Proof Systems and Proof Complexity

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Automated Reasoning and Satisfiability
September 24, 2025

Certificates

What makes a problem hard?

Certificate angle: can one efficiently check an alleged solution?

Consider chess: does white begin and win?

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Is searching for the solution harder than verifying a given solution?

	4		3					
						7	9	
			6					
			1	4		5		
9							1	
2								6
				7	2			
	5					8		
				9				

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Intuition: yes!

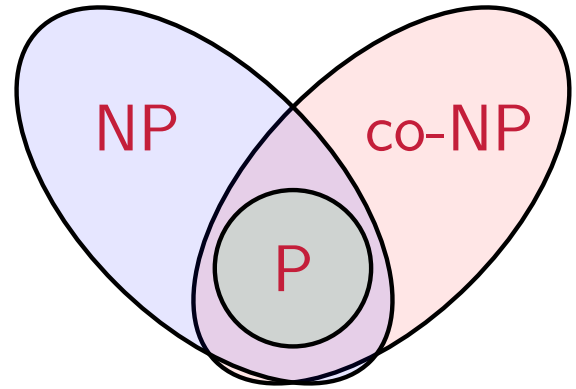
However, many problems for which we can efficiently check a solution turn out to be easy in practice.

1	4	7	3	8	9	2	6	5
5	8	6	2	1	4	7	9	3
3	9	2	6	5	7	1	8	4
8	7	3	1	4	6	5	2	9
9	6	4	7	2	5	3	1	8
2	1	5	9	3	8	4	7	6
6	3	8	5	7	2	9	4	1
7	5	9	4	6	1	8	3	2
4	2	1	8	9	3	6	5	7

Certificates and Complexity

Complexity classes of decision problems:

P : efficiently computable answers.
NP : efficiently checkable yes-answers.
co-NP : efficiently checkable no-answers.



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Solving the $\mathsf{P} \stackrel{?}{=} \mathsf{NP}$ question is worth \$1,000,000 [Clay MI '00].

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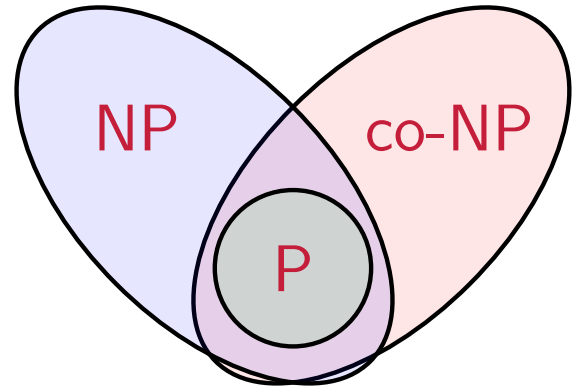

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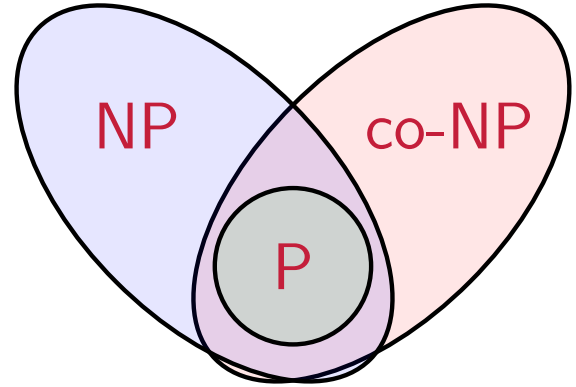

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What about co-NP?

How to find short proofs for interesting problems efficiently?

Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Conclusions

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- Certifying unsatisfiability is not so easy:
  - If a formula has n variables, there are 2^n possible assignments.

  ↪ Checking whether every assignment falsifies the formula is costly.

  - More compact certificates of unsatisfiability are desirable.

    ↪ Proofs

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- Example: Resolution proofs
  - A resolution proof is a sequence $C_1, \ldots, C_m$ of clauses.
  - Every clause is either contained in the formula or derived from two earlier clauses via the resolution rule:

$$\frac{C \vee x \qquad \overline{x} \vee D}{C \vee D}$$

  - C_m is the empty clause (containing no literals), denoted by $\bot$.
  - There exists a resolution proof for every unsatisfiable formula.

Resolution Proofs

- Example: $\Gamma = (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{z}) \wedge (x \vee \overline{y}) \wedge (\overline{u} \vee y) \wedge (u)$
- Resolution proof:
  $(\overline{x} \vee \overline{y} \vee z), (\overline{z}), (\overline{x} \vee \overline{y}), (x \vee \overline{y}), (\overline{y}), (\overline{u} \vee y), (\overline{u}), (u), \bot$

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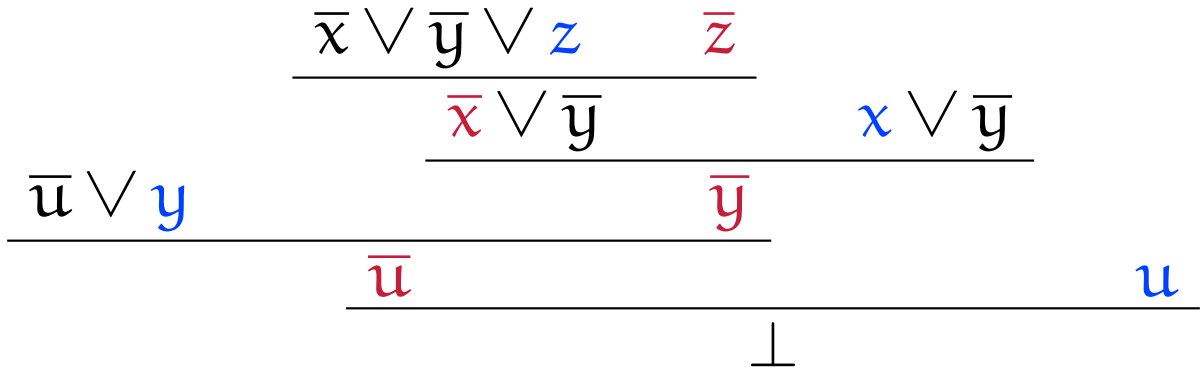

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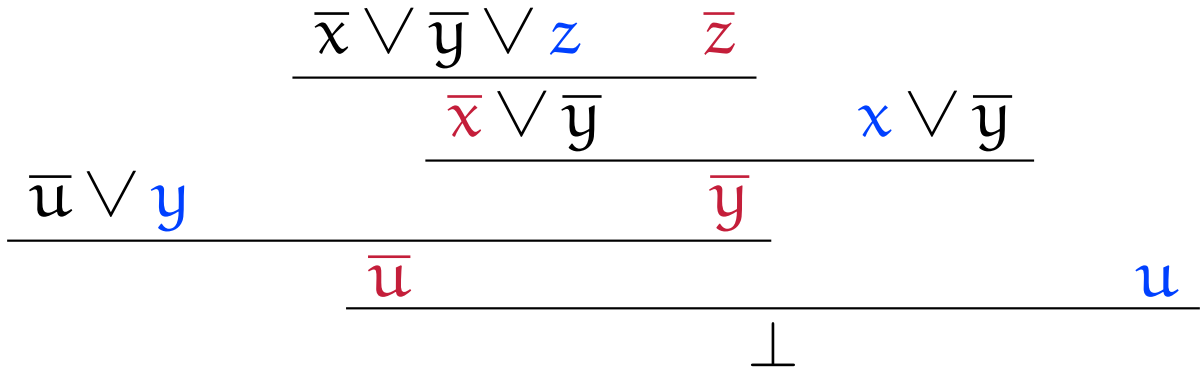
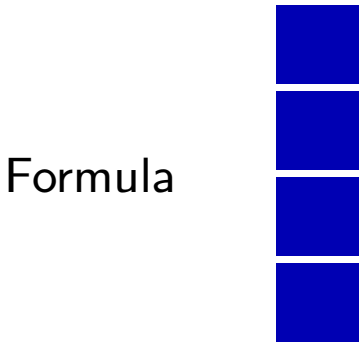


- Drawbacks of resolution:
  - For many seemingly simple formulas, there are only resolution proofs of exponential size.
  - State-of-the-art solving techniques are not succinctly expressible.

