MITSUBISHI ELECTRIC RESEARCH LABORATORIES
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Compressive Sensing in Practice

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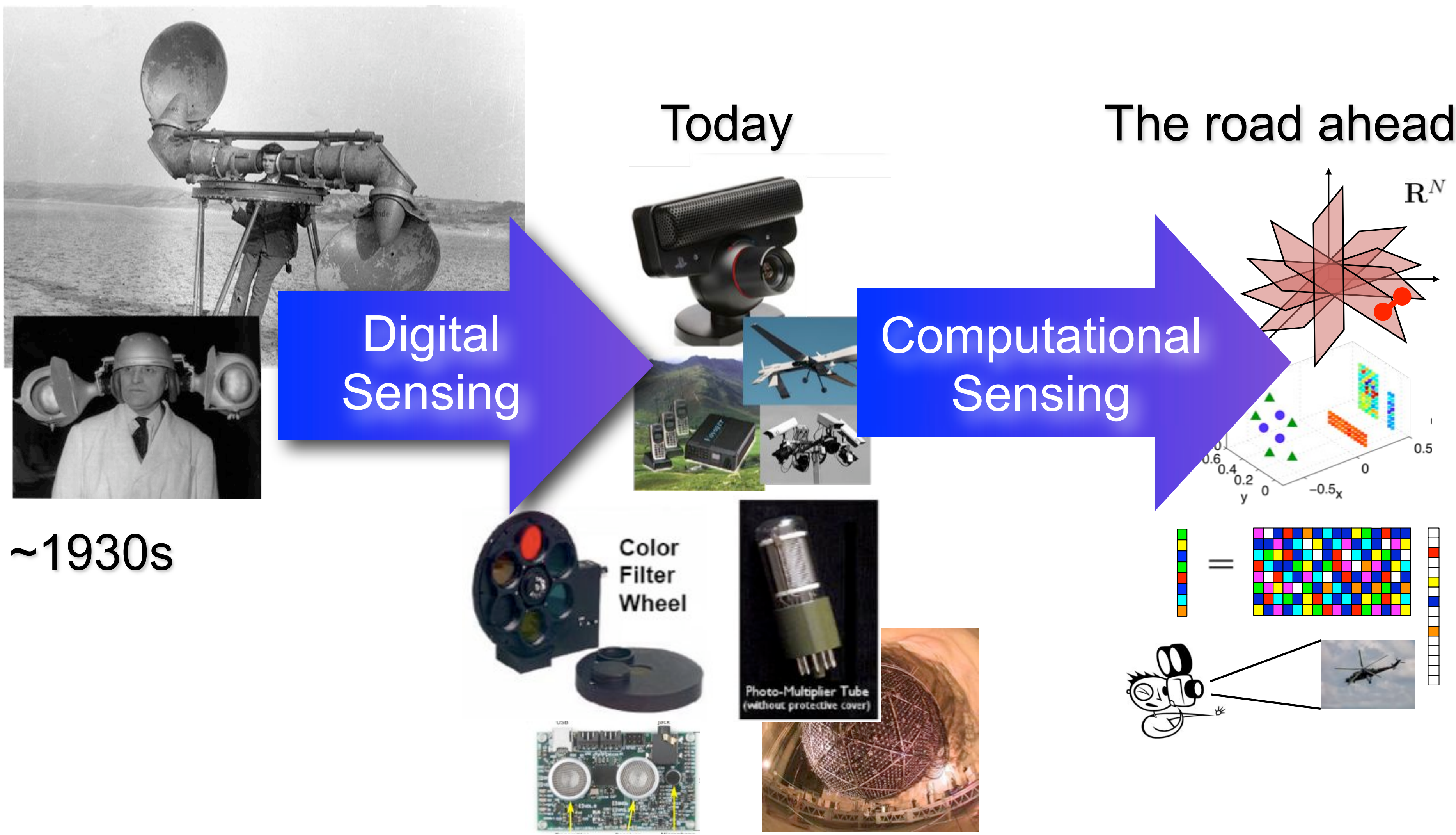
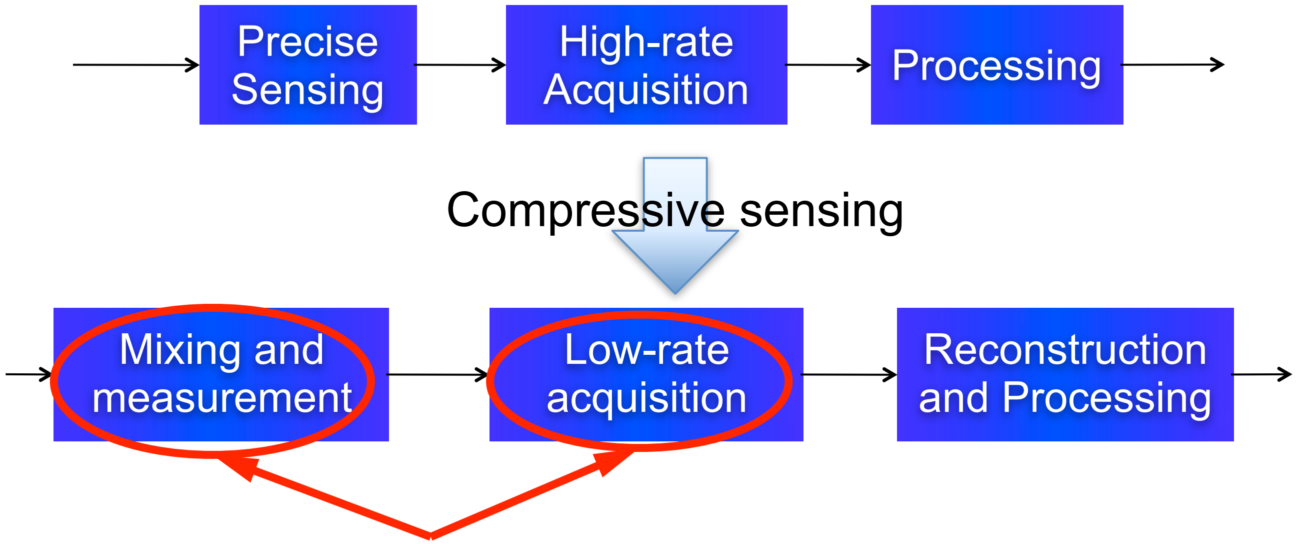
Evolution of Sensing



Sensing Pipeline Paradigm Change



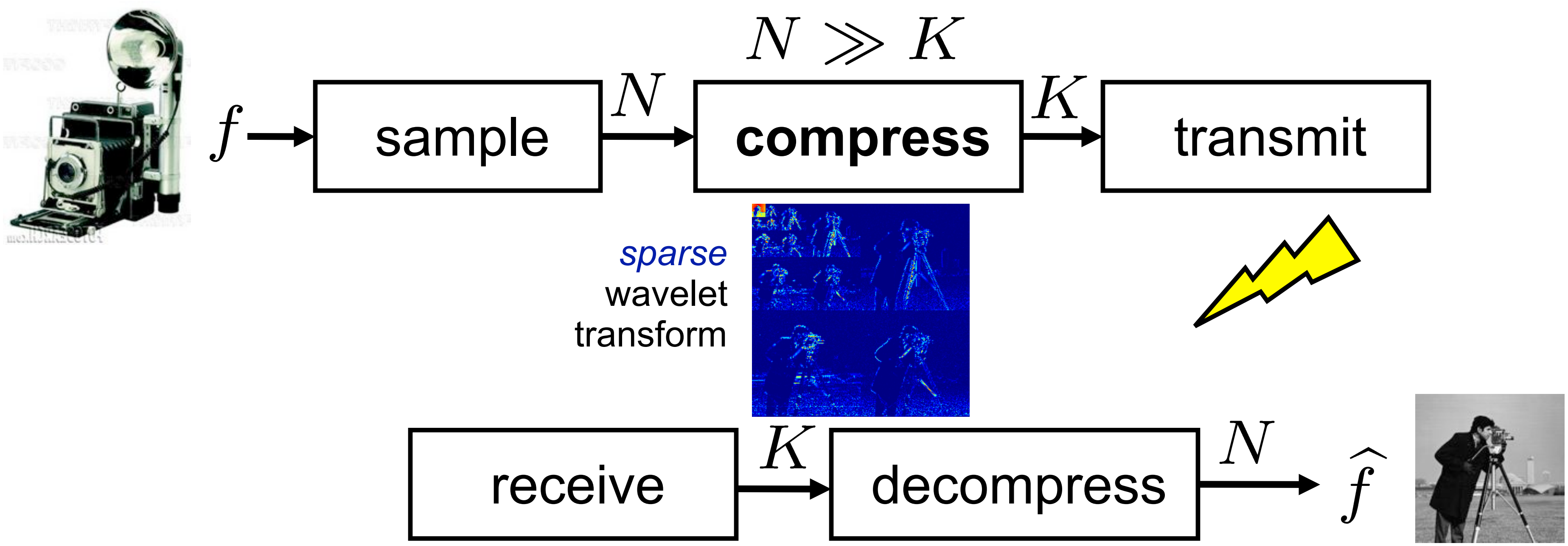
Goal: exploit mixing to **simplify sensor or improve sensor** specifications (e.g., sensor speed, A/D conversion rate, measured bandwidth/resolution)

- Compressive sensing has significantly improved our sensing capability
- Two fundamental Compressive Sensing research aspects
  - Hardware modifications for efficient acquisition
  - Signal/scene models and reconstruction algorithms

CS AT A GLANCE

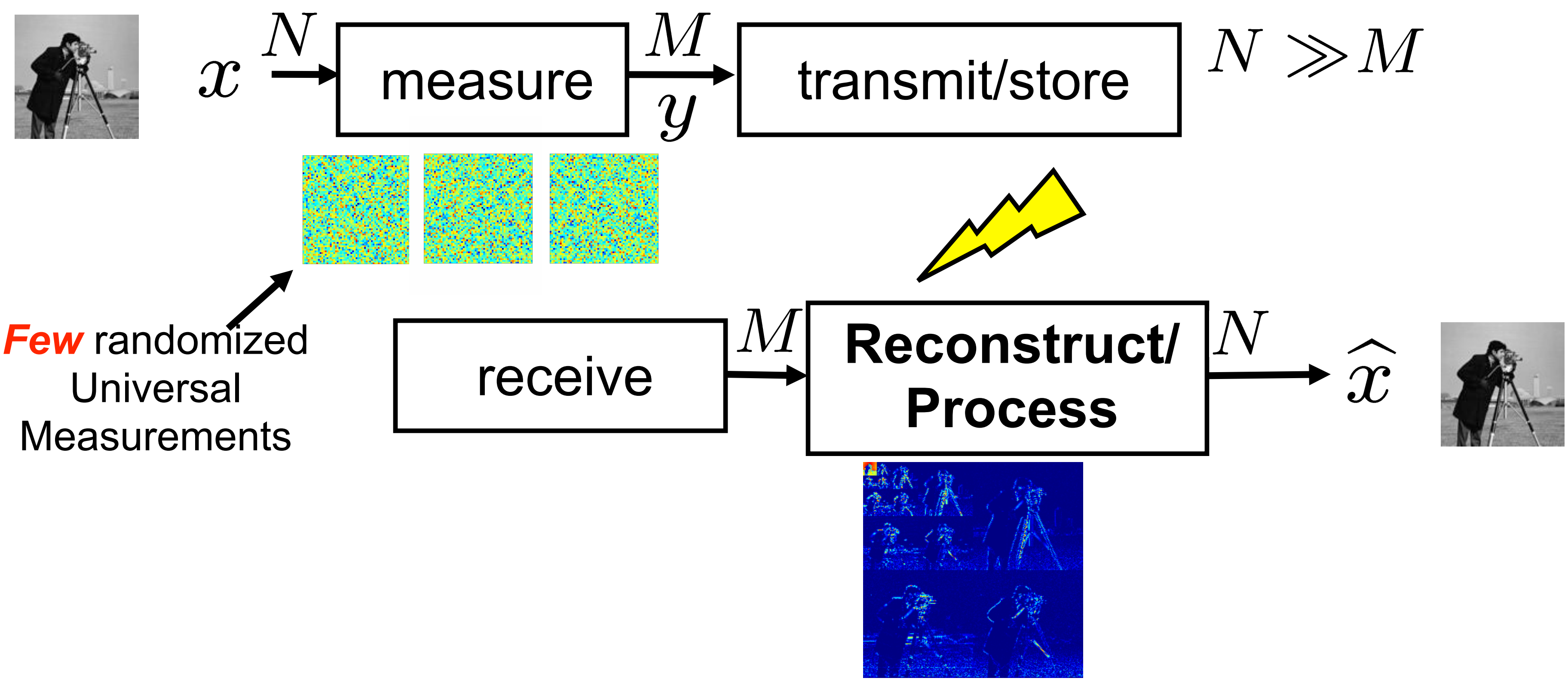
Sensing by Sampling

- Long-established paradigm for digital data acquisition
  - ***sample*** data (A-to-D converter, digital camera, …)
  - ***compress*** data (signal-dependent, nonlinear)
  - ***bottleneck*** to performance of modern acquisition systems



