The Logic of Bunched Implications

Presentation by Robert J. Simmons
Separation Logic, April 6, 2009

Big Picture

$$\frac{P \Rightarrow P' \quad \{P'\}\ C\ \{R'\} \quad R' \Rightarrow R}{\{P\}\ C\ \{R\}}$$

- Valid formula of separation logic

  for all s, h,
  $s,h \models P \Rightarrow P'$

- A valid implication in BI is valid in sep. logic

- Not *complete* (example: pure propositions)

Proofs and provability

- Separation logic has a *model*
  - Valid props. true in all heaps/stores
- Both BI and Boolean BI have *axioms*
  - Valid props. are provable from the axioms
- BI has a *proof theory*
  - Valid props. have closed derivations with no hypotheses
  - Allows *hypothetical reasoning* (to prove A $\Rightarrow$ B assume A, prove B)

Outline: Today

- The logic of bunched implications
  - Substructural logics and bunchy contexts
  - Natural deduction for BI
  - Sequent calculus (& logic prog.) for BI

Outline: Wednesday

- (Maybe) a little more logic programming
- Other logics of bunched implication
  - Boolean BI
    An assertion language for separation logic
  - Classical BI
    Not compatible with separation logic!

Linear Logic (multiplicative)

$$\frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}$$

Constructive Logic (additive)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Linear Logic (multiplicative)

$$\frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B}}{A, B \vdash A \otimes B} \qquad \dfrac{}{B \vdash B}}{A, B, B \vdash (A \otimes B) \otimes B}$$

Constructive Logic (additive)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\dfrac{\dfrac{\dfrac{}{A; B; B \vdash A} \qquad \dfrac{}{A; B; B \vdash B}}{A; B; B \vdash A \wedge B} \qquad \dfrac{}{A; B; B \vdash B}}{A; B; B \vdash (A \wedge B) \wedge B}$$

Contraction

$$\frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$A, B \nvdash (A \otimes B) \otimes B$$

$$\dfrac{\dfrac{\dfrac{}{A; B \vdash A} \qquad \dfrac{}{A; B \vdash B}}{A; B \vdash A \wedge B} \qquad \dfrac{}{A; B \vdash B}}{A; B \vdash (A \wedge B) \wedge B}$$

Weakening

$$\frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$A, B, B, C \nvdash (A \otimes B) \otimes B$$

$$\dfrac{\dfrac{\dfrac{}{A; B; C \vdash A} \qquad \dfrac{}{A; B; C \vdash B}}{A; B; C \vdash A \wedge B} \qquad \dfrac{}{A; B; C \vdash B}}{A; B; C \vdash (A \wedge B) \wedge B}$$

Exchange

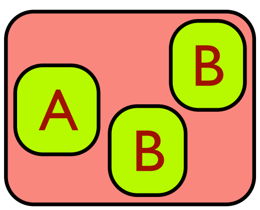
$$\frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$B, B, A \vdash (A \otimes B) \otimes B$

$B, A, B \vdash (A \otimes B) \otimes B$

$A, B, B \vdash (A \otimes B) \otimes B$

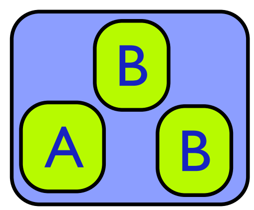


$\vdash (A \otimes B) \otimes B$

$B; B; A \vdash (A \wedge B) \wedge B$

$B; A; B \vdash (A \wedge B) \wedge B$

$A; B; B \vdash (A \wedge B) \wedge B$



$\vdash (A \wedge B) \wedge B$

Linear Logic: Additive and Multiplicative Zones

$$\frac{\Delta, A \vdash B \rightarrow C}{\Delta \vdash A \multimap B \rightarrow C}$$

Linear Logic:
Additive and Multiplicative Zones

Additive zone ↘ Multiplicative zone ↘ Conclusion ↘

$$\cfrac{\cfrac{\Gamma, B; \Delta, A \vdash C}{\Gamma; \Delta, A \vdash B \to C}}{\Gamma; \Delta \vdash A \multimap B \to C}$$

Bunched implication: Additive and Multiplicative Bunches

Additive combination

Multiplicative combination

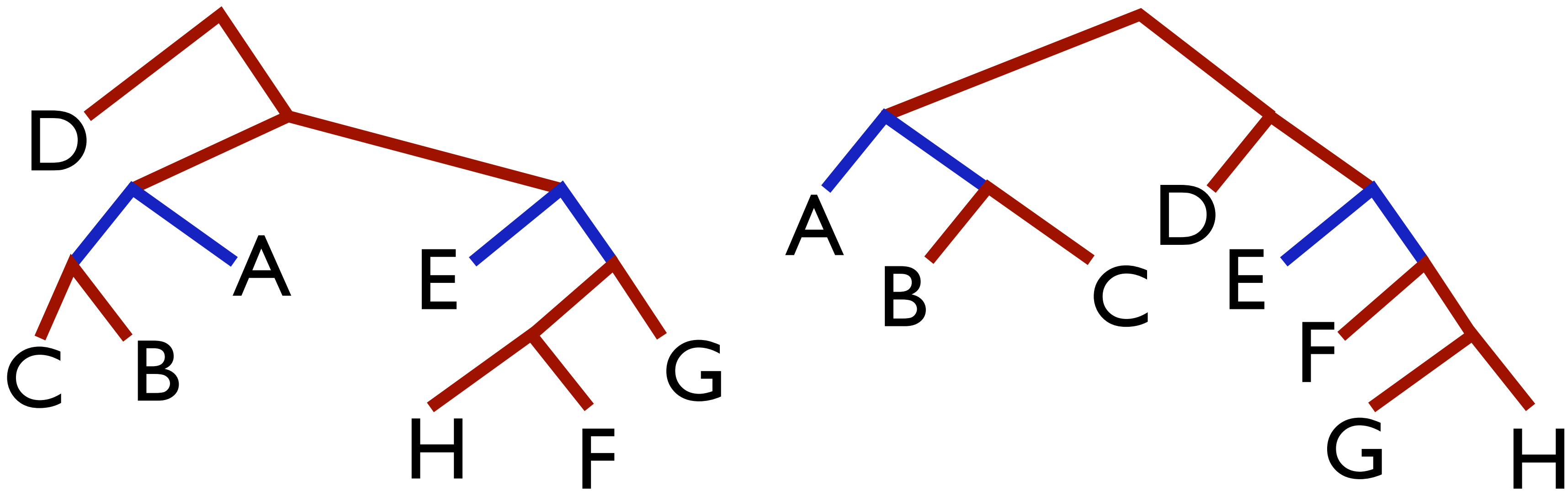
Conclusion

$$\dfrac{\dfrac{(\Gamma ; A), B \vdash C}{\Gamma ; A \vdash B \mathrel{-\!\!*} C}}{\Gamma \vdash A \to B \mathrel{-\!\!*} C}$$