Clausal Proofs

Reduce the size of the proof by only storing added clauses



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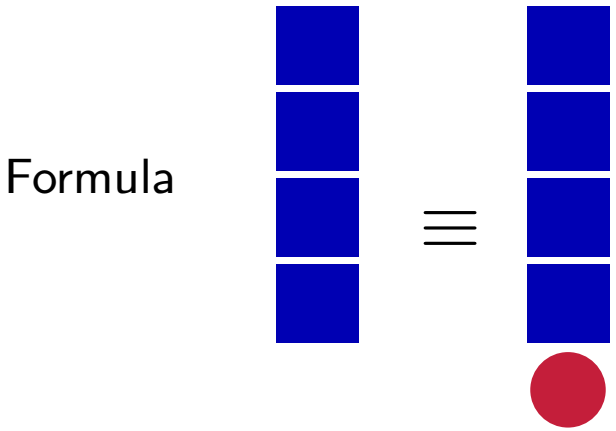

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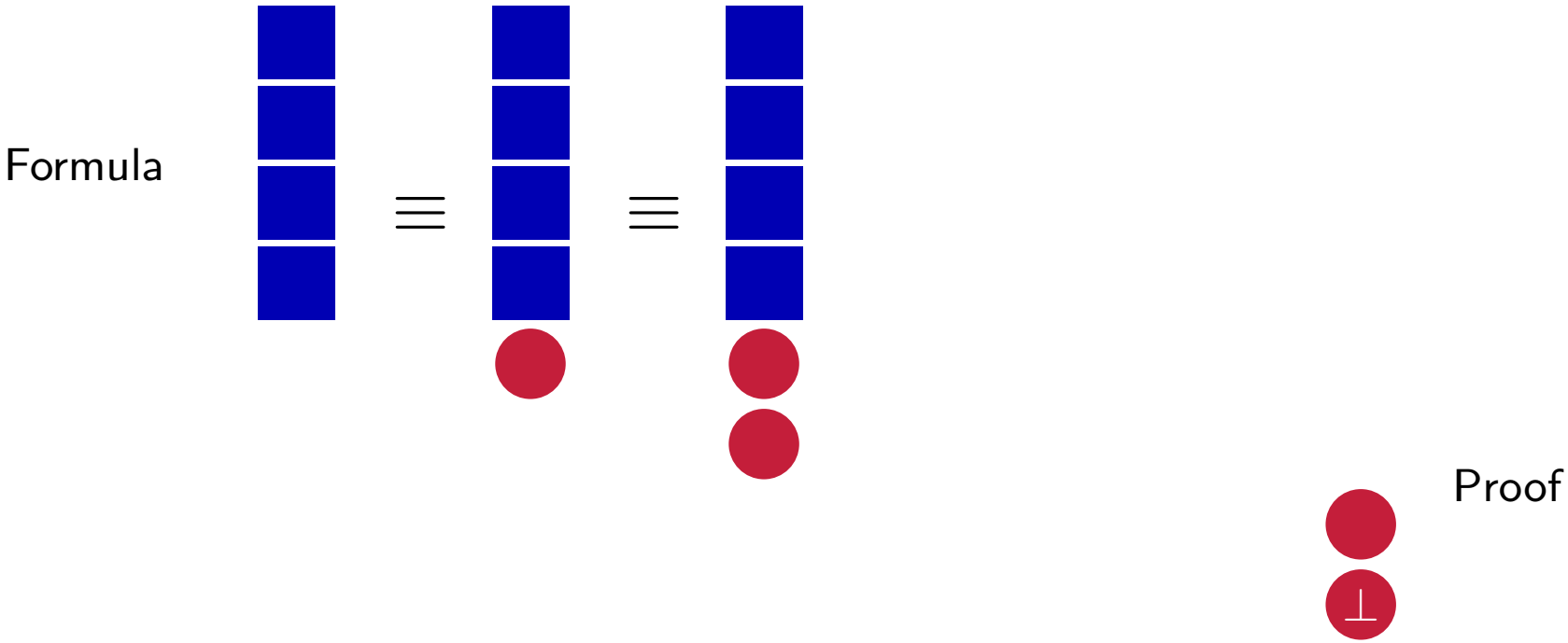

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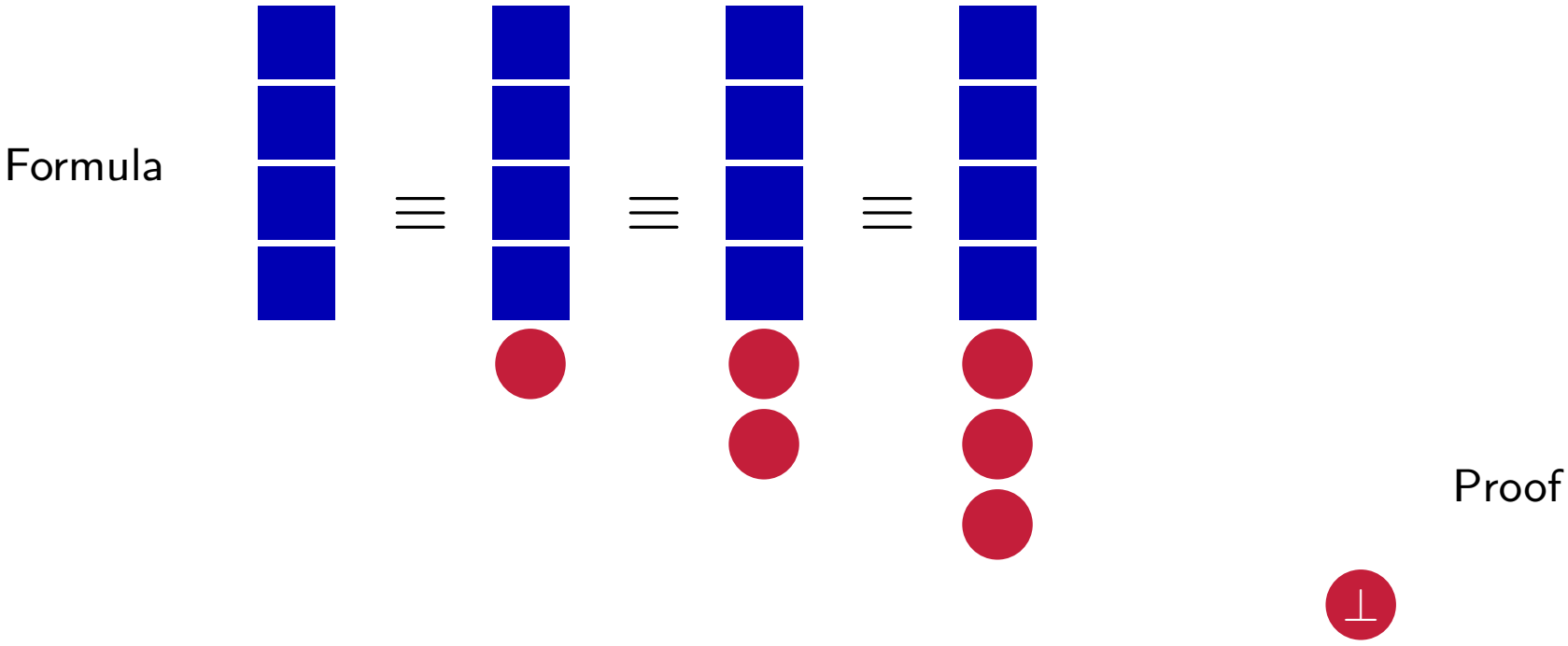