Compressive Sensing (CS) [Candés, Romberg, Tao; Donoho]

- New signal acquisition method
  – Samples and compresses in **one simple step**
  – Uses **computation** to reconstruct signal

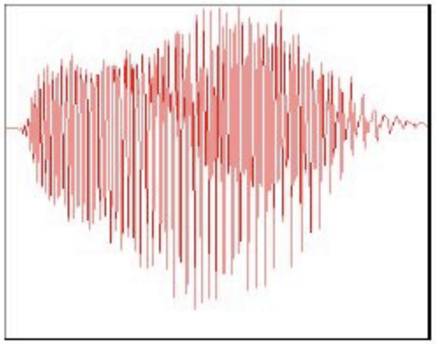


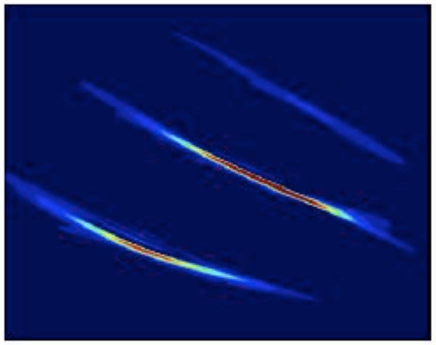
Signal Structure: Sparsity

N
pixels

$K \ll N$
large
wavelet
coefficients

N
wideband
signal
samples

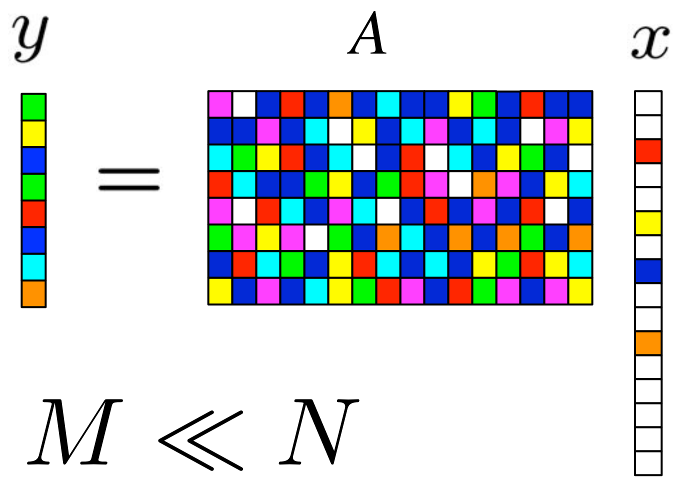




$K \ll N$
large
Gabor
coefficients

Compressed Sensing Measurement Model [Candes et al]

$y = A x$



M measurements

$N \times 1$ sparse signal

K nonzero entries

$$K < M \ll N$$

- x is K-sparse or K-compressible
- A **random**, satisfies a *restricted isometry property (RIP)*

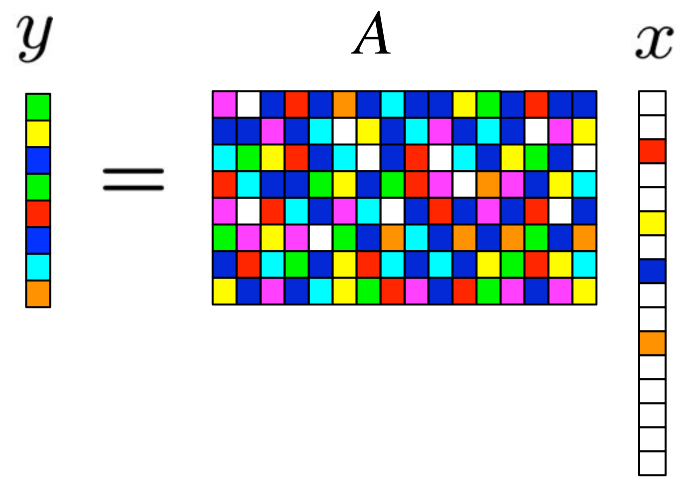
  A has RIP of order $2K$ with constant δ

  If there exists δ s.t. for all $2K$-sparse x:

  $$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\| \leq (1+\delta)\|\mathbf{x}\|_2^2$$

- $M=O(K \log N/K)$
- A also has small *coherence* $\mu \triangleq \max_{i \neq j} |\langle \mathbf{a}_i, \mathbf{a}_j \rangle|$

Compressed Sensing Measurement Model [Candes et al]



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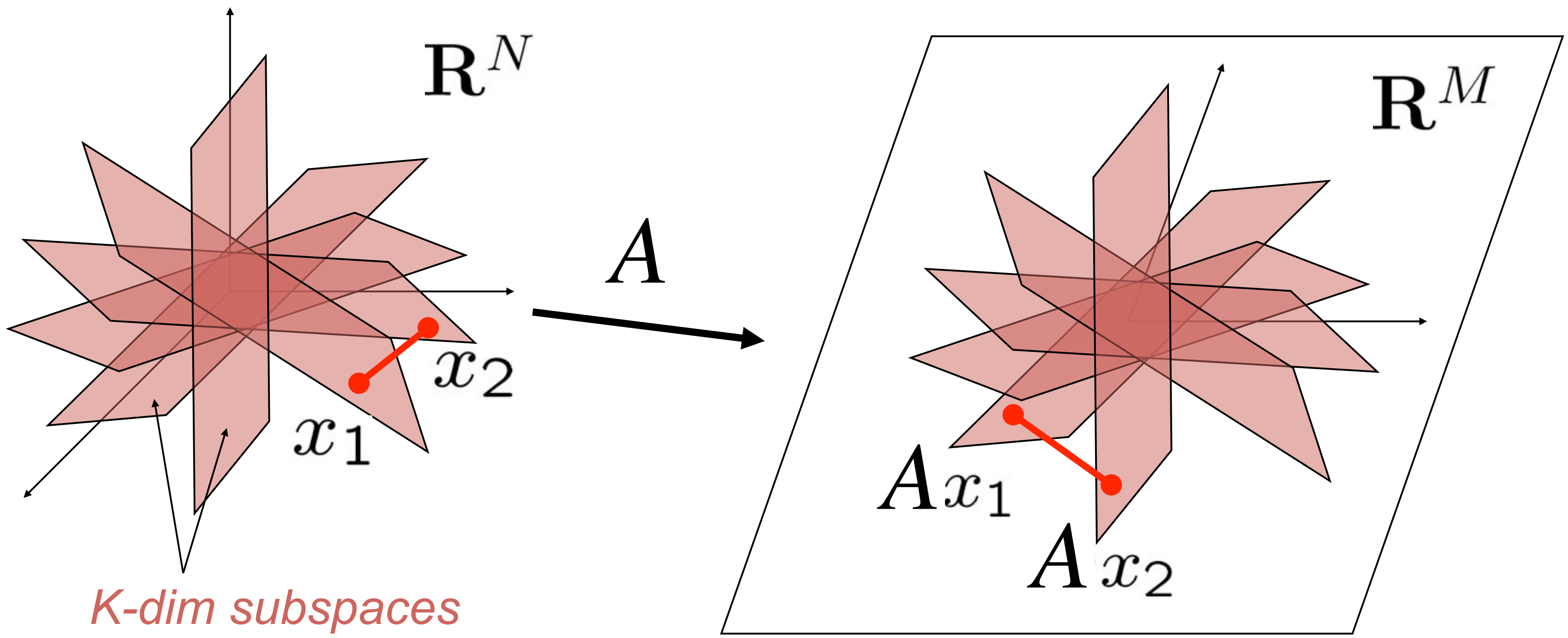
RIP/Stable Embedding

- An information preserving projection A preserves the **geometry** of the set of sparse signals



Restricted Isometry Property

$$(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2$$

CS RECONSTRUCTION

CS Reconstruction

- Reconstruction using **sparse approximation**:
  - Find sparsest $\mathbf{x}$ such that $\mathbf{y} \approx \mathbf{A}\mathbf{x}$

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ s.t. } \mathbf{y} \approx \mathbf{A}\mathbf{x}$$

- **Convex optimization** approach:
  - Minimize l_1 norm: e.g.,

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} \approx \mathbf{A}\mathbf{x}$$

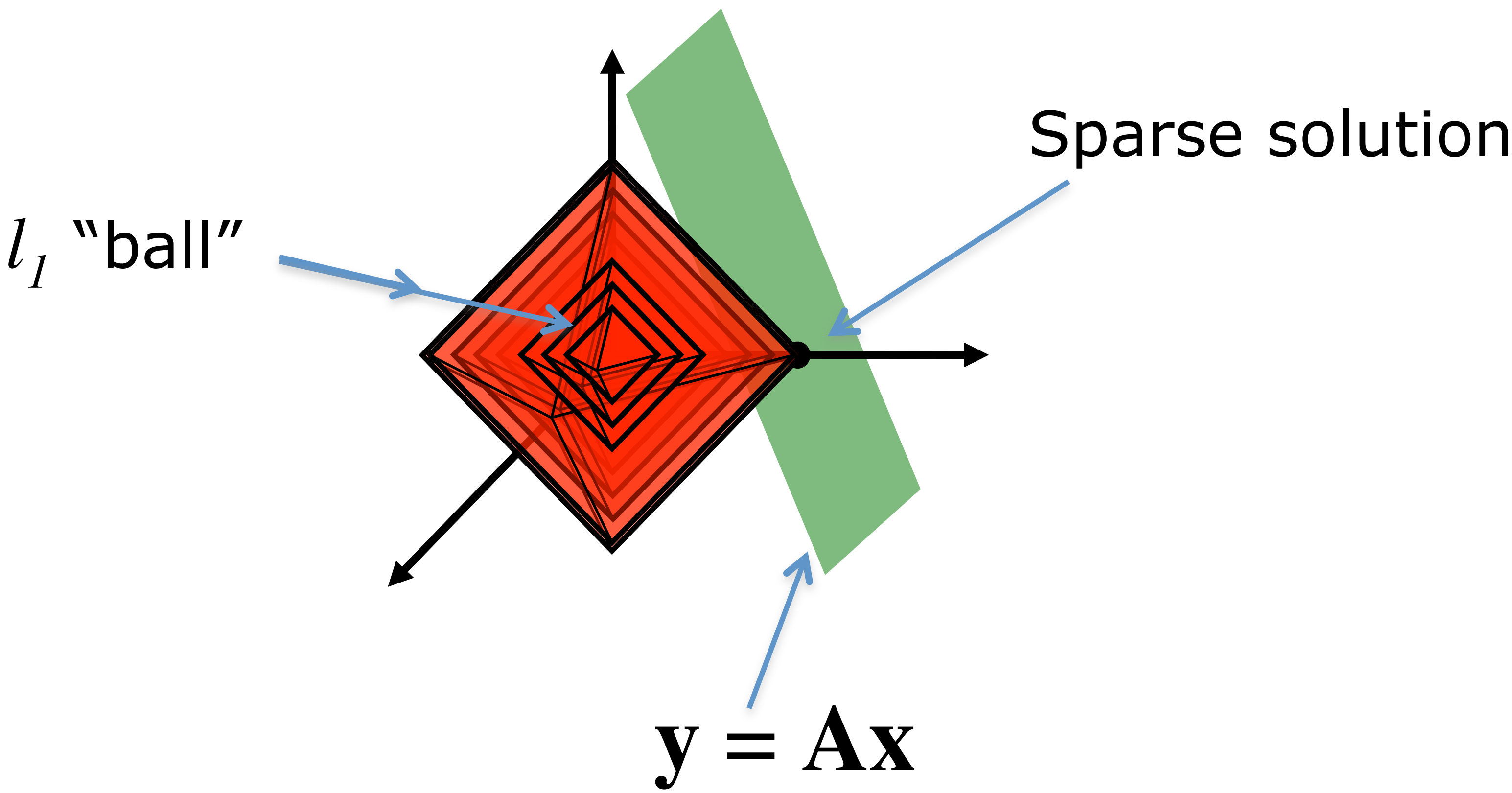
- **Greedy algorithms** approach:
  - Minimize $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2$ such that $\mathbf{x}$ is sparse