Additive and Multiplicative Bunches

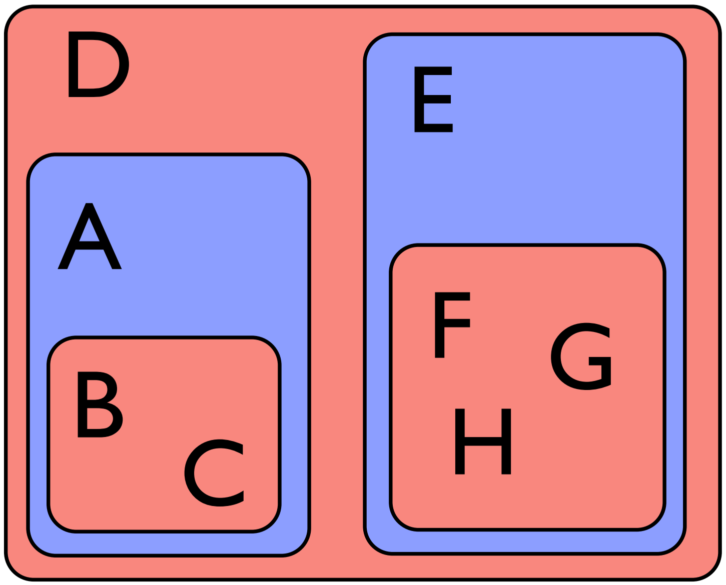
(A; (B, C)), (D, (E; (F, (G, H)))) ⊢ Z

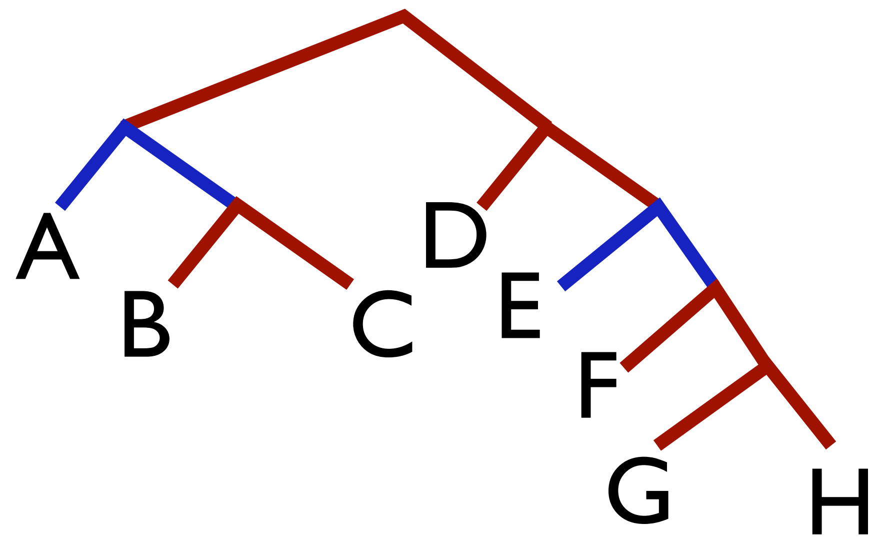
D, (((C, B); A), (E; ((H, F), G))) ⊢ Z



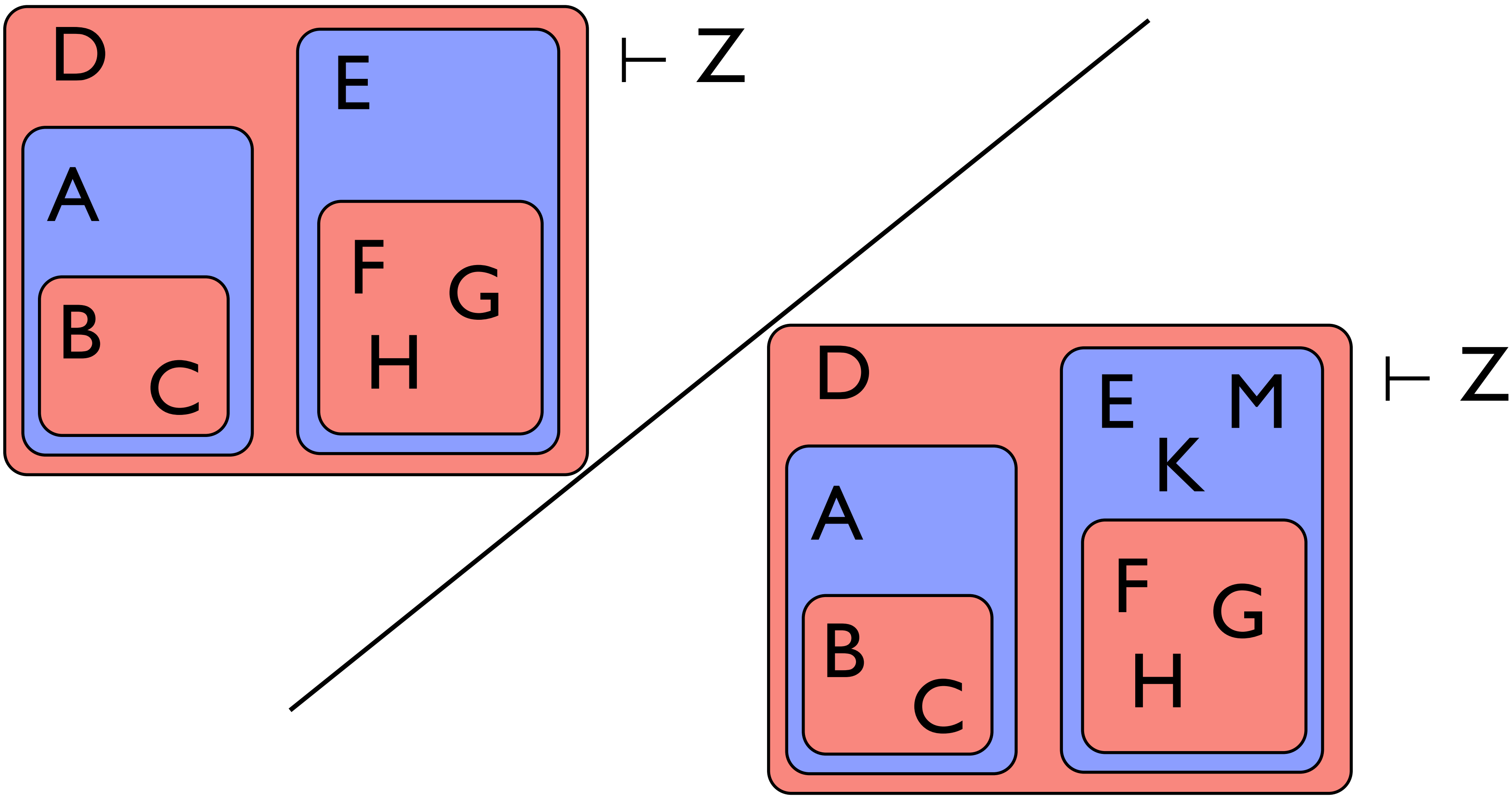
Additive and Multiplicative Bunches

$(A; (B, C)), (D, (E; (F, (G, H)))) \vdash Z$

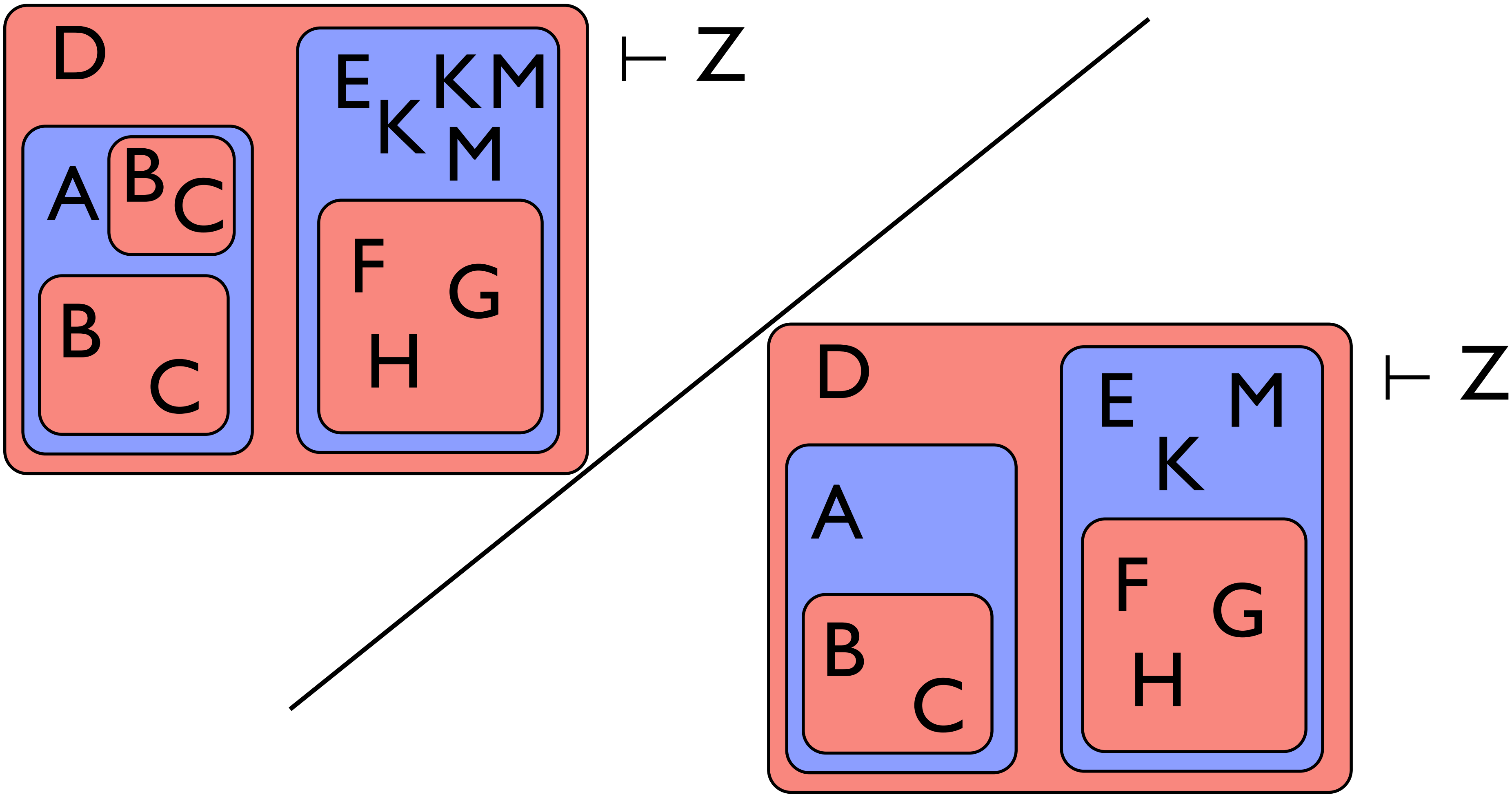




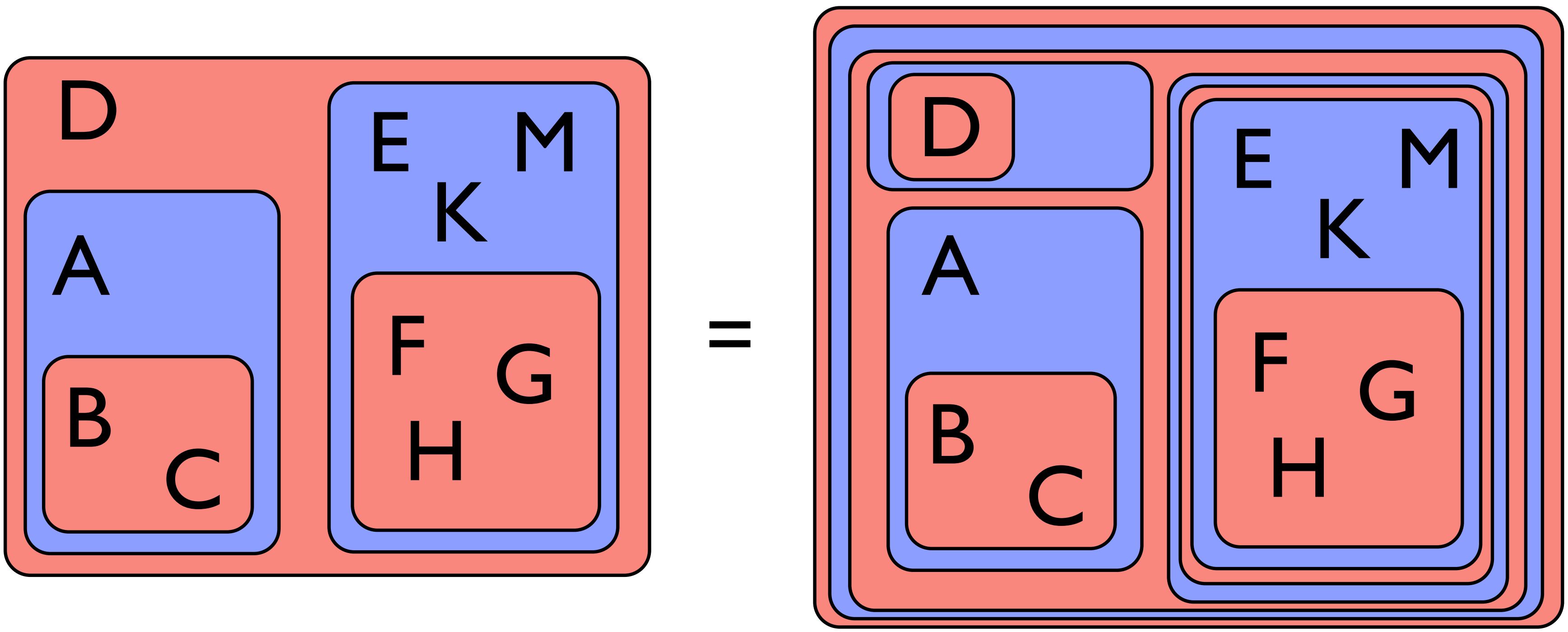
Additive and Multiplicative Bunches
Weakening



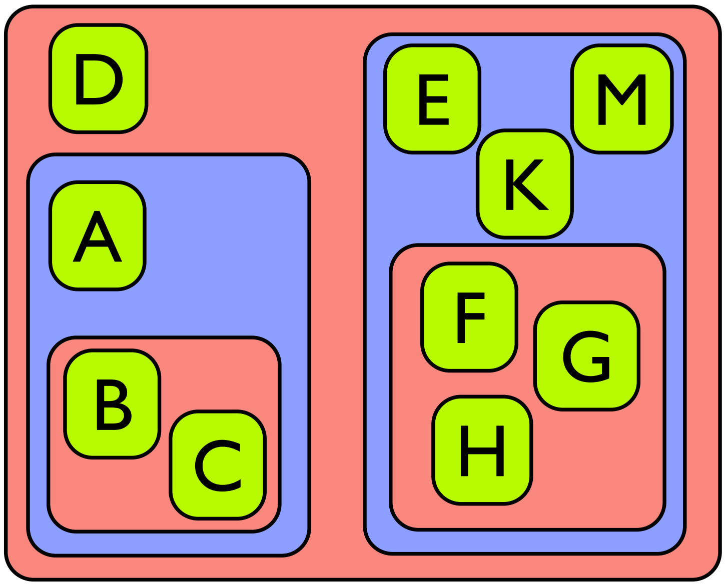
Additive and Multiplicative Bunches