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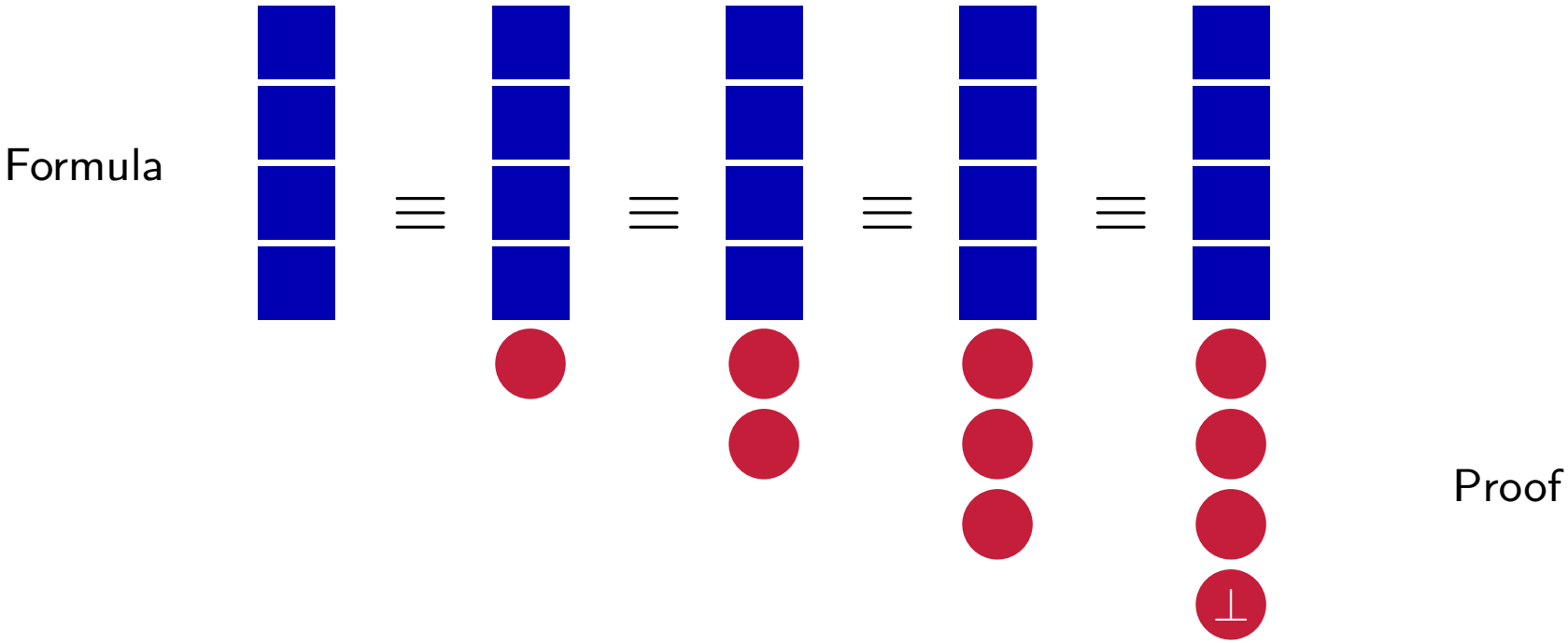

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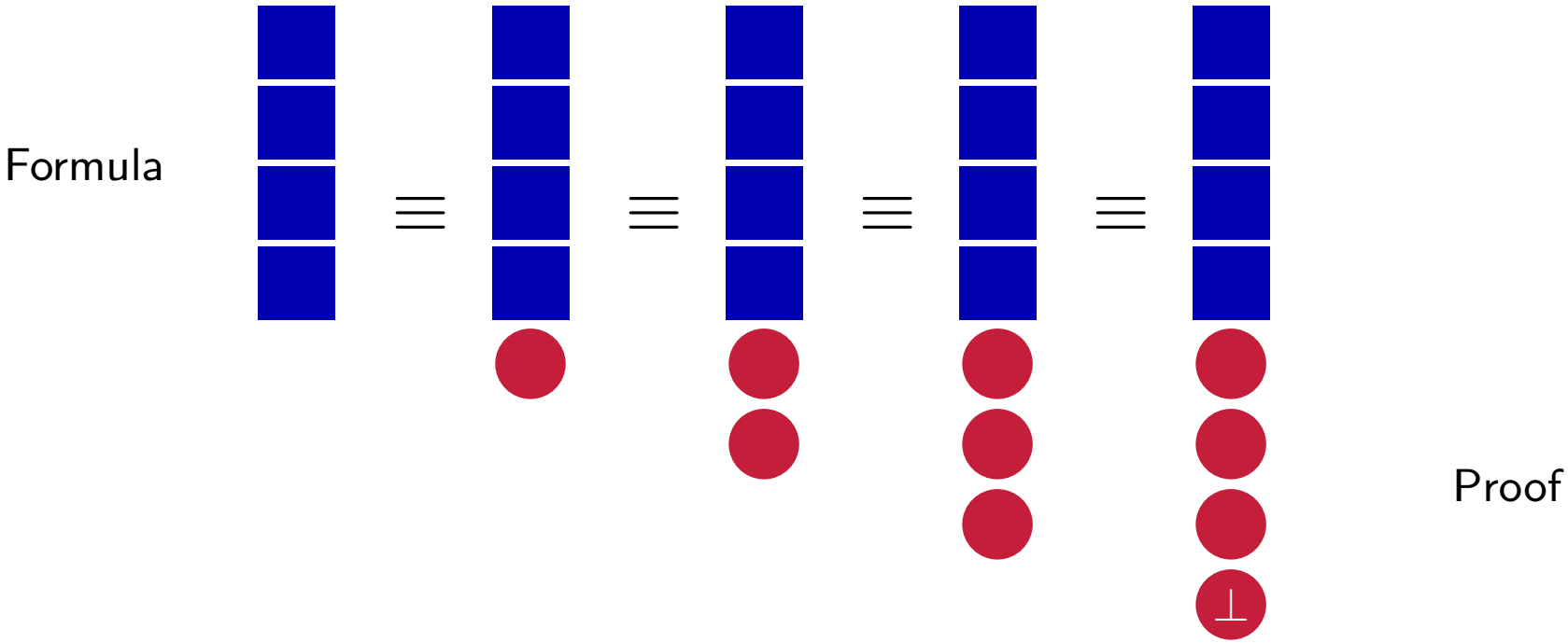