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \text{ s.t. } \|\mathbf{x}\|_0 \leq K$$

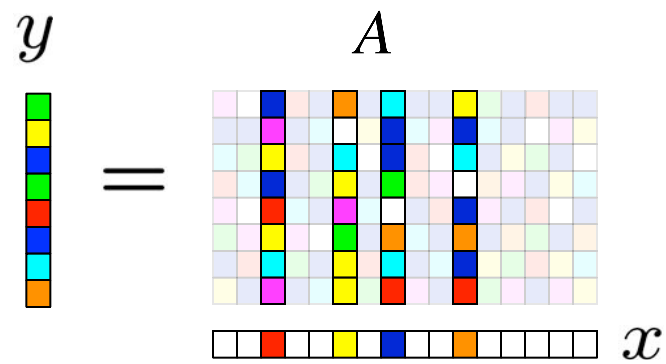
  - MP, OMP, ROMP, StOMP, CoSaMP, …
  - AndrewMP, PYAMP (Pick Your Acronym Matching Pursuit)

Why l_1 relaxation works

$$\min \; \| \mathbf{x} \|_1 \; \text{s.t.} \; \mathbf{y} \approx \mathbf{A}\mathbf{x}$$



Greedy Pursuits Core Idea



- $\boldsymbol{y}$ highly correlated with $\boldsymbol{A}$ at locations where $\boldsymbol{x}$ is high
- $\boldsymbol{A}^T\boldsymbol{y}$ provides a good idea of these locations
  - This is why low coherence is important

$$\mu \triangleq \max_{i \neq j} |\langle \mathbf{a}_i, \mathbf{a}_j \rangle|$$

  - $\boldsymbol{A}^T\boldsymbol{y}$ referred to as *proxy* for $\boldsymbol{x}$. It is also the gradient of $\|\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}\|_2^2$.

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \text{ s.t. } \|\mathbf{x}\|_0 \leq K$$

- General Strategy:
  - Identify locations
  - Invert the system only on those locations

SPARSITY-CONSTRAINED FUNCTION MINIMIZATION

Problem Formulation

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} f(\mathbf{x}) \ \text{ s.t. } \|\mathbf{x}\|_0 \leq K$$

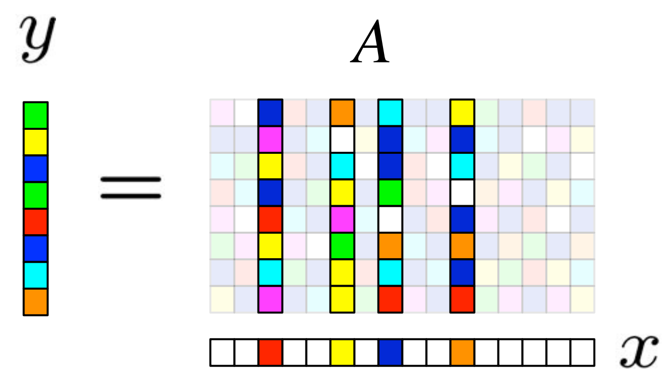
- **Objective: minimize an arbitrary cost function**

- **Applications:**
  - Sparse logistic regression
  - Quantized and saturation-consistent Compressed Sensing
  - De-noising and Compressed Sensing with non-gaussian noise models

- **Questions:**
  - What **algorithms** can we use?
  - What **functions** can we minimize?
  - What are the **conditions** on $f(\mathbf{x})$?
  - What **guarantees** can we provide?

Commonalities in Sparse Recovery Algorithms

- Most greedy and l_1 algorithms have several common steps:
  - **Maintain** a current **estimate**
  - **Compute** a **residual**
  - **Compute** a gradient, **proxy**, correlation, or some other name
  - **Update estimate** based on proxy
  - **Prune** (soft or hard threshold)
  - **Iterate**



- Key step: proxy/correlation $\mathbf{A}^T(\mathbf{y}-\mathbf{A}\mathbf{x})$
  - This is the **gradient** of $f(\mathbf{x})=\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$
  - Can we substitute it with the general gradient $\nabla f(\mathbf{x})$?

YES

We can provide **strong guarantees!**

We can generalize the **RIP!**

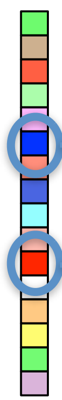
GraSP (Gradient Subspace Purŝuit) [w/ Bahmani, Raj]

State Variables: Signal estimate, **x** support estimate: *T*

Initialize estimate and support: $\hat{\mathbf{x}}=0$, $T=\mathrm{supp}(\hat{\mathbf{x}})$

Compute Gradient at Current Estimate

$$\nabla f(\widehat{\mathbf{x}}) = \mathbf{g}$$



Select location of largest *2K* gradient directions

$$\mathrm{supp}(\mathbf{g}|_{2K})$$

Add to support set

$$\Omega = \mathrm{supp}(\mathbf{g}|_{2K}) \cup T$$

Minimize over support

$$\mathbf{b} = \arg\min_{\mathbf{x}} f(\mathbf{x})$$
$$\text{s.t. } \mathbf{x}_{\Omega^c} = 0$$

Truncate result

$$\widehat{\mathbf{x}} = \mathbf{b}|_K$$
$$T = \mathrm{supp}\left(\mathbf{b}|_K\right)$$

Iterate using residual

GraSP Properties

- Iteration Guarantee:

$$\|\widehat{\mathbf{x}}^{(i)} - \mathbf{x}^{\star}\|_2 \leq 2^{-i}\|\mathbf{x}^{\star}\|_2 + \frac{4\left(2+\sqrt{\frac{3}{2}}\right)}{\epsilon}\|\nabla f\left(\mathbf{x}^{\star}\right)|_{\mathcal{I}}\|_2$$

Error in i^{th} iteration

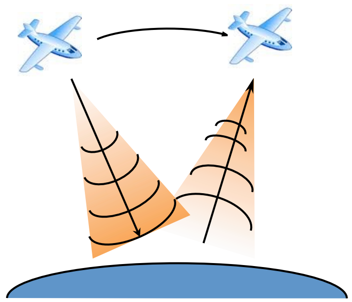
Except for a fixed approximation error

Reduces by half in each iteration

- Connections to Compressive Sensing
  - CS uses $f(\mathbf{x})=\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$
  - General conditions on $f(\mathbf{x})$ (**SHP**) that reduce to the **RIP**
  - **GraSP** reduces to **CoSaMP**
  - Reconstruction guarantees reduce to classical CS guarantees

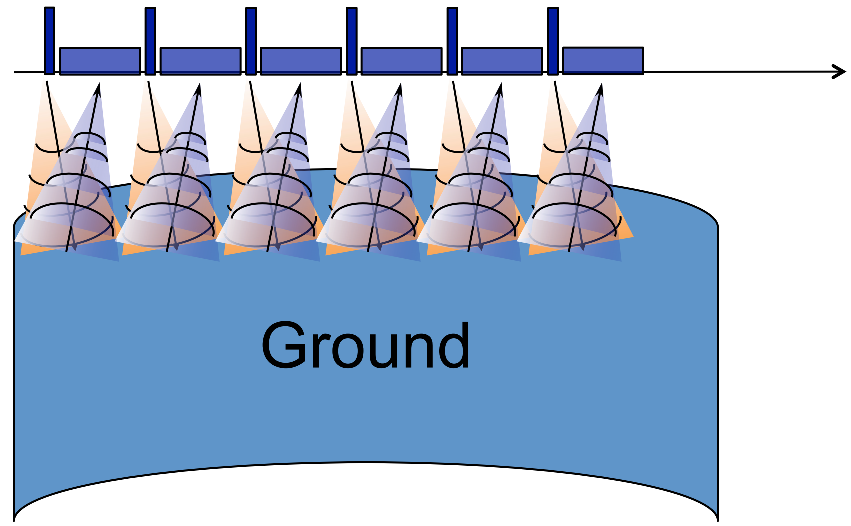
SYNTHETIC APERTURE RADAR

SAR Acquisition Model



Received Data: y

SLC Image: x

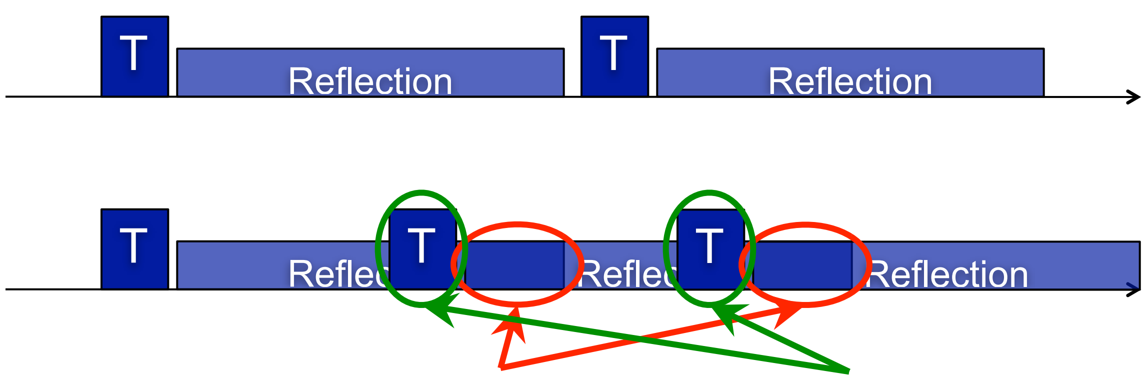


SAR Acquisition Linear Equation: $y = A\,x$

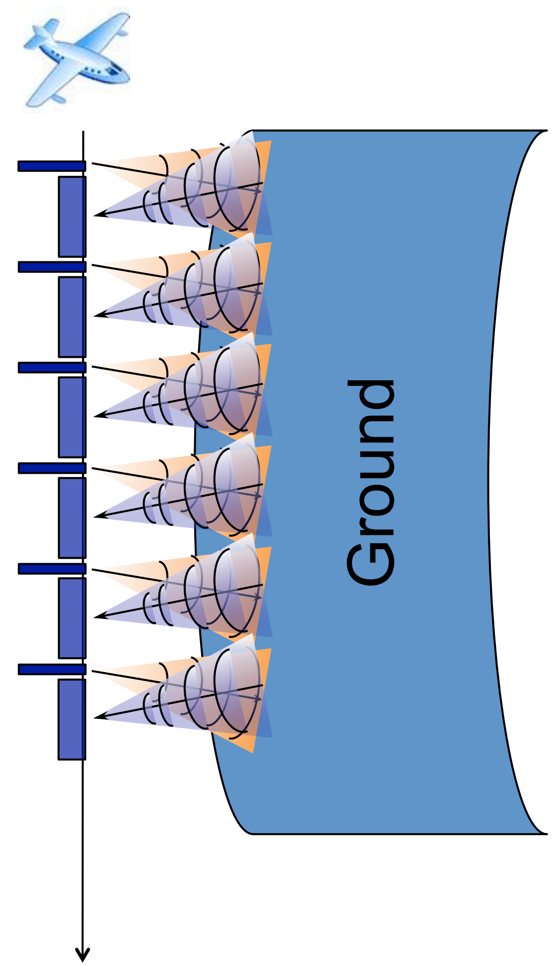
- SAR Acquisition follows linear model
- Acquisition function (A) determined by SAR parameters
  - Pulse shape/rate
  - Doppler bandwidth (beamwidth)
  - Moving platform trajectory
- Image formation: given y determine x.