Contraction



Additive and Multiplicative Bunches
"Commutative monoid equations"

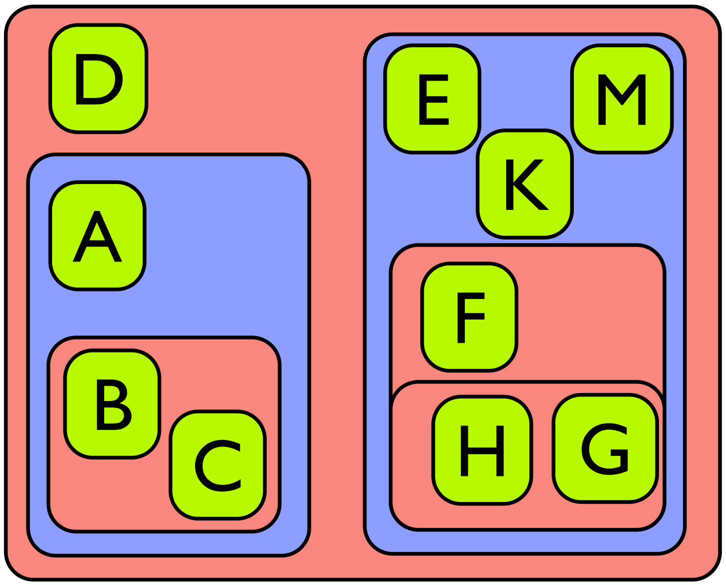


Additive and Multiplicative Bunches
Formulas stand alone



$$A = A; \varnothing_a = A, \varnothing_m$$

Additive and Multiplicative Bunches
Contexts with a missing piece



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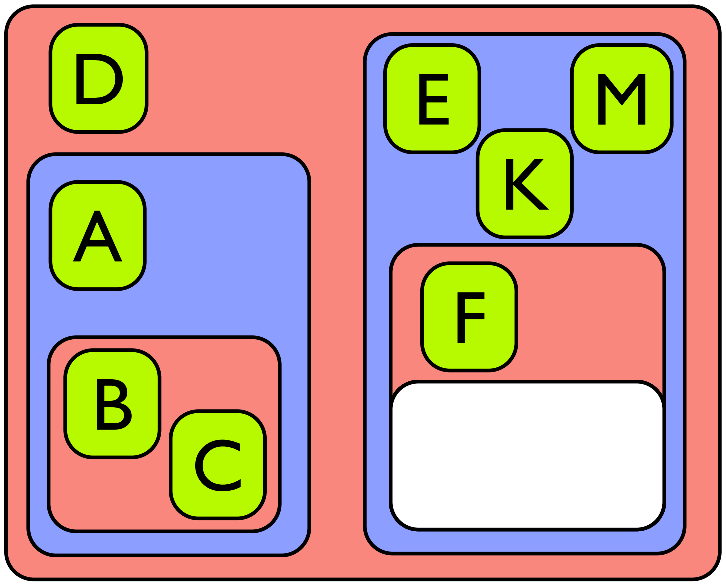
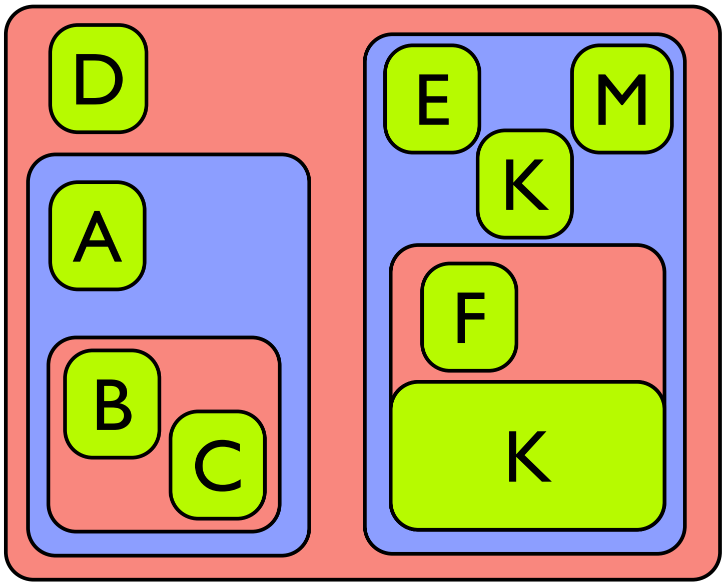
Additive and Multiplicative Bunches
Contexts with a missing piece