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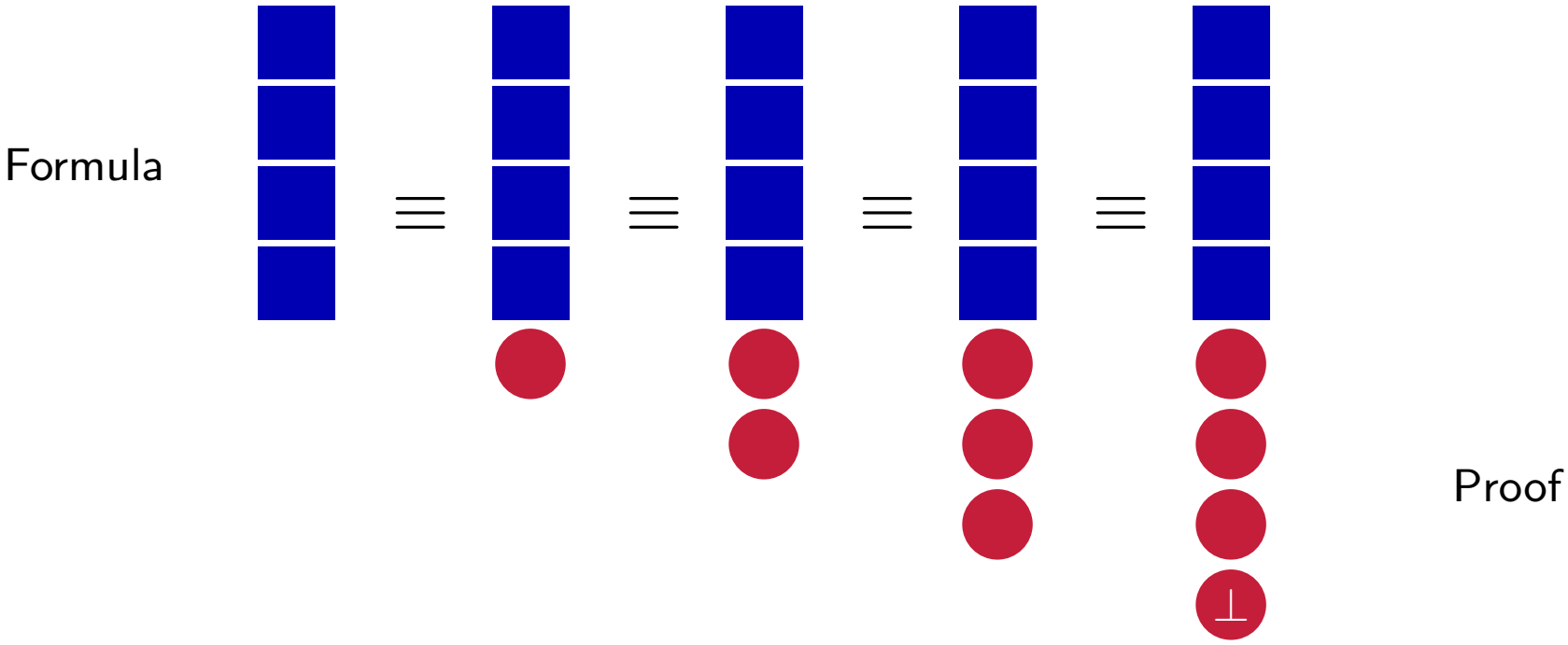

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- Checking redundancy should be efficient.

➥ Idea: Only add clauses that fulfill an efficiently checkable redundancy criterion.

Reverse Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let Γ be a formula. A clause C is implied by Γ via UP (denoted by $\Gamma \vdash_1 C$) if UP on $\Gamma \wedge \neg C$ results in a conflict.

Example

$$\Gamma = (a \vee b \vee \overline{c}) \wedge (\overline{a} \vee \overline{b} \vee c) \wedge (b \vee c \vee \overline{d}) \wedge (\overline{b} \vee \overline{c} \vee d) \wedge (a \vee c \vee d) \wedge (\overline{a} \vee \overline{c} \vee \overline{d}) \wedge (\overline{a} \vee b \vee d) \wedge (a \vee \overline{b} \vee \overline{d})$$

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Proofs of Unsatisfiability

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Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \vee x \qquad \overline{x} \vee D}{C \vee D}\ (\text{RES}) \qquad \frac{A \qquad A \rightarrow B}{B}\ (\text{MP})$$

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➥ Inference rules reason about the presence of facts.

- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

  ➥ This leads to interference-based proof systems.

Early work on reasoning beyond resolution

The early SAT decision procedures used the Pure Literal rule [Davis and Putnam 1960; Davis, Logemann and Loveland 1962]:

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Extended Resolution (ER) [Tseitin 1966]

- Combines resolution with the Extension rule:

$$\frac{x \notin \Gamma \qquad \overline{x} \notin \Gamma}{(x \vee \overline{a} \vee \overline{b}) \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee b)}\ \text{(ER)}$$

- Equivalently, adds the definition $x := \mathrm{AND}(a, b)$
- Can be considered the first interference-based proof system
- Is very powerful: No known lower bounds

Short Proofs of Pigeon Hole Formulas [Cook 1967]

Can $n+1$ pigeons be in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \le p \le n+1} (x_{1,p} \vee \cdots \vee x_{n,p}) \wedge \bigwedge_{1 \le h \le n,\, 1 \le p < q \le n+1} (\overline{x}_{h,p} \vee \overline{x}_{h,q})$$

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However, these proofs require introducing new variables:

- Hard to find such proofs automatically
- Existing ER approaches produce exponentially large proofs
- How to get rid of this hurdle? First approach: blocked clauses...

Blocked Clauses [Kullmann 1999]

Definition (Blocked Clause)

A clause $(C \vee x)$ is a blocked on x w.r.t. a CNF formula Γ if for every clause $(D \vee \overline{x}) \in \Gamma$, resolvent $C \vee D$ is a tautology.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{c}) \wedge (\overline{a} \vee c)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{c}$.
Third clause is blocked by c

Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

Blocked Clause Addition and Blocked Clause Elimination

The Blocked Clause proof system (BC) combines the resolution rule with the addition of blocked clauses.

- BC generalizes ER [Kullmann 1999]
- Recall

$$\frac{x \notin \Gamma \qquad \overline{x} \notin \Gamma}{(x \vee \overline{a} \vee \overline{b}) \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee b)} \text{ (ER)}$$

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Blocked clause elimination used in preprocessing and inprocessing

- Simulates many circuit optimization techniques
- Removes redundant Pythagorean Triples

DRAT: An Interference-Based Proof System

- DRAT is a popular interference-based proof system
- DRAT allows adding RATs (defined below) to a formula.
  - It can be efficiently checked if a clause is a RAT.
  - RATs are not necessarily implied by the formula.
  - But RATs are redundant: their addition preserves satisfiability.
- DRAT also allows clause deletion
  - Initially introduced to check proofs more efficiently
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Definition (Resolution Asymmetric Tautology)

A clause $(C \vee x)$ is a resolution asymmetric tautology (RAT) on x w.r.t. a CNF formula Γ if for every clause $(D \vee \overline{x}) \in \Gamma$, $C \vee D$ is implied by Γ via unit-propagation, i.e., $\Gamma \vdash_1 C \vee D$.