Classical SAR pulse timing

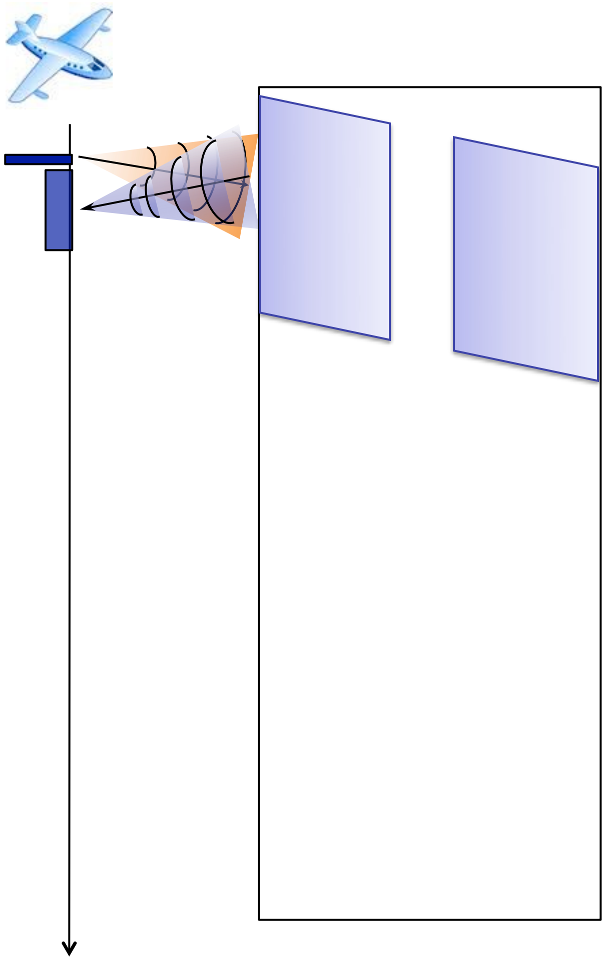
- SAR beamwidth (Doppler bandwidth) dictates azimuth resolution
  - The higher the bandwidth, the better.
- Higher Doppler bandwidth requires higher PRF
- Reflection duration depends on range length
  - Reflection interference limits maximum PRF
  - Increasing PRF reduces the range we can image



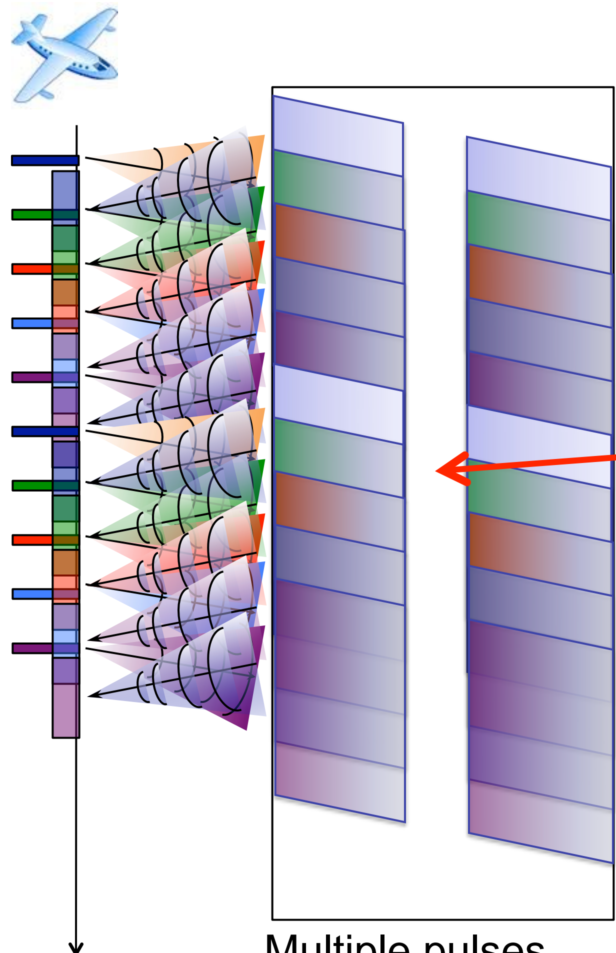
Higher PRF causes **interference** and **missing data**



Ground Coverage: Uniform Pulsing, High PRF



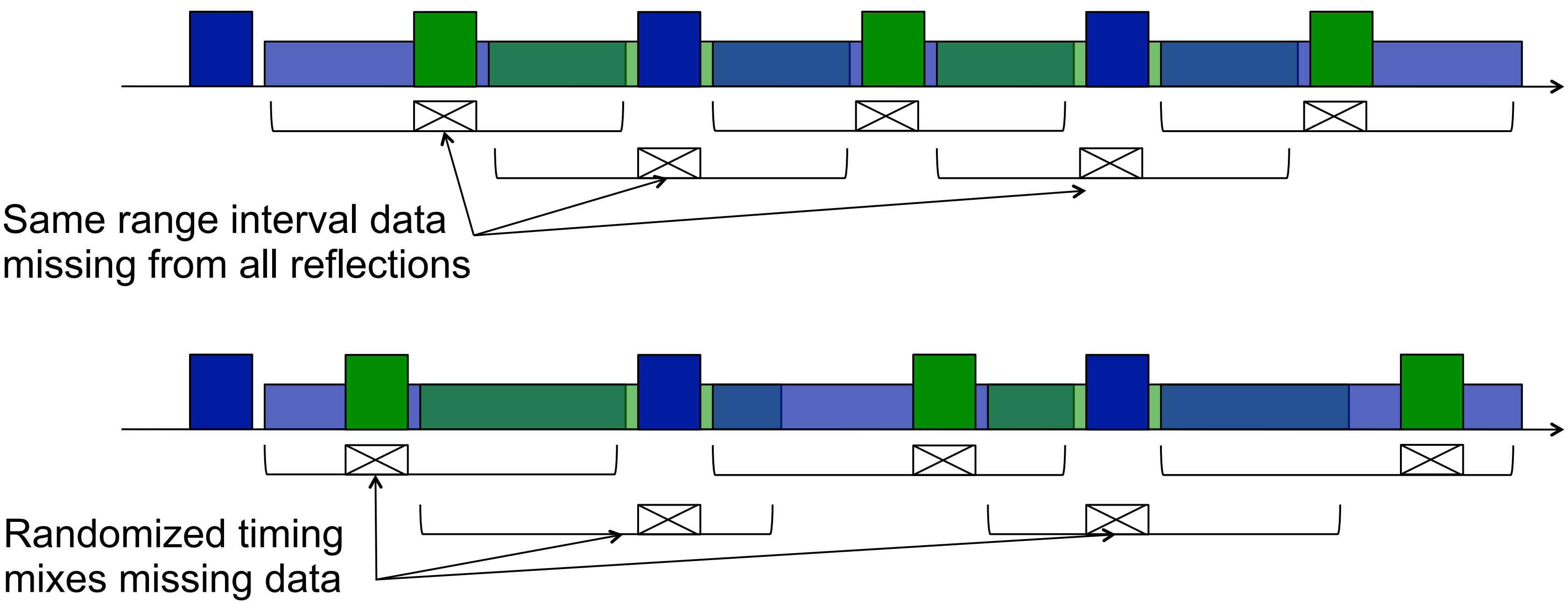
Single pulse w/ missing data



Multiple pulses, uniformly spaced

SAR pulsing and timing [w/ Liu]

- Issue: missing data always in the same range interval
  - Produces black spots in the image
  - Even robust algorithms cannot fill in with such pattern of missing data
  - Ideally, missing data should be in different interval for every azimuth line
- Solution: Randomized pulsing interval



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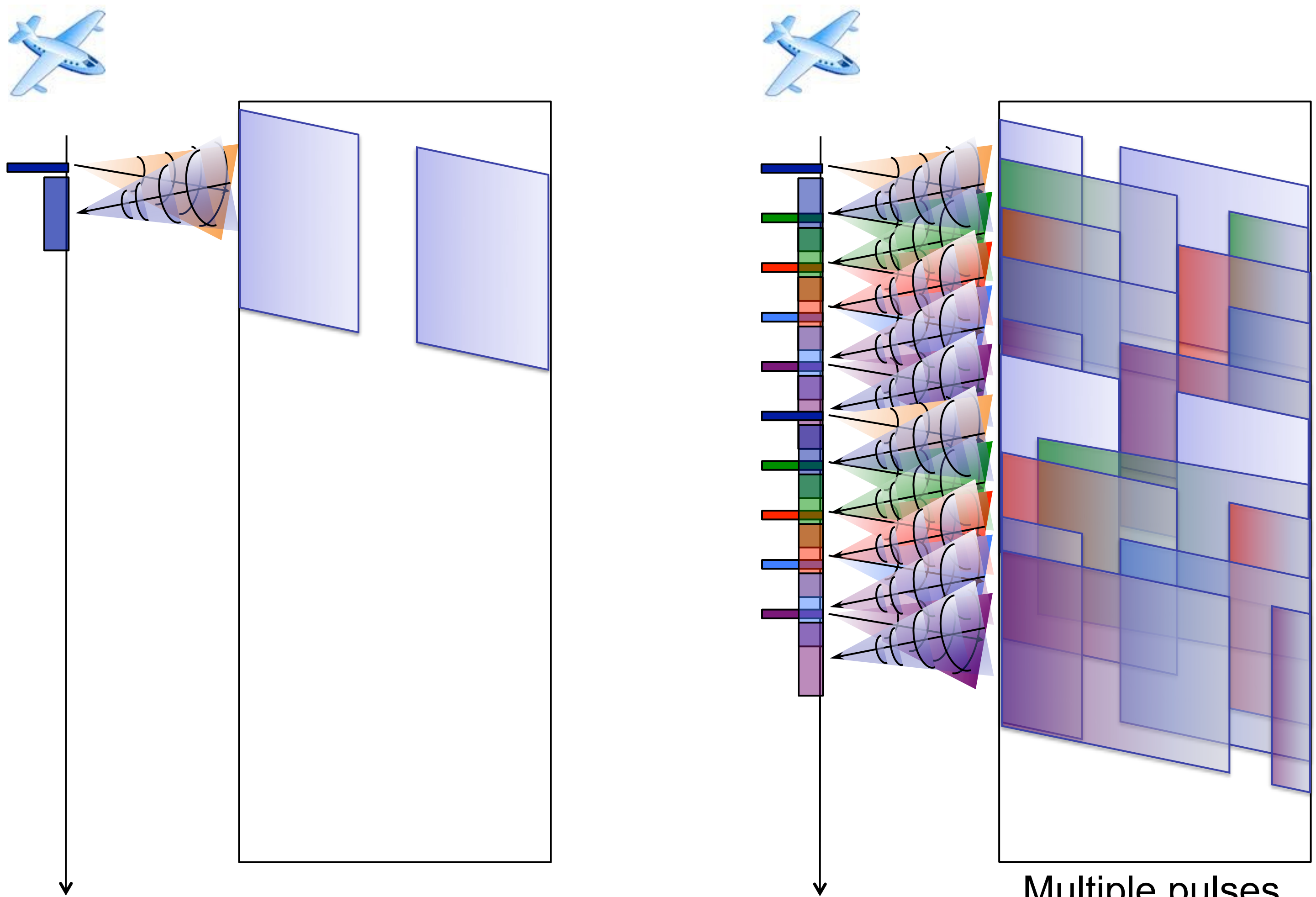
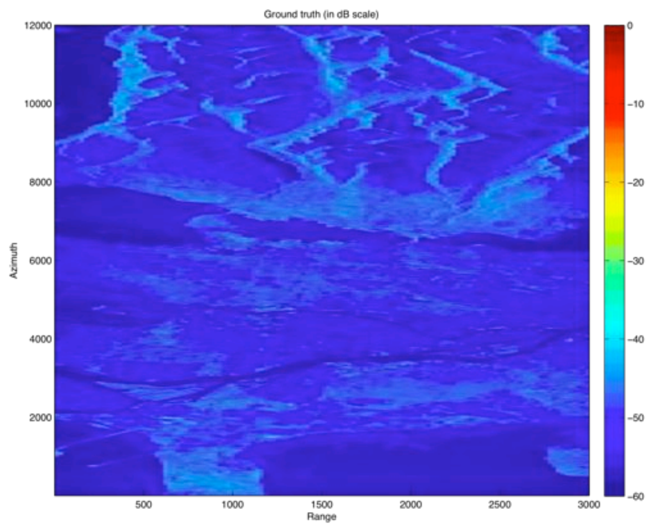