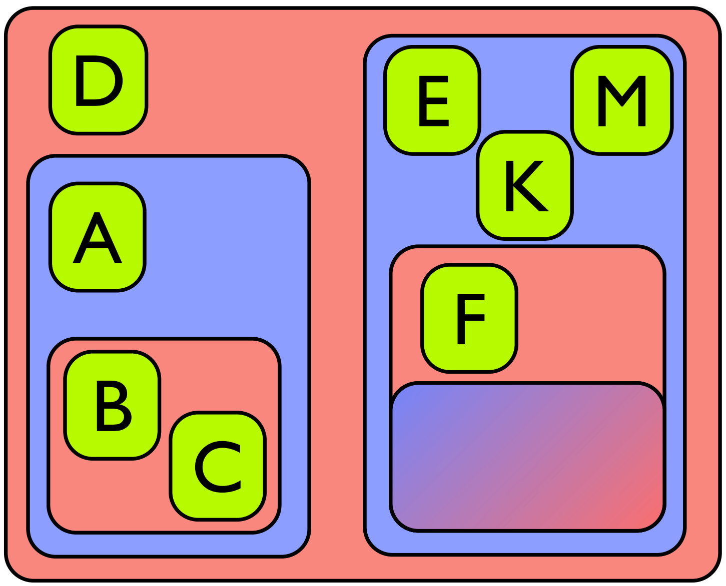
$\Gamma(-)$

Additive and Multiplicative Bunches
Contexts with a missing piece



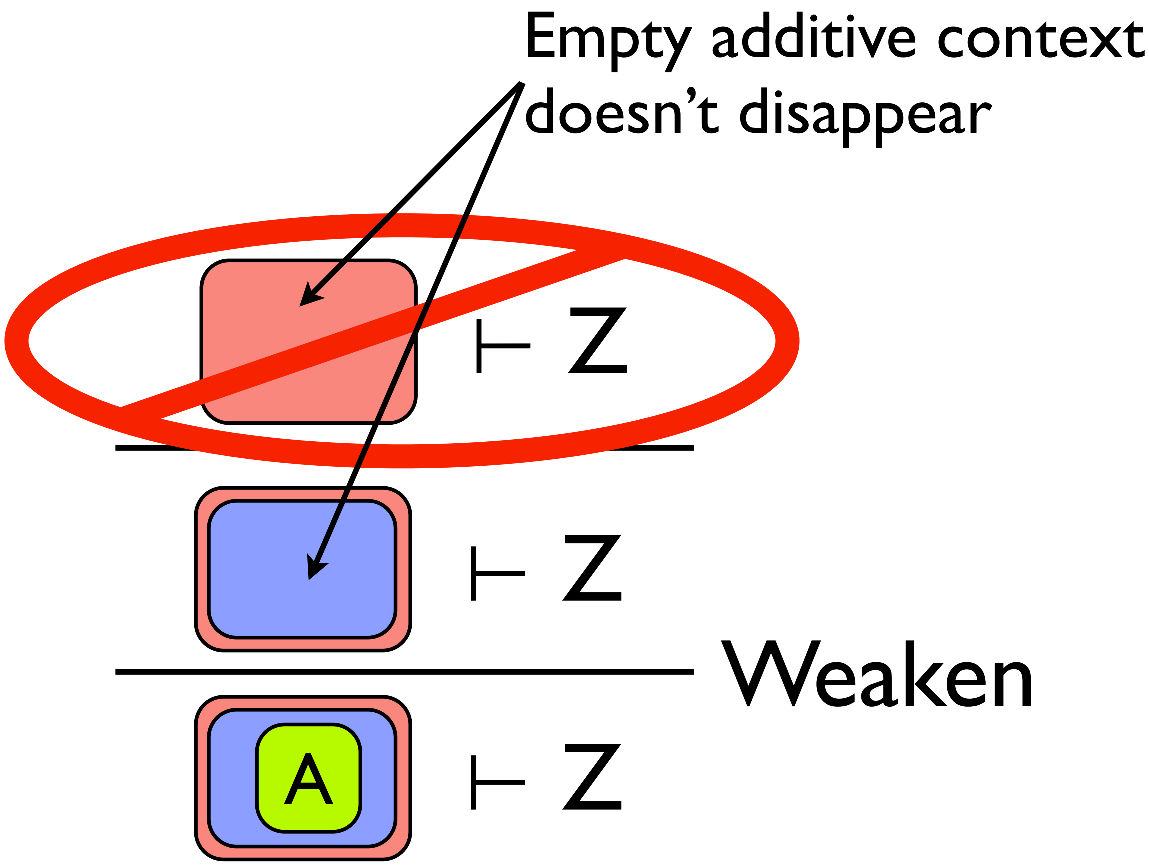
$\Gamma(K)$

Additive and Multiplicative Bunches
Contexts with a missing piece



$\Gamma(\Delta)$

Additive and Multiplicative Bunches ???



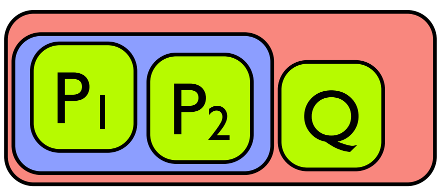
Natural Deduction for BI

$$\frac{}{\varnothing_m \vdash I} \quad \frac{}{\varnothing_a \vdash \top} \quad \frac{}{A \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \qquad \frac{\Delta \vdash A \wedge B \quad \Gamma(A; B) \vdash C}{\Gamma(\Delta) \vdash C} \wedge E$$

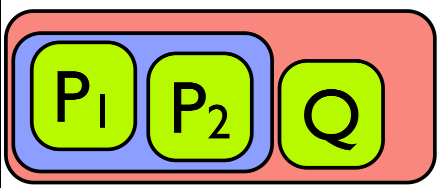
$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A * B} *I \qquad \frac{\Delta \vdash A * B \quad \Gamma(A, B) \vdash C}{\Gamma(\Delta) \vdash C} *E$$

Natural Deduction for BI

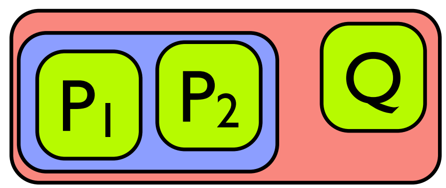


$\vdash (P_1 * Q) \wedge (P_2 * Q)$

Natural Deduction for BI



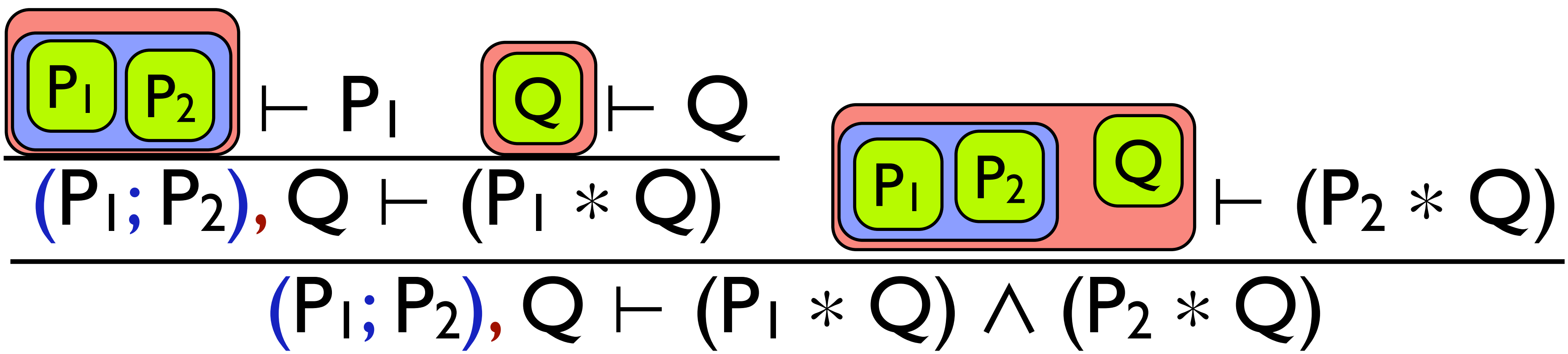
$\vdash (P_1 * Q)$



$\vdash (P_2 * Q)$

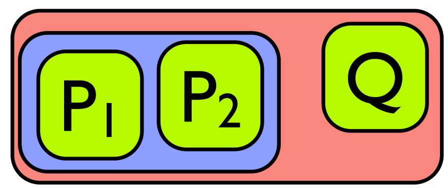
$$\overline{\qquad (P_1 ; P_2), Q \vdash (P_1 * Q) \wedge (P_2 * Q) \qquad}$$