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Redundant Clauses

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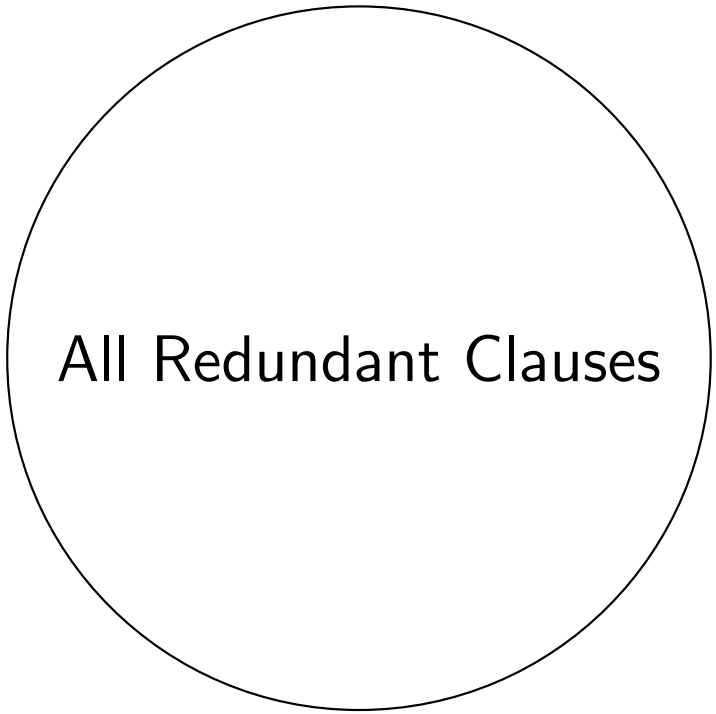

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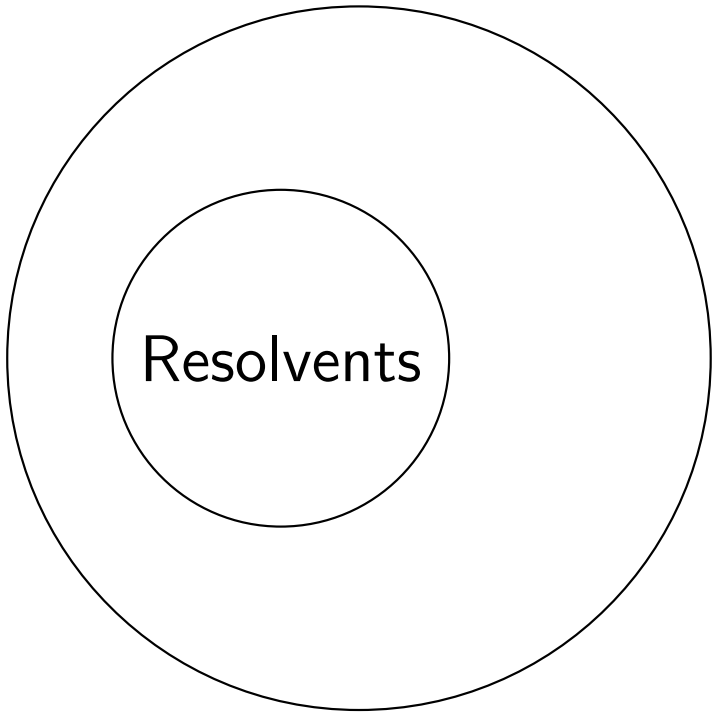

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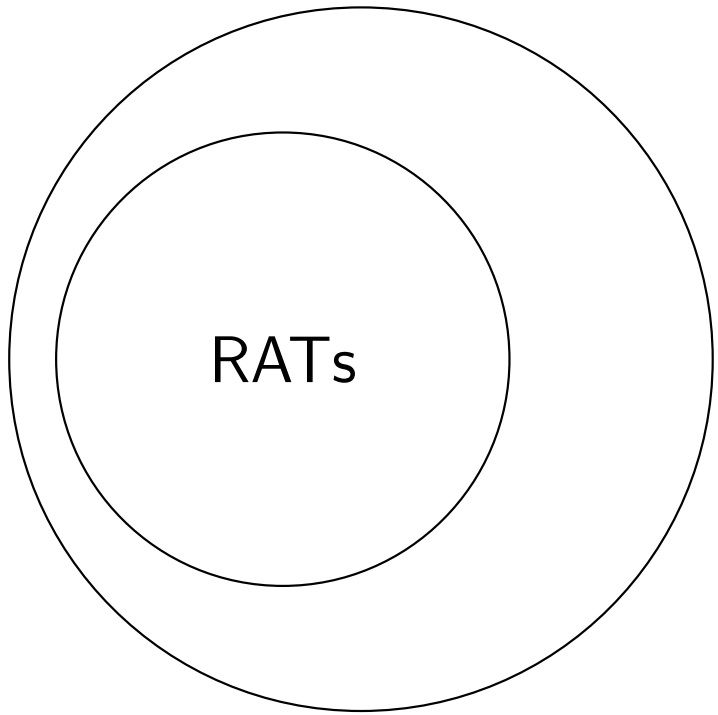

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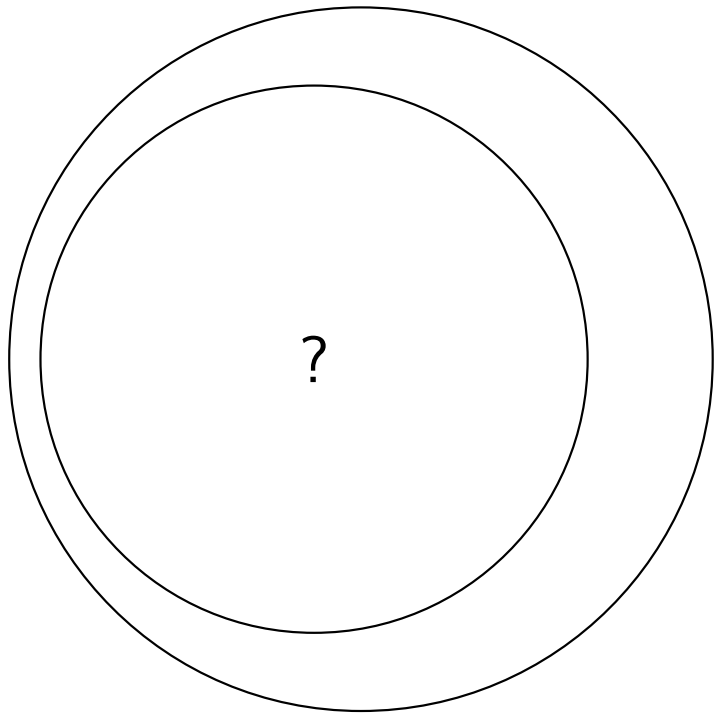
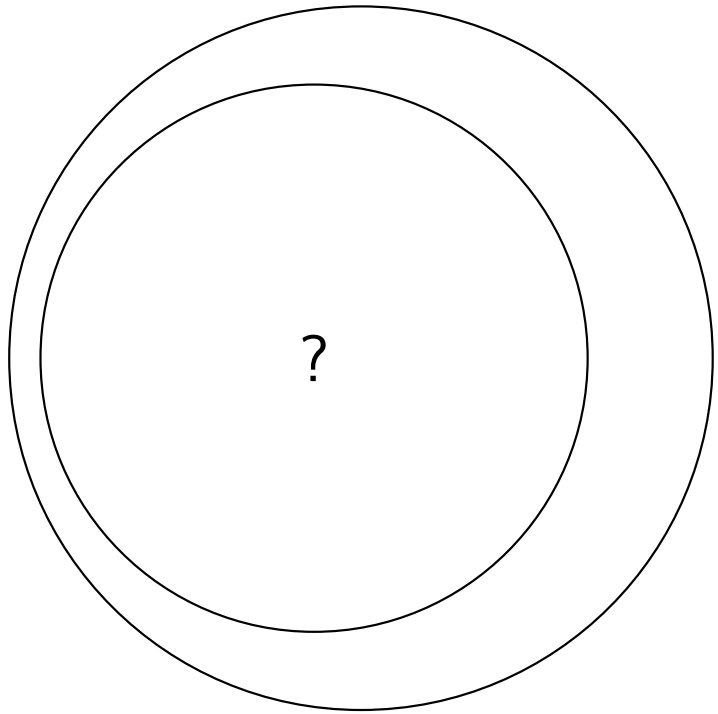