Single pulse w/ missing data

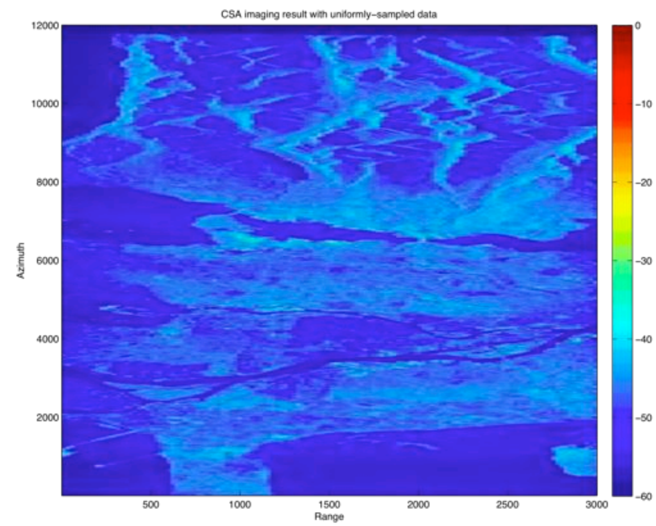
Multiple pulses,
Non-uniformly spaced

Simulation results

True Ground Reflectivity



Uniform pulsing, Small PRF, Small Doppler Bandwidth



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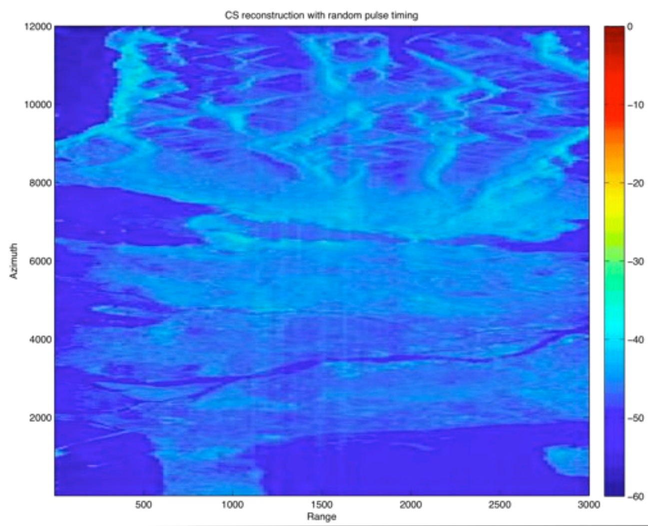

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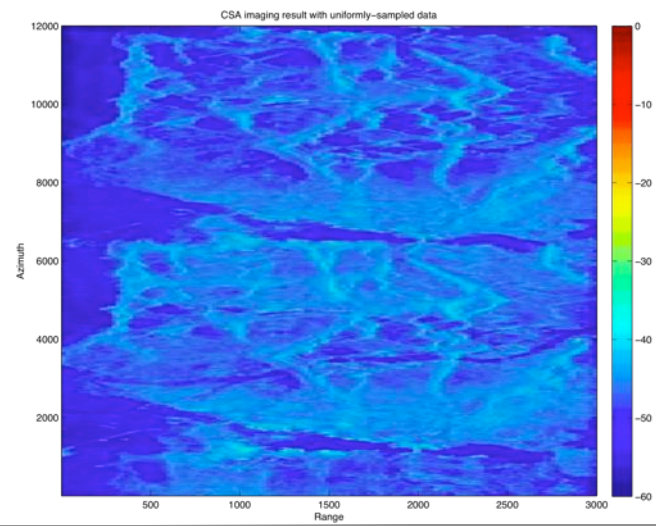

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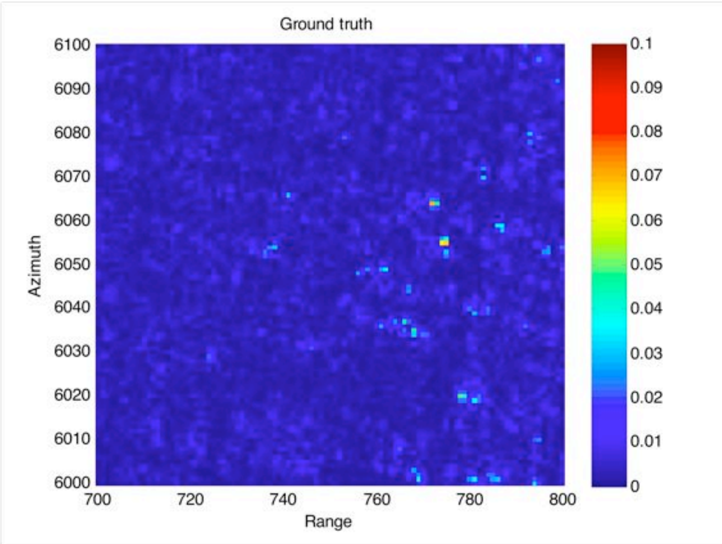

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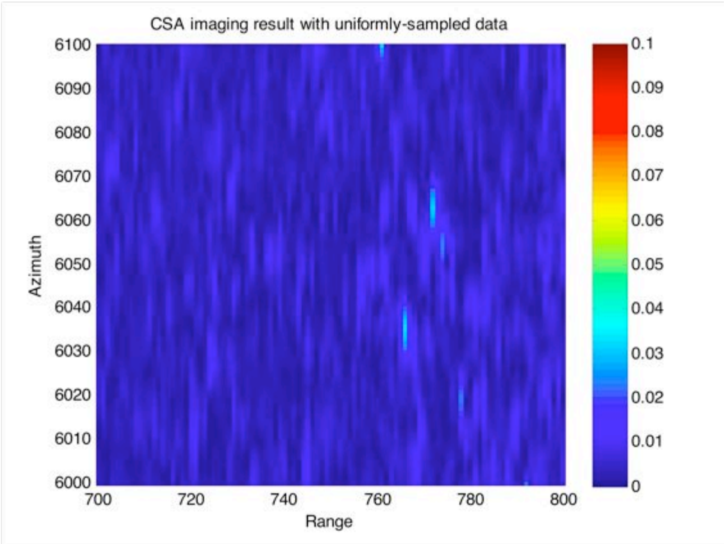

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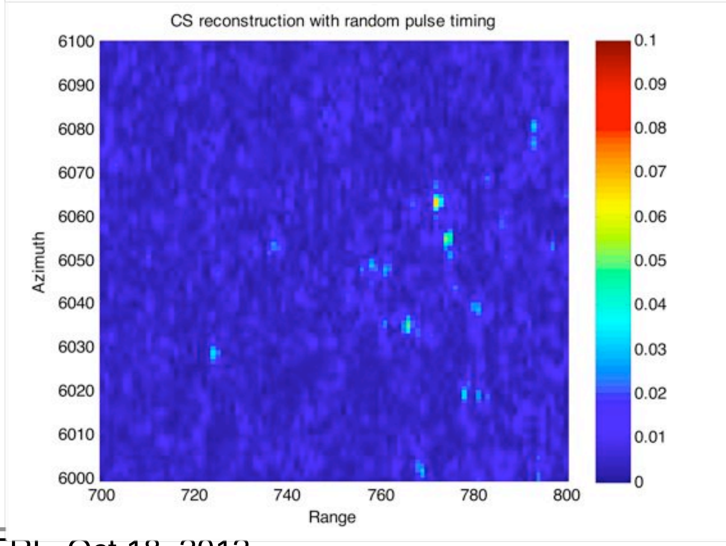

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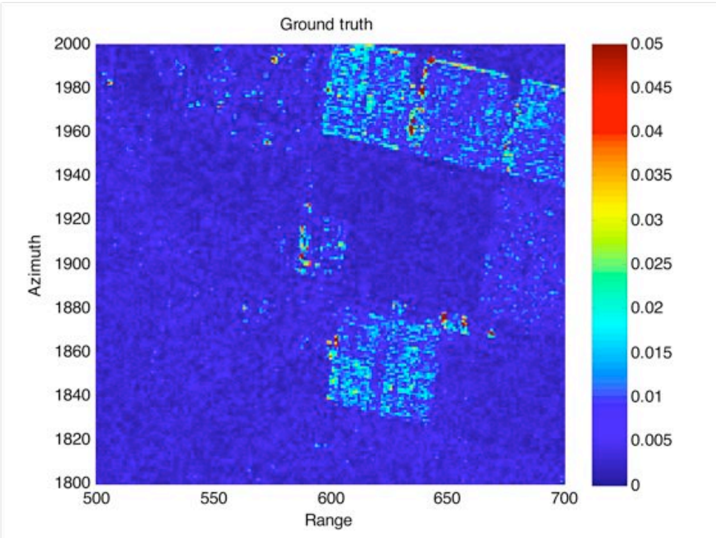

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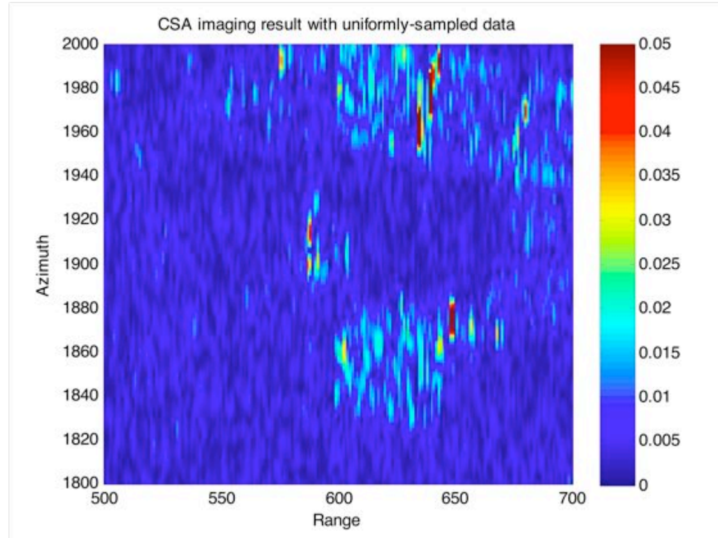

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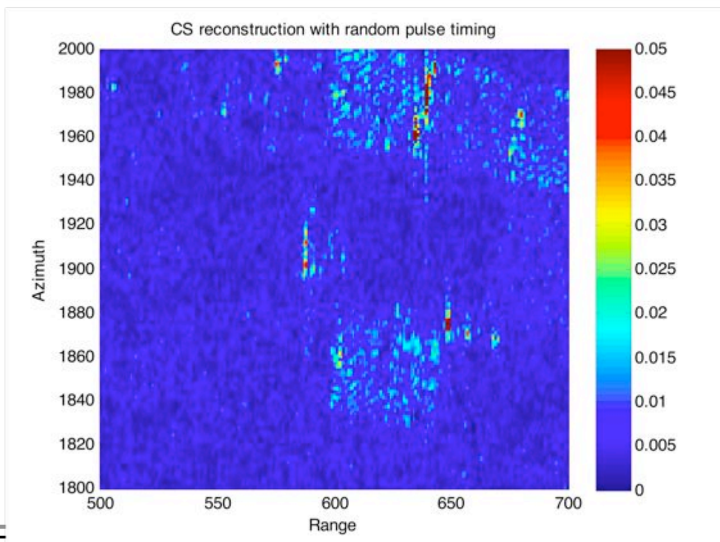
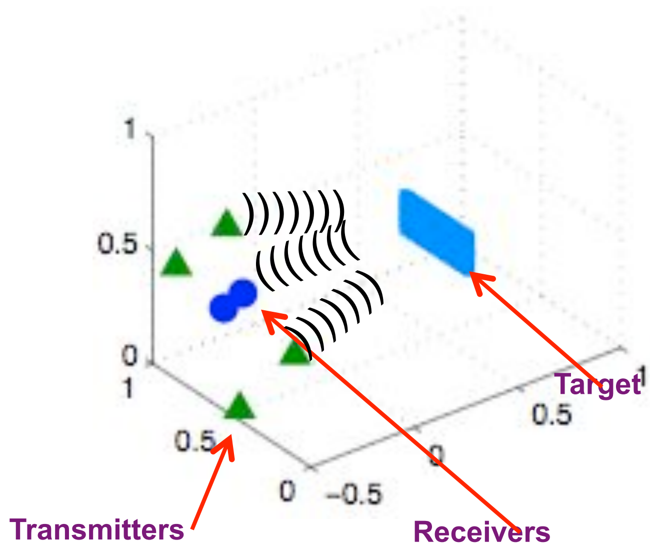


DEPTH SENSING

Depth Sensing

- Coherent Active Depth Sensing
  - Ultrasonic, mmWave, other modalities
- Goal: Illuminate the scene and sense reflections



Scene Reflectivity

Everything in **front of target is zero**

Everything **behind target** invisible (i.e. **zero**)

Scene is sparse!