Natural Deduction for BI



Natural Deduction for BI

$$\dfrac{\dfrac{\dfrac{}{P_1 \vdash P_1}}{(P_1; P_2) \vdash P_1} \qquad \dfrac{}{Q \vdash Q}}{(P_1; P_2), Q \vdash (P_1 * Q)}$$



$\vdash (P_2 * Q)$

$$\overline{(P_1; P_2), Q \vdash (P_1 * Q) \wedge (P_2 * Q)}$$

Natural Deduction for BI

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P_1 \vdash P_1}}{(P_1; P_2) \vdash P_1} \quad \dfrac{}{Q \vdash Q}}{(P_1; P_2), Q \vdash (P_1 * Q)} \qquad \dfrac{\dfrac{\dfrac{}{P_2 \vdash P_2}}{(P_1; P_2) \vdash P_2} \quad \dfrac{}{Q \vdash Q}}{(P_1; P_2), Q \vdash (P_2 * Q)}}{(P_1; P_2), Q \vdash (P_1 * Q) \wedge (P_2 * Q)}$$

Natural Deduction for BI

$$\frac{\Delta \vdash A * B \qquad \Gamma(A, B) \vdash C}{\Gamma(\Delta) \vdash C} \quad \Rightarrow \quad \frac{P_1 * Q_1 \vdash P_1 * Q_1 \qquad P_1, Q_1 \vdash P_2 * Q_2}{P_1 * Q_1 \vdash P_2 * Q_2}$$

$$\dfrac{\dfrac{}{P_1 * Q_1 \vdash P_1 * Q_1} \qquad \dfrac{\dfrac{\vdots}{P_1 \vdash P_2} \qquad \dfrac{\vdots}{Q_1 \vdash Q_2}}{P_1, Q_1 \vdash P_2 * Q_2}}{P_1 * Q_1 \vdash P_2 * Q_2}$$

Natural Deduction for BI

$$\frac{\Delta \vdash A * B \qquad \Gamma(A, B) \vdash C}{\Gamma(\Delta) \vdash C} \quad \Rightarrow \quad \frac{(P_1 \wedge P_2) * Q \vdash (P_1 \wedge P_2) * Q \qquad (P_1 \wedge P_2), Q \vdash (P_1 * Q) \ldots}{(P_1 \wedge P_2) * Q \vdash (P_1 * Q) \ldots}$$

$$\dfrac{\dfrac{}{(P_1 \wedge P_2) * Q \vdash (P_1 \wedge P_2) * Q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P_1 \wedge P_2 \vdash P_1 \wedge P_2} \qquad \dfrac{\dfrac{}{P_1 \vdash P_1}}{P_1 ; P_2 \vdash P_1}}{P_1 \wedge P_2 \vdash P_1} \qquad \dfrac{}{Q \vdash Q}}{(P_1 \wedge P_2), Q \vdash P_1 * Q} \qquad \begin{array}{c}\vdots \\ (P_1 \wedge P_2), Q \vdash P_2 * Q\end{array}}{(P_1 \wedge P_2), Q \vdash (P_1 * Q) \wedge (P_2 * Q)}}{(P_1 \wedge P_2) * Q \vdash (P_1 * Q) \wedge (P_2 * Q)}$$

Natural Deduction for BI

$$\frac{\Gamma ; A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma , A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B} \qquad \frac{\Gamma \vdash A \mathrel{-\!\!*} B \quad \Delta \vdash A}{\Gamma , \Delta \vdash B}$$

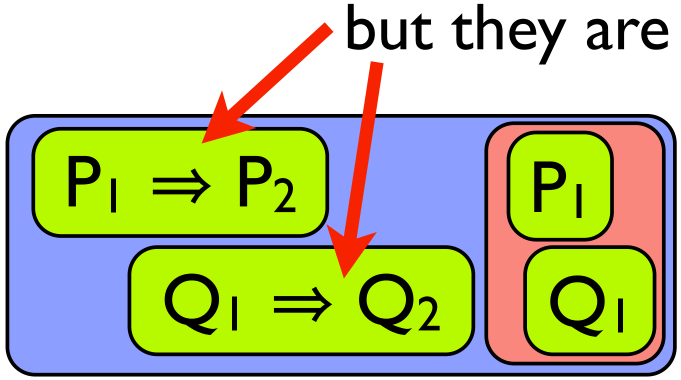
Natural Deduction for BI

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\varnothing_a \vdash P_0 * P_1 \Rightarrow P_2}}{P_0, P_1 \vdash P_0 * P_1 \Rightarrow P_2} \qquad \dfrac{\dfrac{}{P_0 \vdash P_0} \quad \dfrac{}{P_1 \vdash P_1}}{P_0, P_1 \vdash P_0 * P_1}}{P_0, P_1 \vdash P_2}}{\dfrac{\varnothing_a; P_0 \vdash P_1 \mathrel{-\!\!*} P_2}{\varnothing_a \vdash P_0 \Rightarrow P_1 \mathrel{-\!\!*} P_2}}
$$

Natural Deduction for BI

We want these to be *valid* props., but they are "stuck in a bunch"

$$\dfrac{\begin{array}{c}\vdots\\ P_1 \vdash P_2\end{array} \quad \begin{array}{c}\vdots\\ Q_1 \vdash Q_2\end{array}}{P_1 * Q_1 \vdash P_2 * Q_2}$$



$\vdash (P_2 * Q_2)$

$$\dfrac{}{P_1 * Q_1 \vdash P_1 * Q_1}$$

$$(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) \vdash (P_2 * Q_2)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(P_1 * Q_1 \vdash P_1 * Q_1) \quad (P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) \vdash (P_2 * Q_2)}{(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1 * Q_1) \vdash (P_2 * Q_2)}}{(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2) \vdash (P_1 * Q_1) \Rightarrow (P_2 * Q_2)}}{\varnothing_a; (P_1 \Rightarrow P_2) \vdash (Q_1 \Rightarrow Q_2) \Rightarrow (P_1 * Q_1) \Rightarrow (P_2 * Q_2)}}{\varnothing_a \vdash (P_1 \Rightarrow P_2) \Rightarrow (Q_1 \Rightarrow Q_2) \Rightarrow ((P_1 * Q_1) \Rightarrow (P_2 * Q_2))}$$

Sequent Calculus for BI

$$\dfrac{P_1 * Q_1 \vdash P_1 * Q_1 \qquad (P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) \vdash (P_2 * Q_2)}{(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1 * Q_1) \vdash (P_2 * Q_2)}$$

Sequent Calculus for BI

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A * B} {*R} \qquad \frac{\Gamma(A, B) \vdash C}{\Gamma(A * B) \vdash C} {*L}$$

$$\frac{(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1, Q_1) \vdash (P_2 * Q_2)}{(P_1 \Rightarrow P_2); (Q_1 \Rightarrow Q_2); (P_1 * Q_1) \vdash (P_2 * Q_2)}$$

Sequent Calculus for BI

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A * B} *R \qquad \frac{\Gamma(A, B) \vdash C}{\Gamma(A * B) \vdash C} *L$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R \qquad \frac{\Gamma(A; B) \vdash C}{\Gamma(A \wedge B) \vdash C} \wedge L$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2} \vee R_i \qquad \frac{\Gamma(A) \vdash C \qquad \Gamma(B) \vdash C}{\Gamma(A \vee B) \vdash C} \vee L$$

Sequent Calculus for BI

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P_0 \vdash P_0}}{P_0 \vdash P_0 \lor P_1} \qquad \dfrac{}{Q \vdash Q}}{P_0, Q \vdash (P_0 \lor P_1) * Q}}{P_0 * Q \vdash (P_0 \lor P_1) * Q} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P_1 \vdash P_1}}{P_1 \vdash P_0 \lor P_1} \qquad \dfrac{}{Q \vdash Q}}{P_1, Q \vdash (P_0 \lor P_1) * Q}}{P_1 * Q \vdash (P_0 \lor P_1) * Q}}{(P_0 * Q) \lor (P_1 * Q) \vdash (P_0 \lor P_1) * Q}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P_0 \vdash P_0} \qquad \dfrac{}{Q \vdash Q}}{P_0, Q \vdash P_0 * Q}}{P_0, Q \vdash (P_0 * Q) \lor (P_1 * Q)} \qquad \dfrac{\dfrac{\dfrac{}{P_1 \vdash P_1} \qquad \dfrac{}{Q \vdash Q}}{P_1, Q \vdash P_1 * Q}}{P_1, Q \vdash (P_0 * Q) \lor (P_1 * Q)}}{(P_0 \lor P_1), Q \vdash (P_0 * Q) \lor (P_1 * Q)}}{(P_0 \lor P_1) * Q \vdash (P_0 * Q) \lor (P_1 * Q)}$$

Sequent Calculus for BI

$$\frac{\Gamma; A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow R \qquad \frac{\Delta \vdash A \quad \Gamma(\Delta; B) \vdash C}{\Gamma(\Delta; A \Rightarrow B) \vdash C} \Rightarrow L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B} \mathrel{-\!\!*} R \qquad \frac{\Delta \vdash A \quad \Gamma(B) \vdash C}{\Gamma(\Delta, A \mathrel{-\!\!*} B) \vdash C} \mathrel{-\!\!*} L$$