- The new proof systems can give short proofs of formulas that are considered hard.

Redundant Clauses

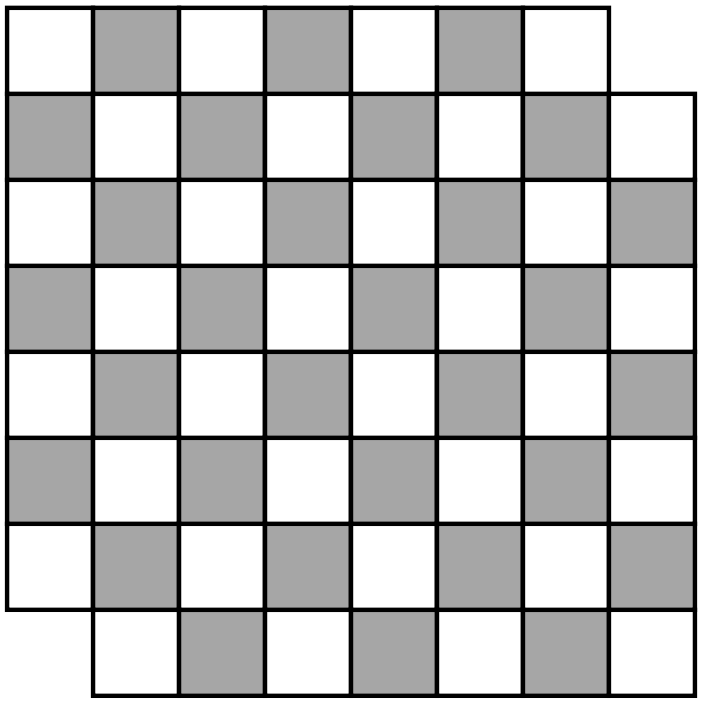
- Strong proof systems allow adding many redundant clauses.

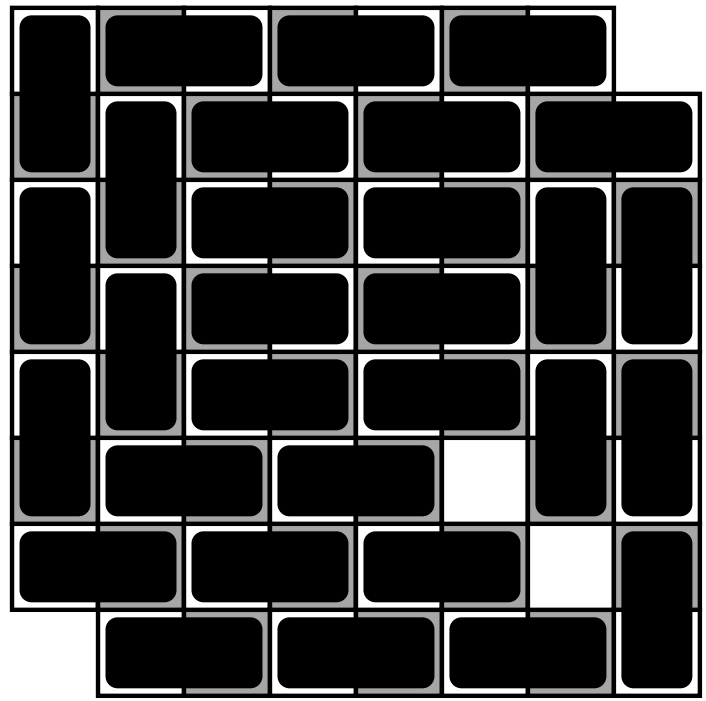


- The new proof systems can give short proofs of formulas that are considered hard.
- Are stronger redundancy notions still efficiently checkable?

Mutilated Chessboards: "A Tough Nut to Crack" [McCarthy]

Can a chessboard be fully covered with dominos after removing two diagonally opposite corner squares?





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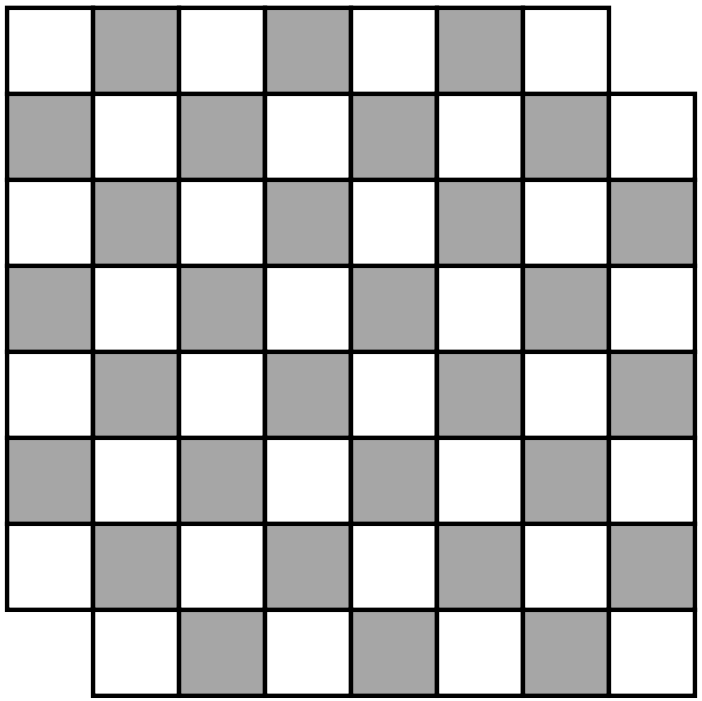
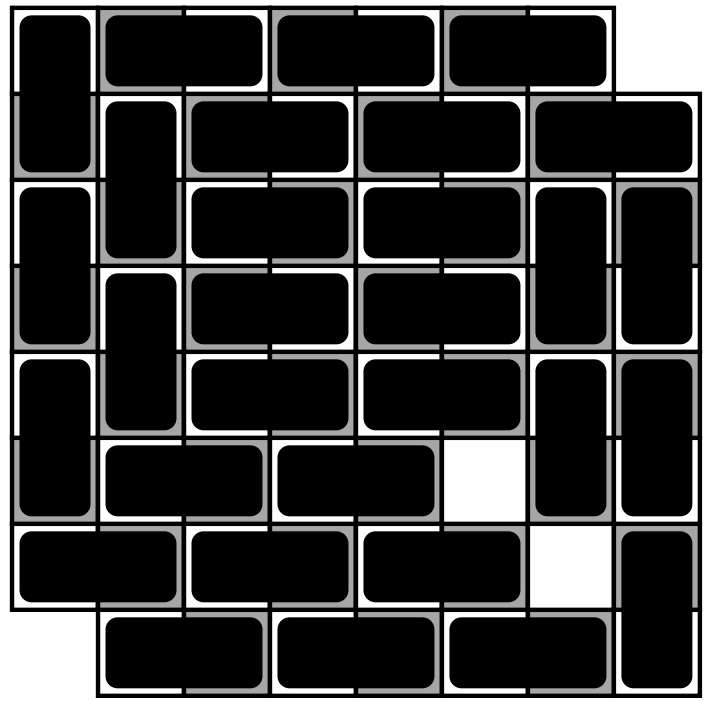