Depth Sensing

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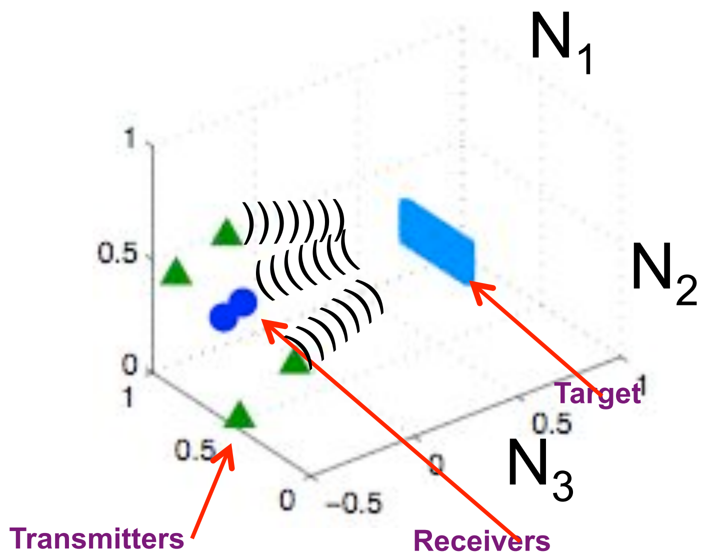


Discretization

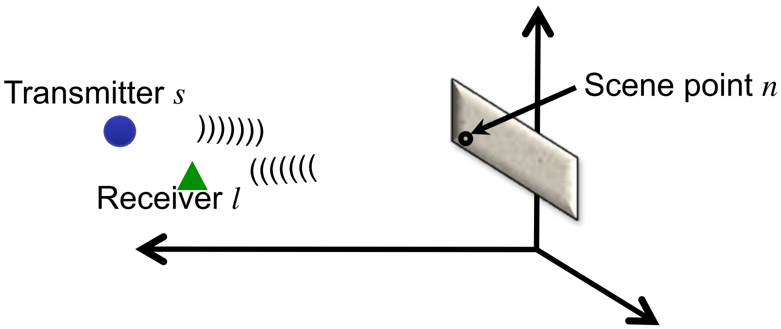
Scene Size: $N = N_1 x N_2 x N_3$

(# of gridpoints in scene)

Sparsity: $K \leq N_1 x N_2 < N$

Scene is sparse!

Modeling



Reflectivity of scene point n (signal of interest):	x_n
Pulse transmitted by transmitter s (freq. domain):	$P_{s,f}$
Signal received by receiver l (freq. domain):	$R_{l,f}$
Distance of transmitter s to scene point n:	$d_{s,n}$
Distance of receiver l to scene point n:	$d_{n,l}$
Speed of sound:	c
Time delay for distance d:	d/c
Time delay from s to l through n:	$\tau_{s,l,n}=(d_{s,n}+d_{n,l})/c$

S transmitters, L receivers, N scene points (scene discretized), F transmitted frequencies

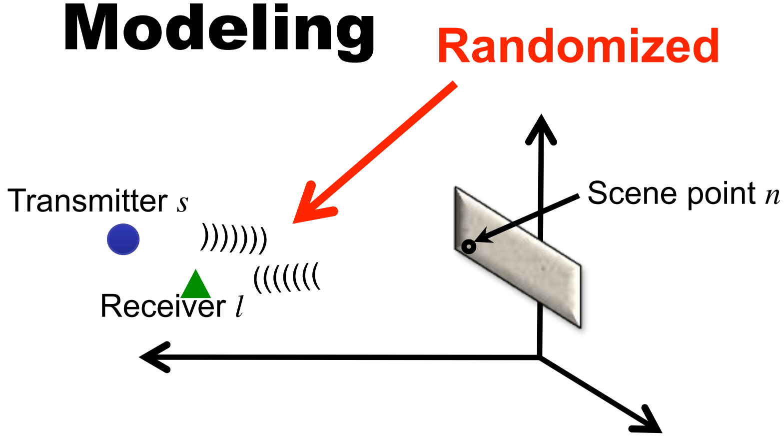
Propagation equation: $$R_{l,f} = \sum_n \left(\sum_s P_{s,f} e^{-j\omega_f \tau_{s,l,m}} \right) x_n$$

Discretizing in Frequency and converting to matrix form:

$$\mathbf{r} = \begin{bmatrix} R_{1,1} \\ \vdots \\ R_{1,F} \\ \vdots \\ R_{l,f} \\ \vdots \\ R_{L,1} \\ \vdots \\ R_{L,F} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \sum_s P_{s,1} e^{-j\omega_1 \tau_{s,1,1}} & \cdots & \sum_s P_{s,1} e^{-j\omega_1 \tau_{s,1,N}} \\ \vdots & \ddots & \vdots \\ \sum_s P_{s,F} e^{-j\omega_F \tau_{s,1,1}} & \cdots & \sum_s P_{s,F} e^{-j\omega_F \tau_{s,1,N}} \\ \vdots & \ddots & \vdots \\ \sum_s P_{s,f} e^{-j\omega_f \tau_{s,l,1}} & \cdots & \sum_s P_{s,f} e^{-j\omega_f \tau_{s,l,N}} \\ \vdots & \ddots & \vdots \\ \sum_s P_{s,1} e^{-j\omega_1 \tau_{s,L,1}} & \cdots & \sum_s P_{s,1} e^{-j\omega_1 \tau_{s,L,N}} \\ \vdots & \ddots & \vdots \\ \sum_s P_{s,F} e^{-j\omega_F \tau_{s,L,1}} & \cdots & \sum_s P_{s,F} e^{-j\omega_F \tau_{s,L,N}} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\omega_f = \frac{2\pi f}{F}$$

Modeling



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S transmitters, L receivers, N scene points (scene discretized), F transmitted frequencies