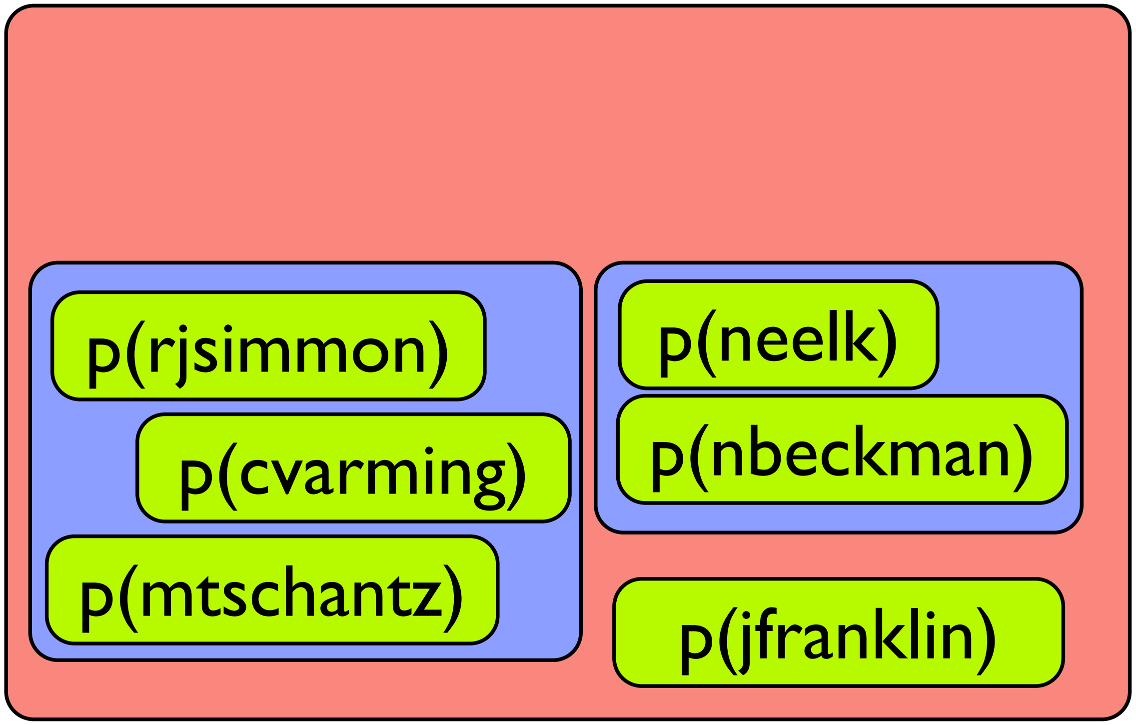
$$\frac{\Gamma \vdash [t/x]A}{\Gamma \vdash \exists x.A} \exists R \qquad \frac{\Gamma([t/x]A) \vdash C}{\Gamma(\forall x.A) \vdash C} \forall R$$

Office Assignments

3 research groups (POP, Plaid, Sec)

SCS wants the 2 students in each office in Gates to be from different research groups

Office Assignments



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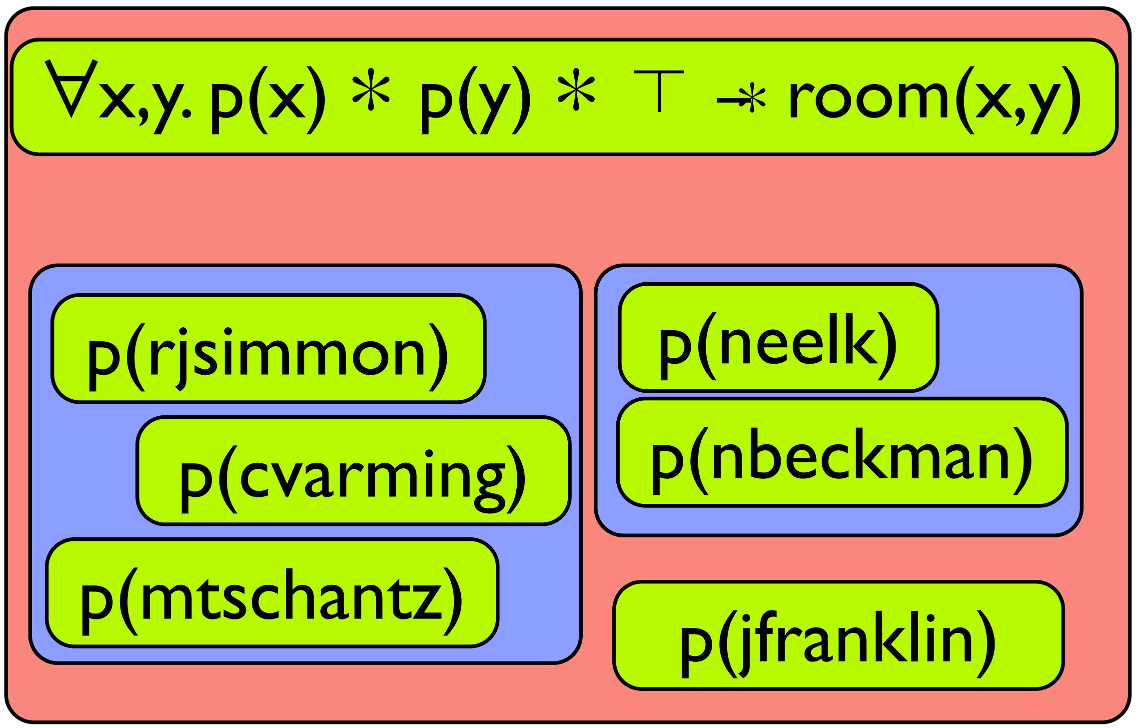
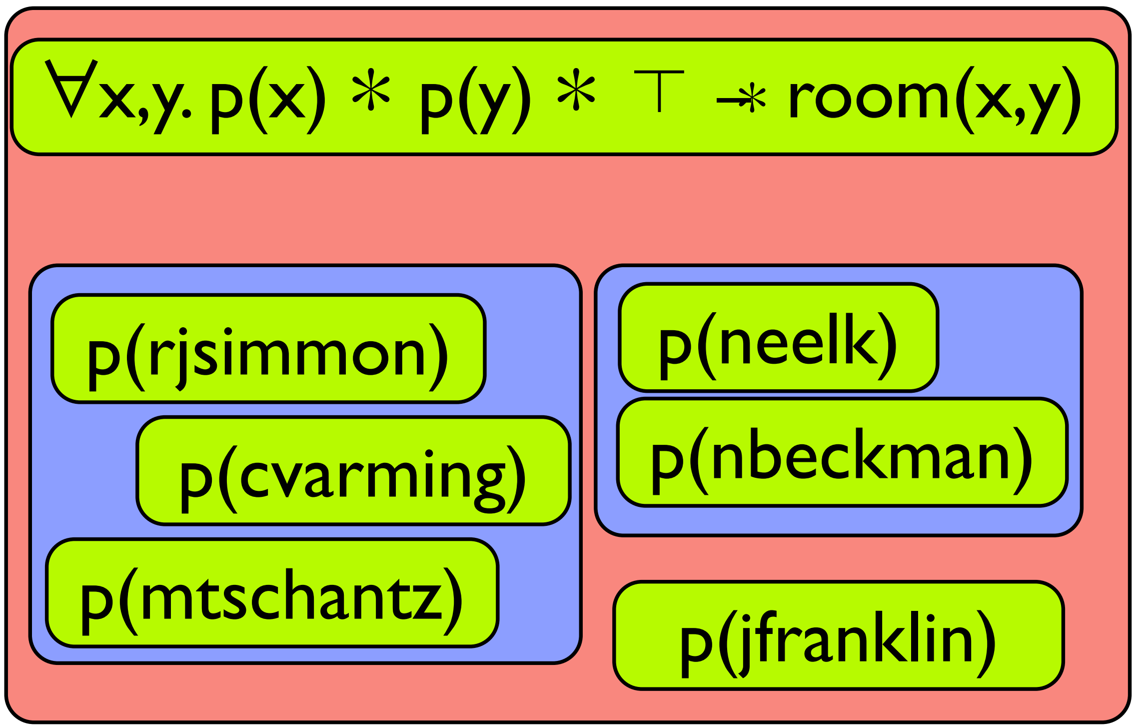


