ \mathit{true} \vdash P\ \mathit{true} \,\},$$

to be the equivalence class of P under mutual entailment, and define $[P] \leq [Q]$ to hold exactly when $P \leq Q$. (This is easily seen to be well-defined.)

Since equivalent propositions have the same truth value, this suggests that we may re-cast the truth-table interpretation in terms of partially ordered sets. Specifically, we define the relation $\leq$ on the set $\{\top, \bot\}$ of truth values to be the least reflexive, transitive, and anti-symmetric binary relation such that $\bot \leq \top$. Then $\bot$ is the least element of the ordering, and $\top$ is the greatest element. With respect to this ordering, the conjunction of two truth values is their greatest lower bound (the smaller of the two), and their disjunction is the least upper bound (the larger of the two).[4] Negation inverts the ordering in the sense that $a \leq b$ iff $\neg b \leq \neg a$.

[1] When $n = 0$, this means $\top\ \mathit{true} \vdash P\ \mathit{true}$.

[2] That is, a reflexive and transitive binary relation.

[3] That is, an anti-symmetric pre-order.

[4] Recall that the *greatest lower bound*, or *meet*, of a and b is the element $a \wedge b$ defined by the conditions $a \wedge b \leq a$, $a \wedge b \leq b$, and if $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$. The *least upper bound*, or *join*, of two elements is defined similarly.

Making use of this order structure on truth values, we may re-interpret the hypothetical judgement P_1 *true*, $\ldots$, P_n *true* $\vdash P$ *true* to mean that $a_1 \wedge \cdots \wedge a_n \leq a$, where P_i denotes a_i and P denotes a. When $n = 0$, this means that $\top \leq a$, where P denotes a. Since $\top$ is the greatest element of the pre-order, this is equivalent to saying that $a = \top$.

This interpretation is an instance of the general concept of a *Boolean algebra*. A Boolean algebra is a partially ordered set $\mathcal{B} = (B, \leq)$ satisfying the following requirements:

1. There is a (unique) *least* element, $\perp \in B$, a (unique) *greatest* element, $\top \in B$.

2. Each pair $a, b \in B$ of elements has a (unique) greatest lower bound, $a \wedge b \in B$, and a (unique) least upper bound, $a \vee b \in B$.

3. For every $a, b \in B$, there exists a (unique) element $a \to b \in B$ (sometimes written b^a and called the *exponential*) such that

$$c \leq a \to b \textit{ iff } c \wedge a \leq b.$$

4. For all $a \in B$, $\neg\neg a \leq a$. (Here we define $\neg a$ to be the element $a \to \perp$ provided by the first and third conditions.)

The first two requirements state that $\mathcal{B}$ is a *lattice*; the third states that it is *Cartesian closed*; the fourth states that it is *stable*. When only the first three requirements are met, the structure forms a *Heyting algebra*. A Heyting algebra is therefore a *Cartesian-closed pre-order*, and a Boolean algebra is a *stable* Heyting algebra. There are Heyting algebras that are not Boolean; the constructive Lindenbaum algebra (discussed below) is a simple example.

Given any Boolean algebra $\mathcal{B}$, the *denotation*, $P^{\mathcal{B}}$, of a proposition P in $\mathcal{B}$ is defined by the following equations:

$$\begin{array}{rcl}
\perp^{\mathcal{B}} & = & \perp \\
\top^{\mathcal{B}} & = & \top \\
(P \wedge Q)^{\mathcal{B}} & = & P^{\mathcal{B}} \wedge Q^{\mathcal{B}} \\
(P \vee Q)^{\mathcal{B}} & = & P^{\mathcal{B}} \vee Q^{\mathcal{B}} \\
(P \supset Q)^{\mathcal{B}} & = & P^{\mathcal{B}} \to Q^{\mathcal{B}} \\
(\neg P)^{\mathcal{B}} & = & P^{\mathcal{B}} \to \perp
\end{array}$$

Note that the symbols on the left are logical connectives, whereas on the right they are part of the structure of $\mathcal{B}$.

Boolean algebras are defined so as to capture the “essence” of classical logic. More precisely, the the relationship between Boolean algebras and classical logic is established by the following theorem.

Theorem 0.1 *P_1 true, $\ldots$, P_n true $\vdash$ P true in classical propositional logic iff $P_1^{\mathcal{B}} \wedge \cdots \wedge P_n^{\mathcal{B}} \leq P^{\mathcal{B}}$ for every Boolean algebra $\mathcal{B} = (B, \leq)$. In particular, P true holds in classical logic iff $\top \leq P^{\mathcal{B}}$ holds in every Boolean algebra $\mathcal{B}$.*

Proof (sketch): In the forward direction we proceed by induction on the derivation of the hypothetical judgement. This is called the *soundness* of the classical logic for the class of Boolean algebras. In the reverse direction we construct a Boolean algebra, called the *Lindenbaum algebra*, $\mathcal{L}$, whose elements are equivalence classes, $[P]$, of propositions P, partially ordered as above. The least element is $[\bot]$, the greatest is $[\top]$, $[P] \wedge [Q] = [P \wedge Q]$, $[P] \vee [Q] = [P \vee Q]$, and $[P] \to [Q] = [P \supset Q]$. These operations are all well-defined, and satisfy the requirements of a Boolean algebra. The Lindenbaum algebra $\mathcal{L}$ is constructed so that $P^{\mathcal{L}} = [P]$, and such that $P_1^{\mathcal{L}} \wedge \cdots \wedge P_n^{\mathcal{L}} \leq P^{\mathcal{L}}$ iff P_1 *true*$, \ldots, P_n$ *true* $\vdash P$ *true*. This is called the *completeness* of classical logic for the class of Boolean algebras. □

A similar story can be told for constructive propositional logic, except for Heyting algebras instead of Boolean algebras.

Theorem 0.2 *P_1 true$, \ldots, P_n$ true $\vdash P$ true in constructive propositional logic iff for every Heyting algebra $\mathcal{H} = (H, \leq)$, $P_1^{\mathcal{H}} \wedge \cdots \wedge P_n^{\mathcal{H}} \leq P^{\mathcal{H}}$. In particular, P true holds in constructive logic iff $\top \leq P^{\mathcal{H}}$ in every Heyting algebra $\mathcal{H}$.*

This theorem is proved in essentially the same manner as the preceding one. The constructive Lindenbaum algebra is a Heyting algebra that is not stable, precisely because $\neg\neg P$ is not equivalent to P in constructive logic.

The utility of the completeness theorems is that we may show that a particular hypothetical judgement is *not* provable in classical or constructive logic by finding a *counter-model*, a Boolean or Heyting algebra, respectively, in which the hypotheses are all true, but the consequent is not. Since every Boolean algebra is a Heyting algebra, one consequence of the completeness theorem for constructive logic is that a proposition P is constructively provable only if it is classically true. Put the other way around, if a proposition is false classically, then it cannot be provable constructively. For example, we cannot prove $\neg(P \vee \neg P)$ is constructive logic, because this proposition denotes $\bot$ in the truth-table interpretation.

Exercises

1. Show that the meet and join of any two elements of a Boolean algebra is uniquely determined.

2. Show that the exponential of any two elements of a Boolean algebra is uniquely determined. That is, if $a \to' b$ also satisfies the requirements of the third axiom, then $a \to' b = a \to b$.

3. Give a precise definition of the Lindenbaum algebra for constructive logic. What logical equivalences do you need to prove for the Lindenbaum algebra to be a Heyting algebra?