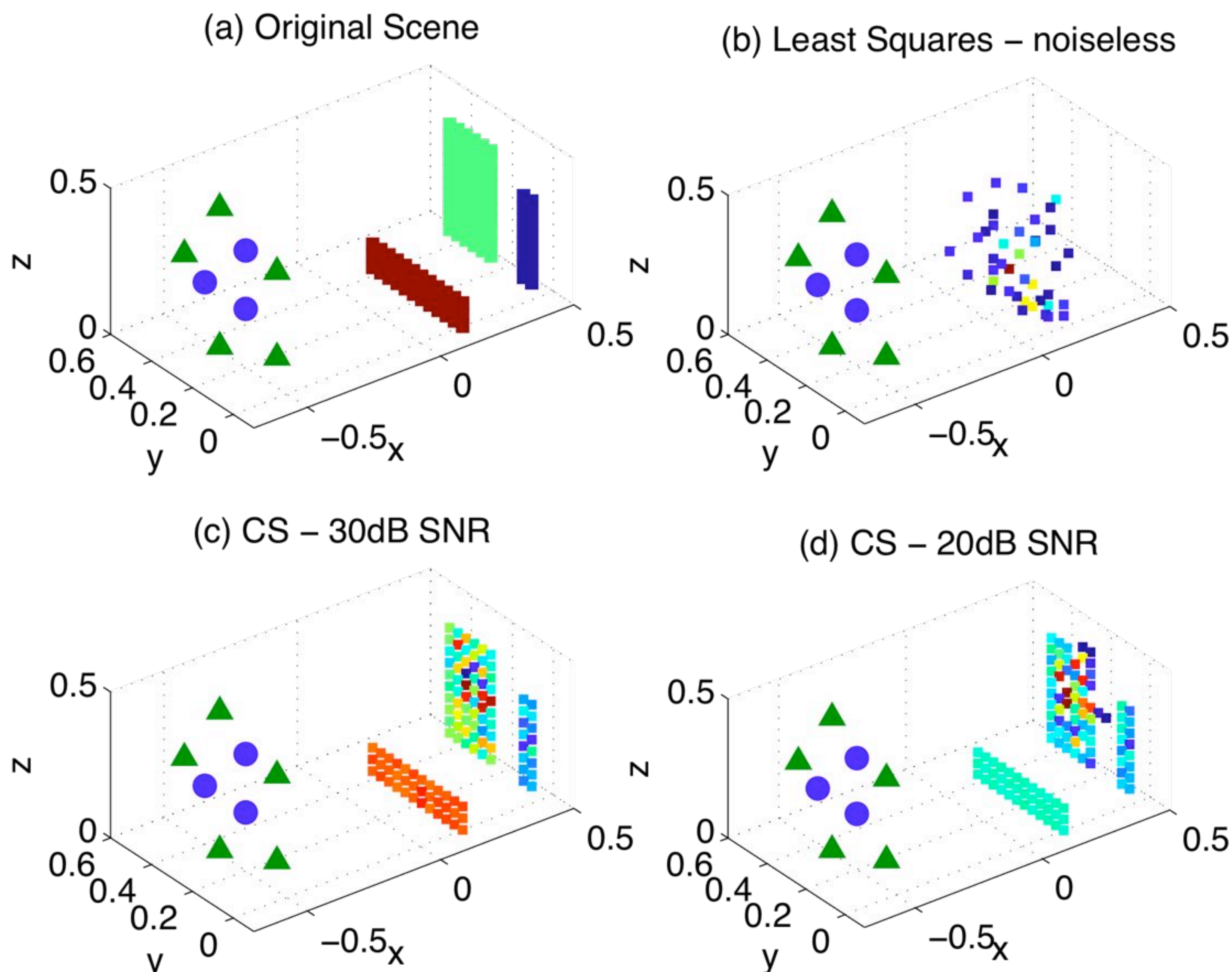
$$\mathbf{r} = \mathbf{A}\mathbf{x}$$

Received data — $\mathbf{r}$

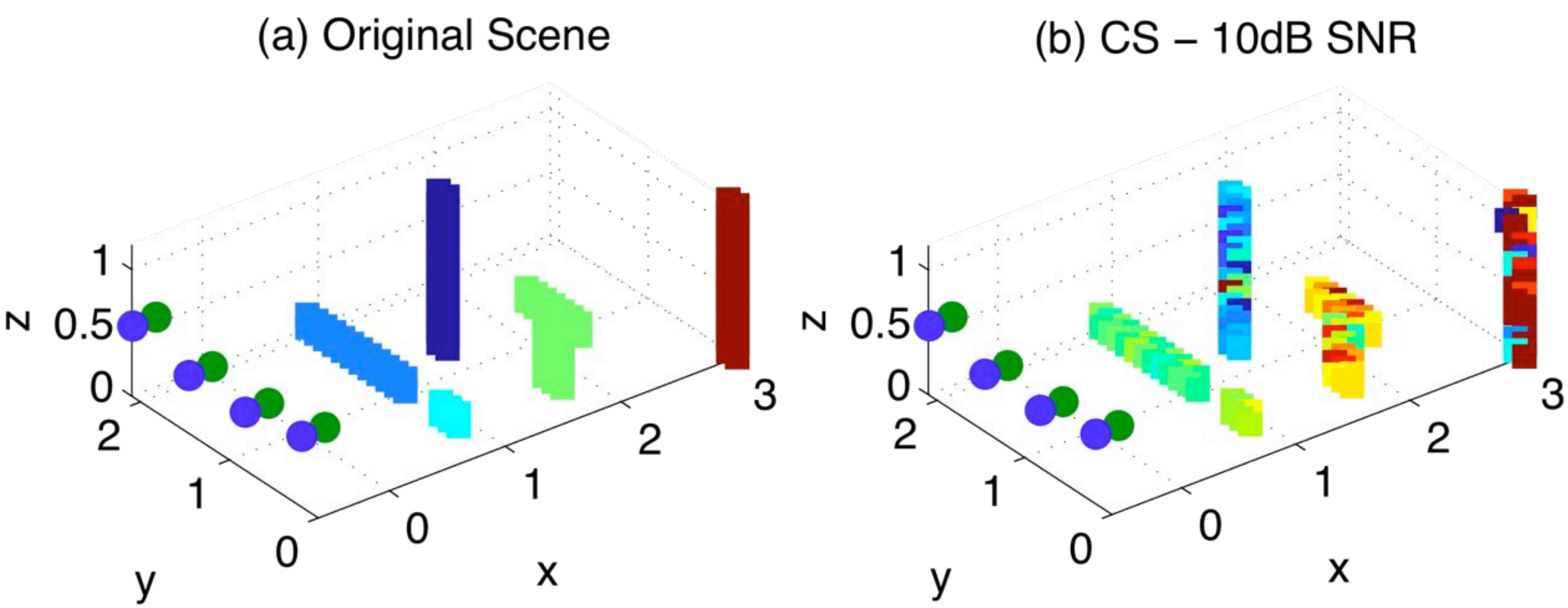
Sensing Matrix (determined by pulse shape) — $\mathbf{A}$

Scene (sparse) — $\mathbf{x}$

Simulation Results: Ultrasonic Array

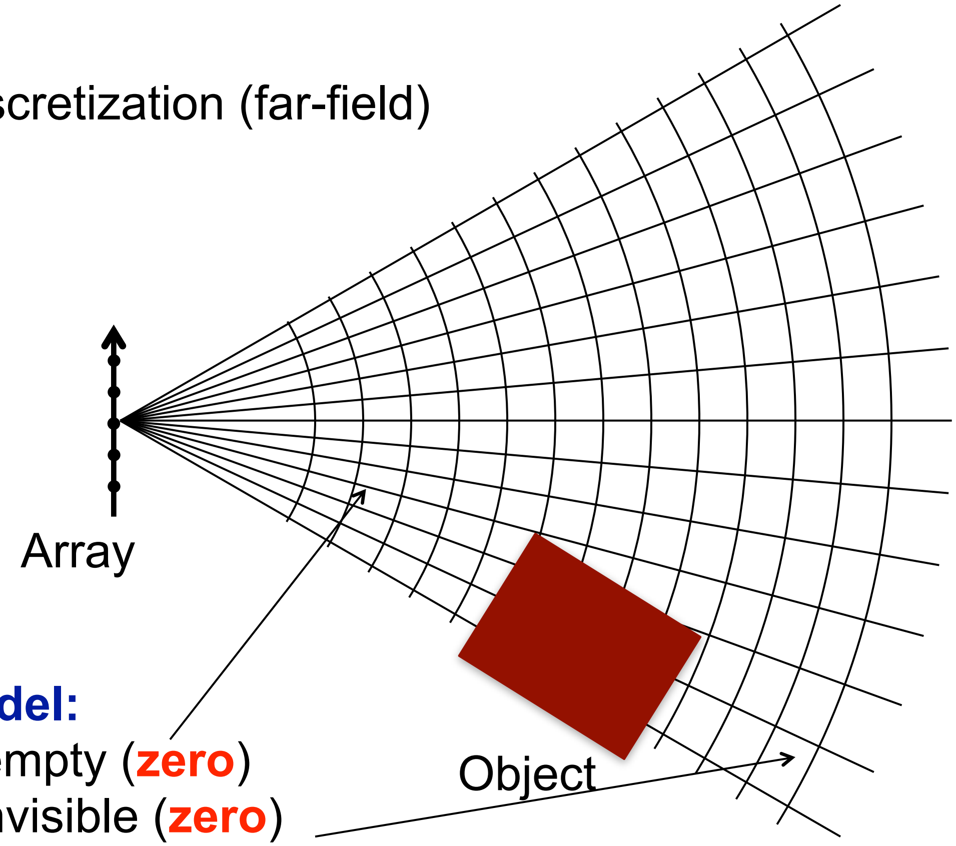


Simulation Results: Virtual Array



Signal Model

Scene discretization (far-field)



Scene (signal) model:
Front of objects is empty (**zero**)
Rear of objects is invisible (**zero**)

Q: Can we exploit the scene model beyond sparsity?

A: YES! Model Based Compressed Sensing [Baraniuk et. al.]

Model Based Compressed Sensing

- Model-based Compressed Sensing [Baraniuk et. al.]
  - Enables model-based reconstruction
  - Modifies existing greedy CS algorithms such as CoSaMP
  - Provides theoretical analysis

- Fundamental operation: Model-based Thresholding
  - Replaces hard thresholding in standard algorithms
  - Enforces model instead of simple sparsity

- Challenge: Determine appropriate thresholding operation

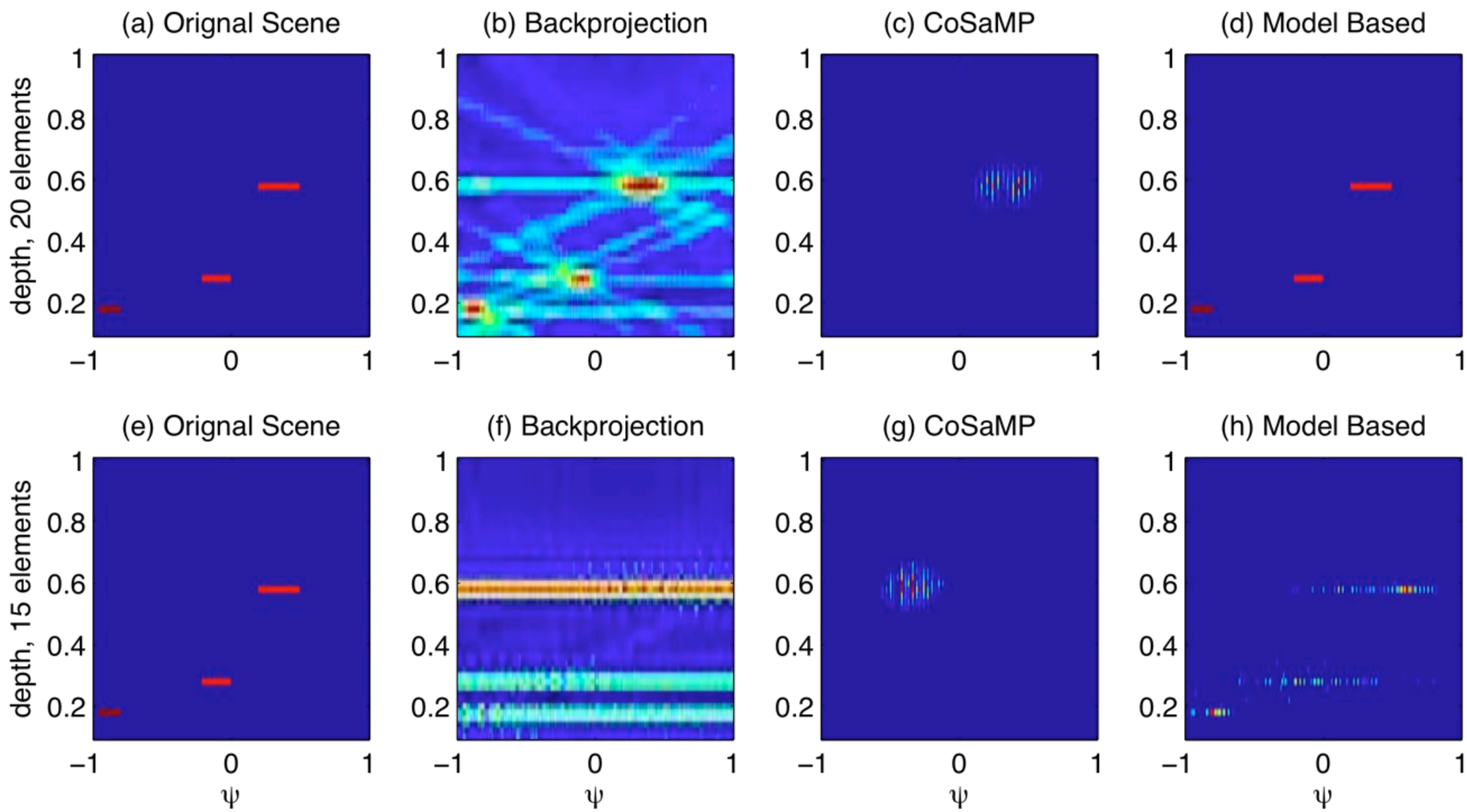
Example: mmWave Radar Simulation

- Operating Frequency: 76-77GHz
  - (Specs from: http://www.mitsubishielectric.com/bu/automotive/advanced_technology/pdf/vol94_tr5.pdf)

- Simulation in 2D-field (easier to visualize results)
  - Assuming uniform linear array
  - We expect 3D results to be similar

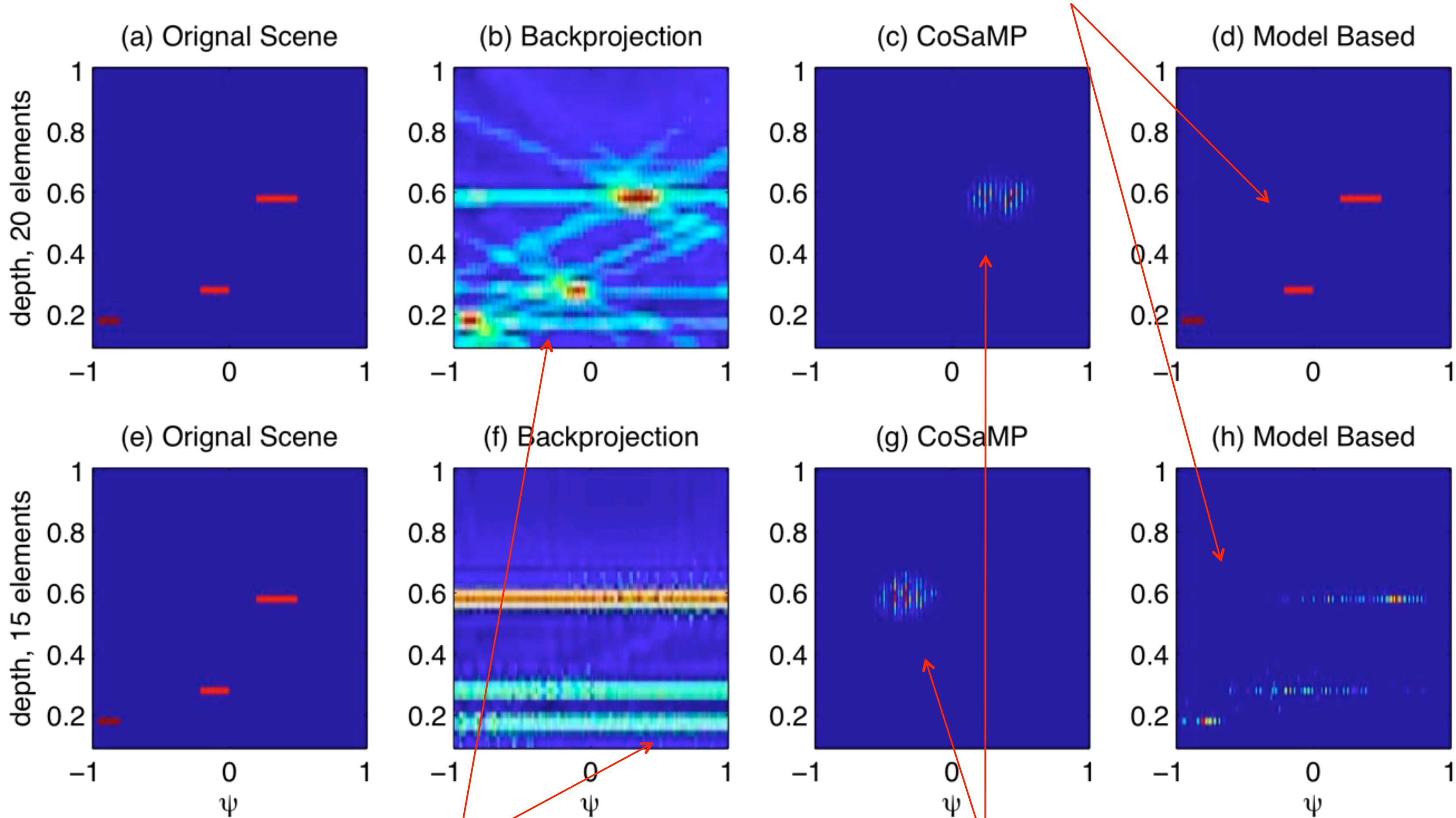
- Compared three approaches
  - Classical backprojection (beamforming)
  - Standard Compressive Sensing
  - Model-based Compressive Sensing