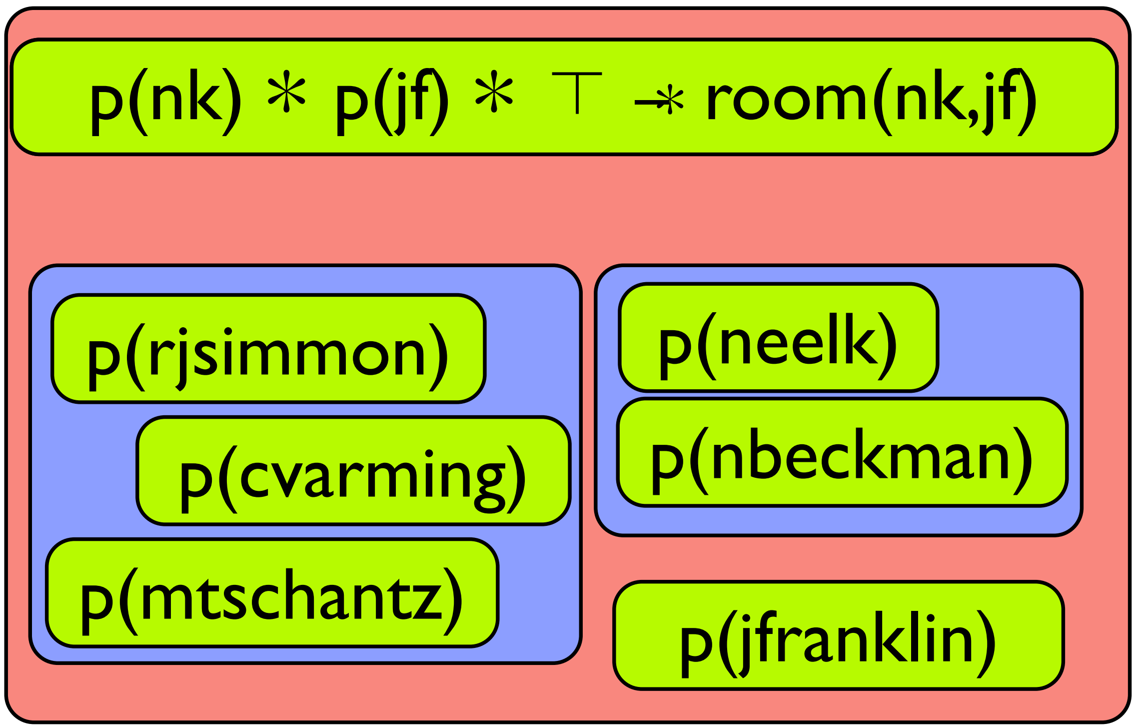
$\vdash \exists x,y.\ room(x,y)$

Office Assignments



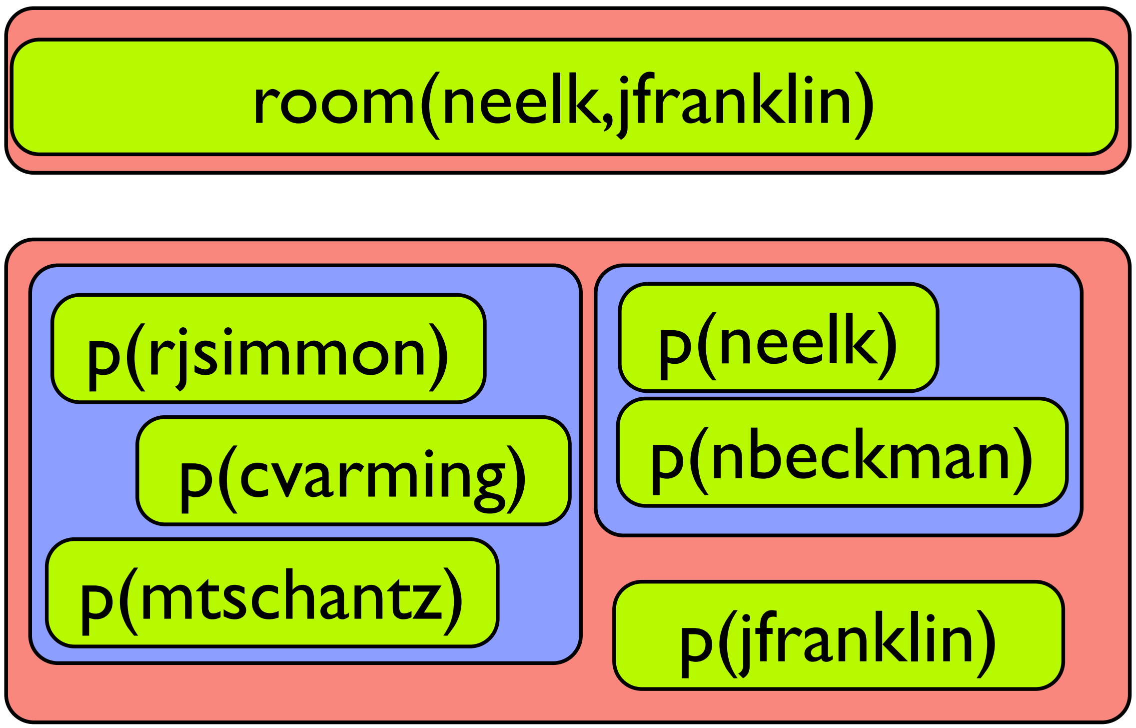
⊢ room(neelk, jfranklin)

Office Assignments



$\vdash$ room(neelk, jfranklin)

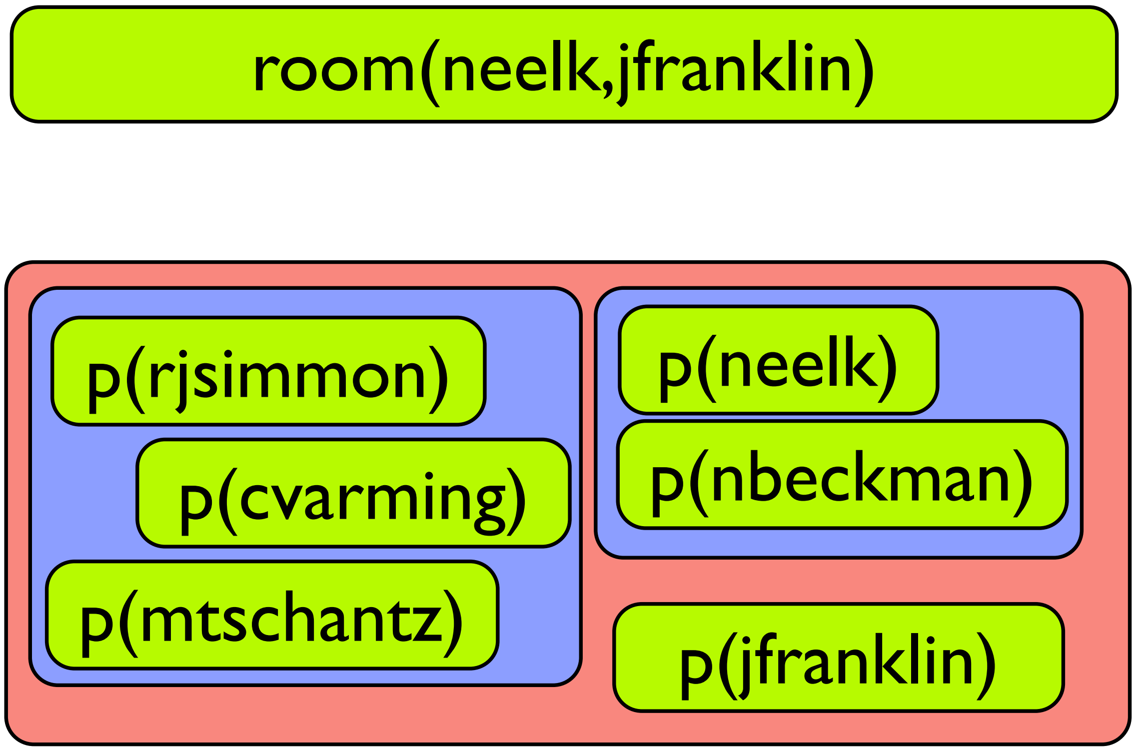
Office Assignments



$\vdash$ room(neelk, jfranklin)

$\vdash$ p(neelk) $*$ p(jfranklin) $*$ $\top$

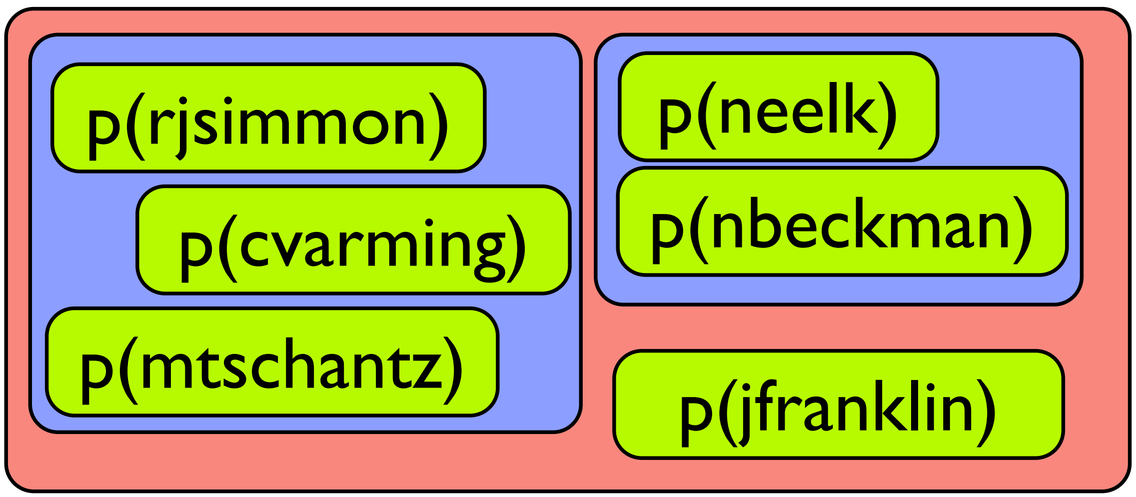
Office Assignments



$\vdash$ room(neelk, jfranklin)