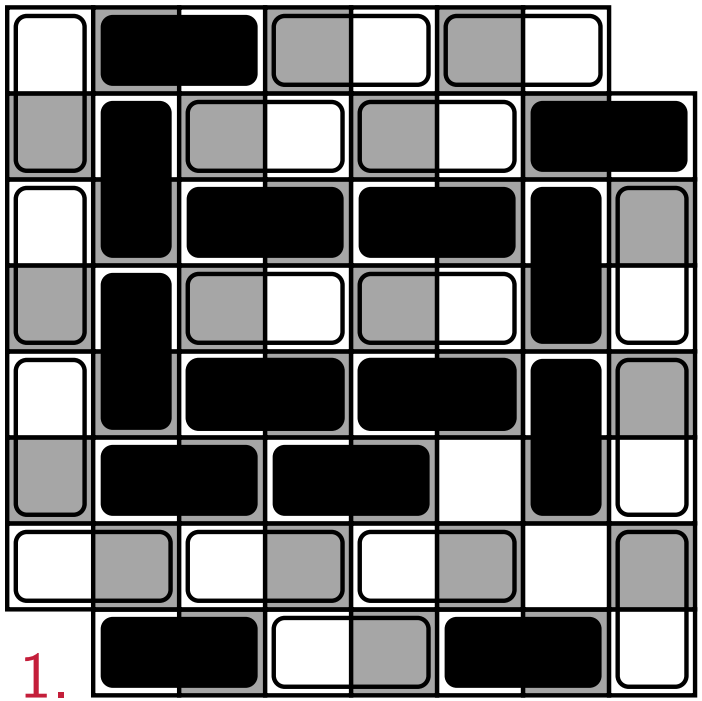


Easy to refute based on the following two observations:

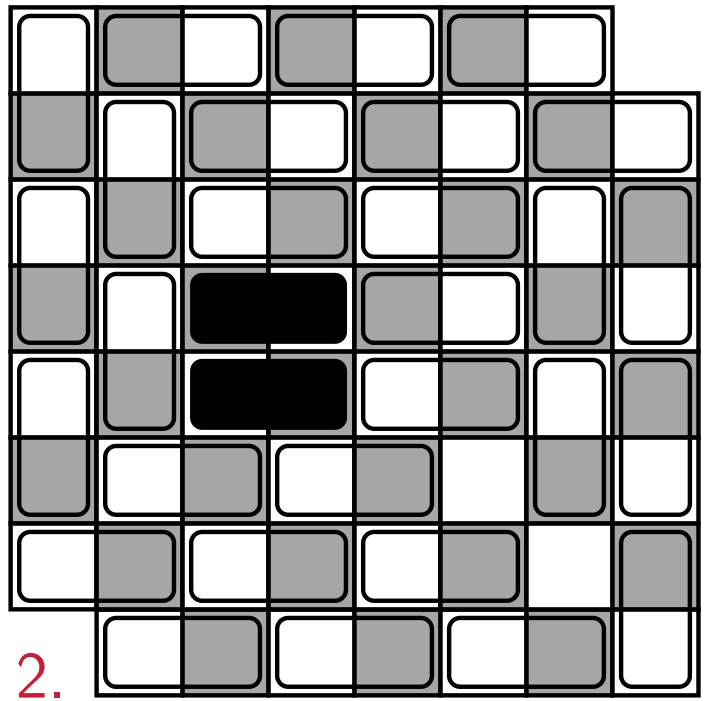
- There are more white squares than black squares; and
- A domino covers exactly one white and one black square.

Without Loss of Satisfaction

One of the crucial techniques in SAT solvers is to generalize a conflicting state and use it to constrain the problem.



1.



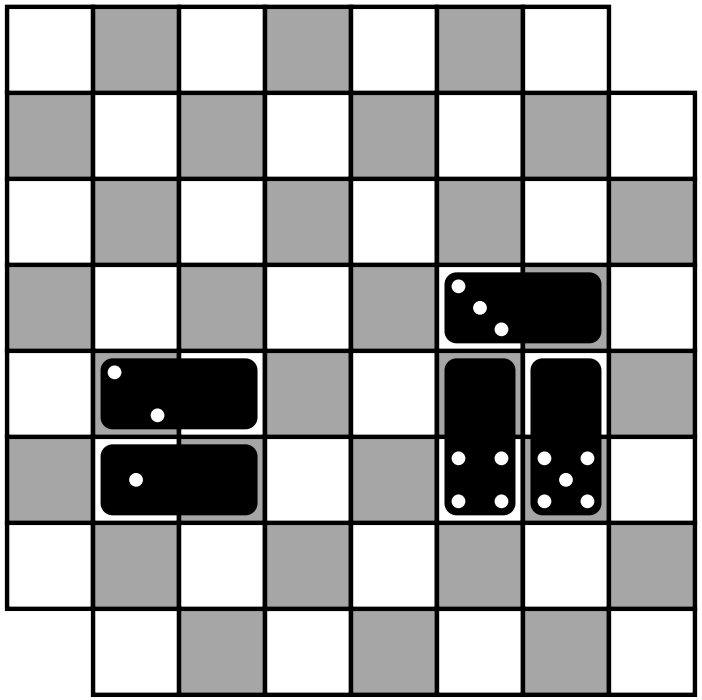
2.

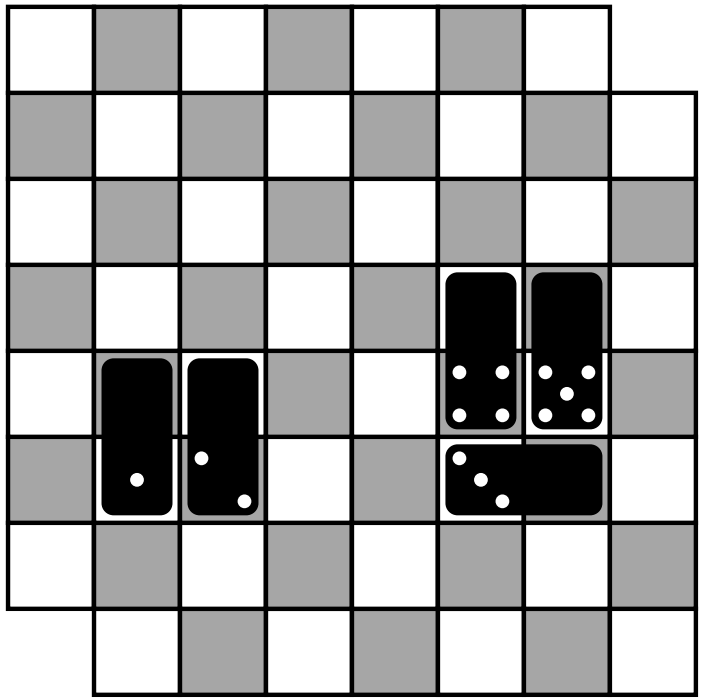
The used proof system can have a big impact on the size:

1. Resolution can only reduce the 30 dominos to 14 (left); and
2. "Without loss of satisfaction" can reduce them to 2 (right).

Mutilated Chessboards: An alternative proof

Satisfaction-Driven Clause Learning (SDCL) is a new solving paradigm that finds proofs in the PR proof system [HKB '17]





SDCL can detect that the above two patterns can be blocked

- This reduces the number of explored states exponentially
- We produced SPR proofs that are linear in the formula size

Redundancy as an Implication

A formula Δ is at least as satisfiable as a formula Γ if $\Gamma \models \Delta$.

Given a formula Γ and assignment α, we denote with $\Gamma|_\alpha$ the reduced formula after removing from Γ all clauses satisfied by α and all literals falsified by α.

Theorem
A clause C is redundant w.r.t. a formula Γ iff there exists an assignment α such that

$$\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\alpha$$

This is the strongest notion of redundancy. However, entailment ($\models$) cannot be checked in polynomial time (assuming P $\neq$ NP), unless bounded.

Checking Redundancy Using Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let Γ be a formula, C a clause, and α the smallest assignment that falsifies C. C is implied by Γ via UP (denoted by $\Gamma \vdash_1 C$) if UP on $\Gamma|_\alpha$ results in a conflict.
- Implied by UP is used in SAT solvers to determine redundancy of learned clauses and therefore $\vdash_1$ is a natural restriction of $\models$.
- We bound $\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\alpha$ by $\Gamma \wedge \neg C \vdash_1 (\Gamma \wedge C)|_\alpha$
- Example:
  $\Gamma = (x \vee y \vee z) \wedge (\overline{x} \vee y \vee z) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{y} \vee z)$ and $\Delta = (z)$. Observe that $\Gamma \models \Delta$, but that $\Gamma \not\vdash_1 \Delta$.

Hand-crafted PR Proofs of Pigeon Hole Formulas

We manually constructed PR proofs of the famous pigeon hole formulas and the two-pigeons-per-hole family.

- The proofs consist only of binary and unit clauses.
- Only original variables appear in the proof.
- All proofs are linear in the size of the formula.

➥ The PR proofs are smaller than Cook's ER proofs.

- All resolution proofs of these formulas are exponential in size.

Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Conclusions

Autarkies

A non-empty assignment α is an autarky for formula Γ if every clause $C \in \Gamma$ that is touched by α is also satisfied by α.

A pure literal and a satisfying assignment are autarkies.

Example

Consider the formula $\Gamma := (x \vee y) \wedge (\overline{x} \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
Assignment $\alpha_1 = \overline{z}$ is an autarky:
$(x \vee y) \wedge (\overline{x} \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$. Assignment $\alpha_2 = x\,\overline{y}\,z$ is an autarky: $(x \vee y) \wedge (\overline{x} \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

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Given an assignment α, $\Gamma|_\alpha$ denotes a formula Γ without the clauses satisfied by α and without the literals falsified by α.

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula Γ.
Then, Γ and $\Gamma|_\alpha$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a conditional autarky for formula Γ if α_{aut} is an autarky for $\Gamma|\alpha_{\text{con}}$.

Example

Consider the formula $\Gamma := (x \vee y) \wedge (\overline{x} \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
Let $\alpha_{\text{con}} = y$ and $\alpha_{\text{aut}} = \overline{x}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\overline{y}$ is a conditional autarky for Γ:

$$\alpha_{\text{aut}} = \overline{x} \text{ is an autarky for } \Gamma|\alpha_{\text{con}} = (\overline{x}) \wedge (\overline{z}).$$

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Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a conditional autarky for formula Γ. Then Γ and $\Gamma \wedge (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\overline{y} \vee \overline{x})$.

Finding Redundant Clauses: The Positive Reduct

Determining whether a clause C is PR w.r.t. a formula Γ is an NP-complete problem.

How to find PR clauses? Encode it in SAT!

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Key Idea: While solving a formula Γ, check whether the positive reduct of Γ and the current assignment α is satisfiable. In that case, prune the branch α.

The Positive Reduct: An Example

Given a formula Γ and a clause C. Let α denote the smallest assignment that falsifies C. The positive reduct of Γ and α, denoted by $p(\Gamma, \alpha)$, is the formula that contains C and all $assigned(D, \alpha)$ with $D \in \Gamma$ and D is satisfied by α.

Example

Consider the formula $\Gamma := (x \vee y) \wedge (\overline{x} \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

Let $C_1 = (\overline{x})$, so $\alpha_1 = x$.
The positive reduct $p(\Gamma, \alpha_1) = (\overline{x}) \wedge (x)$ is unsatisfiable.

Let $C_2 = (x \vee \overline{y})$, so $\alpha_2 = \overline{x}\, y$.
The positive reduct $p(\Gamma, \alpha_2) = (x \vee \overline{y}) \wedge (x \vee y) \wedge (\overline{x} \vee \overline{y})$ is satisfiable.

Pseudo-Code of CDCL (formula Γ)

```
1    α := ∅
2    forever do
3       α := Simplify (Γ, α)
4       if Γ|α contains a falsified clause then
5          C := AnalyzeConflict ()
6          if C is the empty clause then return unsatisfiable
7          Γ := Γ ∪ {C}
8          α := BackJump (C, α)
13      else
14         l := Decide ()
15         if l is undefined then return satisfiable
16         α := α ∪ {l}
```

Pseudo-Code of SDCL (formula Γ)

```
1   α := ∅
2   forever do
3     α := Simplify (Γ, α)
4     if Γ|α contains a falsified clause then
5       C := AnalyzeConflict ()
6       if C is the empty clause then return unsatisfiable
7       Γ := Γ ∪ {C}
8       α := BackJump (C, α)
9     else if p(Γ, α) is satisfiable then
10      C := AnalyzeWitness ()
11      Γ := Γ ∪ {C}
12      α := BackJump (C, α)
13    else
14      l := Decide ()
15      if l is undefined then return satisfiable
16      α := α ∪ {l}
```

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We introduced new redundancy notions for SAT.

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- They are more suitable for mechanized proof search.

SDCL generalizes the well-known CDCL paradigm by allowing to prune branches that are potentially satisfiable:

- Such branches can be found using the positive reduct;
- Pruning can be expressed in the PR proof system;
- Runtime and proofs can be exponentially smaller.