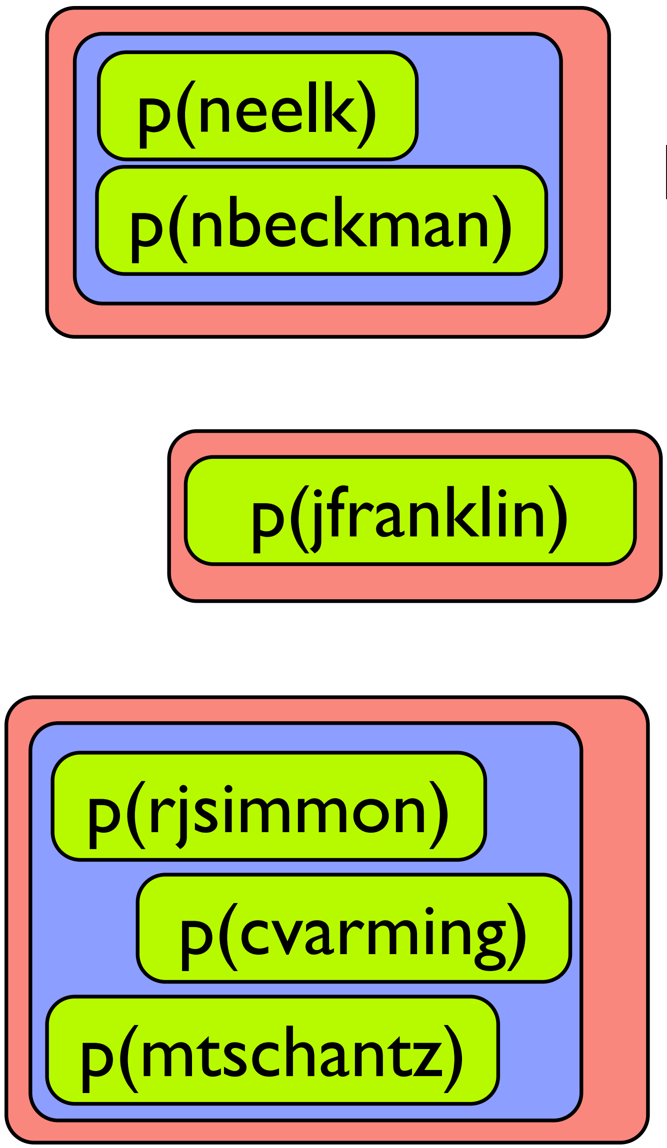
$\vdash$ p(neelk) $*$ p(jfranklin) $*$ $\top$

Office Assignments



$$\vdash p(neelk) * p(jfranklin) * \top$$

Office Assignments

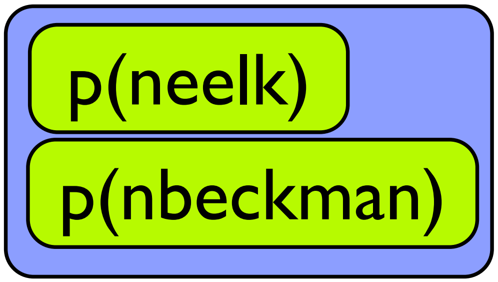


$\vdash$ p(neelk)

$\vdash$ p(jfranklin)

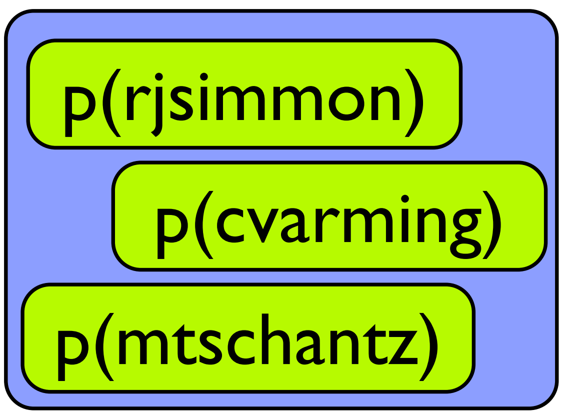
$\vdash \top$

Office Assignments



$\vdash$ p(neelk)

$\vdash$ p(jfranklin)



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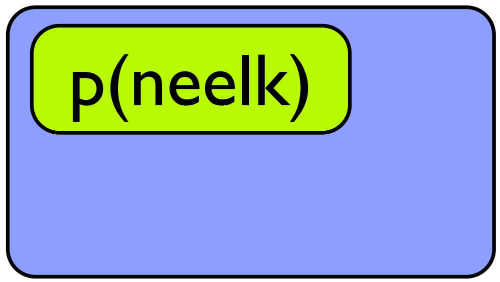
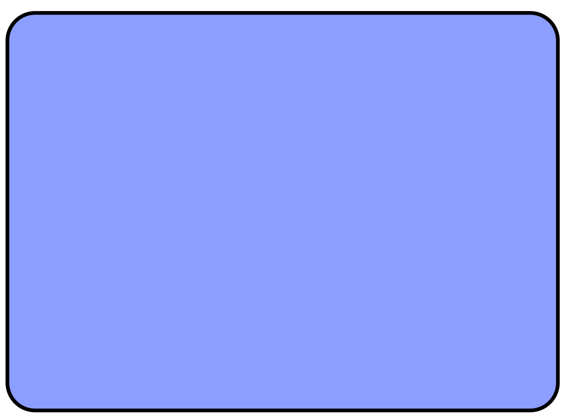
Office Assignments



$\vdash$ p(neelk)

$\vdash$ p(jfranklin)



$\vdash \top$

Office Assignments

p(neelk) $\vdash$ p(neelk)

p(jfranklin) $\vdash$ p(jfranklin)

$\vdash \top$

Office Assignments: Success!

Wrap-up

- BI: Logic sound for sep. logic
  - Axioms of BI have a proof theory
- Not covered: research on a proof theory for separation logic by Neel K.
  - Permits ($A \wedge I \vdash A * A$), unlike BI
- Sequent calculus/logic programming!
- Next time: Classical BI, Boolean BI

The Logic of Bunched Implications

```
@article{DBLP:journals/bsl/OHearnP99,
  author    = {Peter W. O'Hearn and
               David J. Pym},
  title     = {The logic of bunched implications},
  journal   = {Bulletin of Symbolic Logic},
  volume    = {5},
  number    = {2},
  year      = {1999},
  pages     = {215-244},
  ee        = {http://www.math.ucla.edu/$\sim$asl/bsl/0502/0502-003.ps},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}
```

Bunched Logic Programming

```
@inproceedings{DBLP:conf/cade/ArmelinP01,
  author    = {Pablo A. Armel\'{\i}n and David J. Pym},
  title     = {Bunched Logic Programming},
  booktitle = {IJCAR},
  year      = {2001},
  pages     = {289-304},
  ee        = {http://link.springer.de/link/service/series/0558/bibs/2083/20830289.htm},
  crossref  = {DBLP:conf/cade/2001},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}
@proceedings{DBLP:conf/cade/2001,
  editor    = {Rajeev Gor{\'e} and Alexander Leitsch and Tobias Nipkow},
  title     = {Automated Reasoning, First International Joint Conference, IJCAR 2001, Siena, Italy, June 18-23, 2001, Proceedings},
  booktitle = {IJCAR},
  publisher = {Springer},
  series    = {Lecture Notes in Computer Science},
  volume    = {2083},
  year      = {2001},
  isbn      = {3-540-42254-4},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}
```

The Semantics of BI

```
@article{DBLP:journals/tcs/PymOY04,
  author    = {David J. Pym and
               Peter W. O'Hearn and
               Hongseok Yang},
  title     = {Possible worlds and resources: the semantics of BI},
  journal   = {Theor. Comput. Sci.},
  volume    = {315},
  number    = {1},
  year      = {2004},
  pages     = {257-305},
  ee        = {http://dx.doi.org/10.1016/j.tcs.2003.11.020},
  bibsource = {DBLP, http://dblp.uni-trier.de}
}
```