Simulation results – mmWave radar



Simulation results

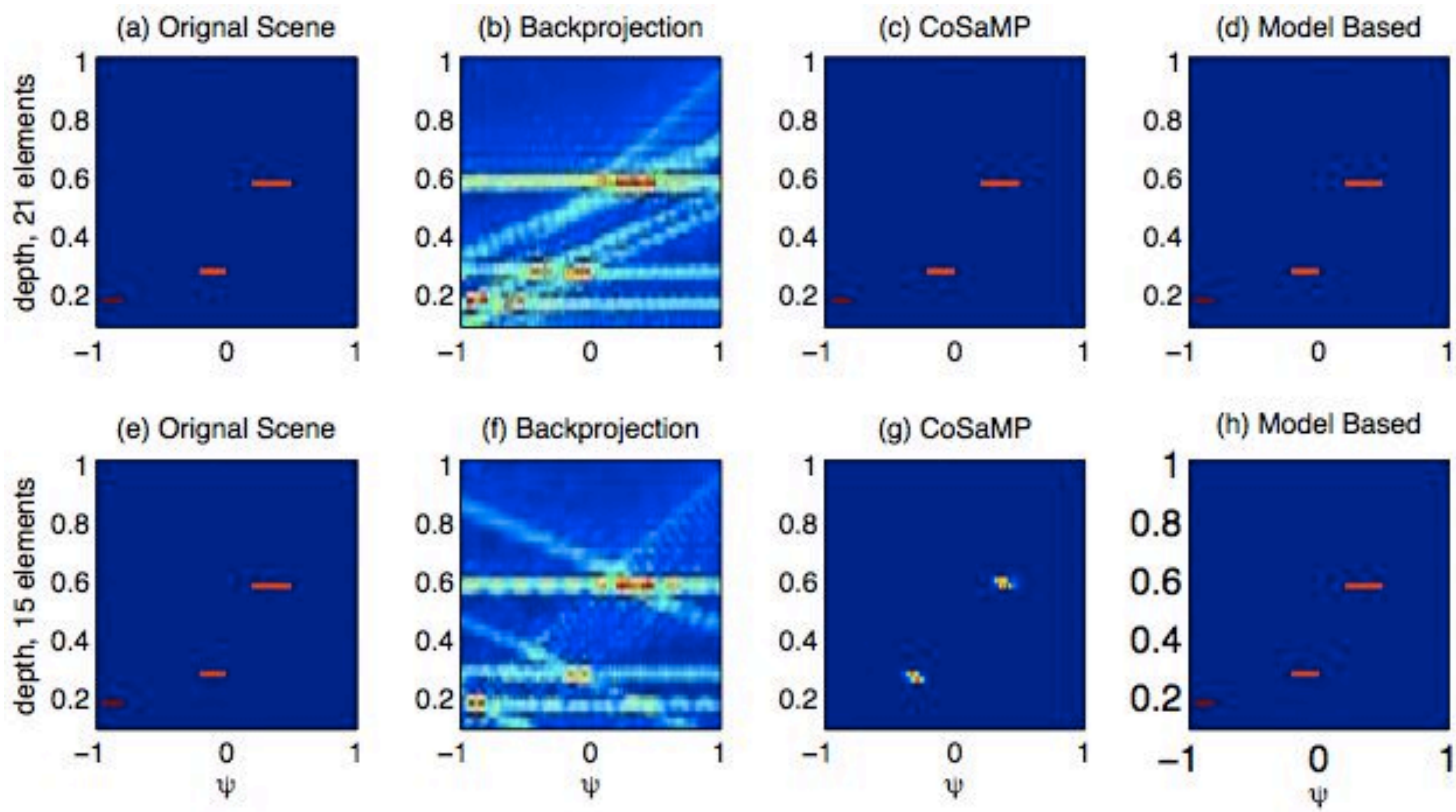
Model enforcement improves reconstruction significantly, even with significant blur



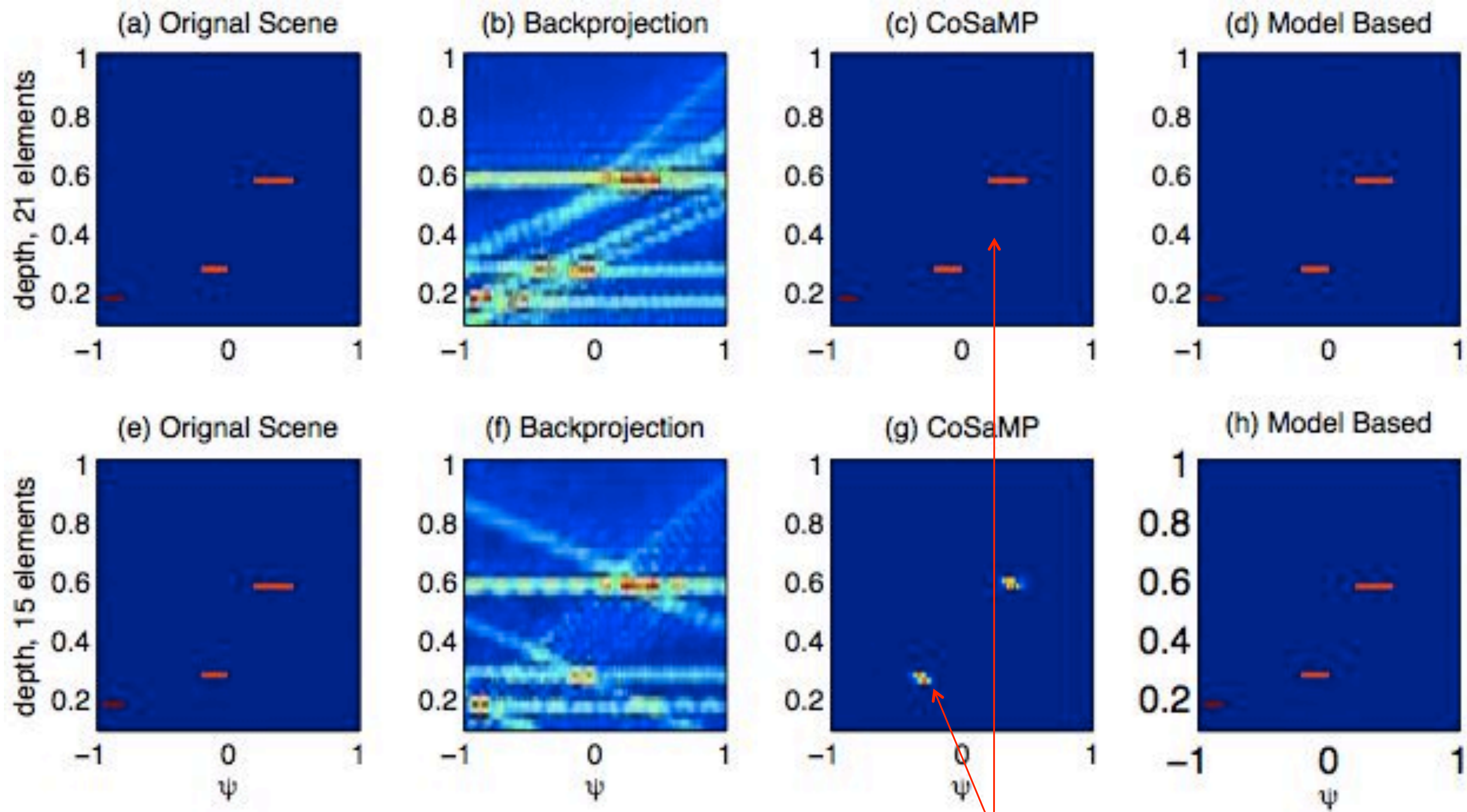
Backprojection (beamforming) exhibits significant blur, especially as array elements are reduced

Blur also confuses Classical CS algorithms

Simulation results – randomized spacing



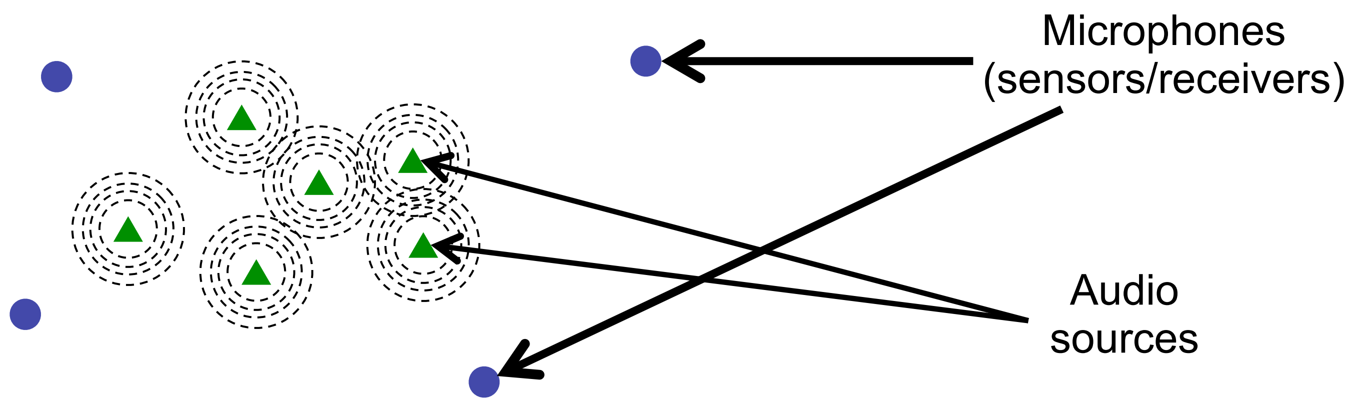
Simulation results – randomized spacing



CoSaMP performance
improves, but not perfect

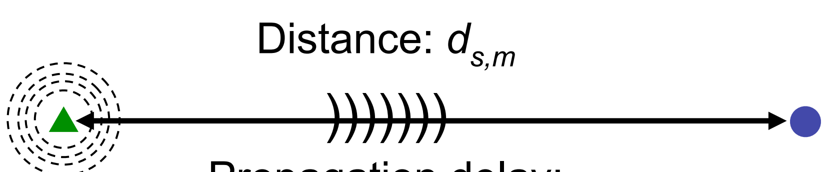
MICROPHONE ARRAYS

Problem at a Glance [w/ Raj, Smaragdis]



- Sources and sensors are wideband (e.g., audio)
- Few sources; source signals not known but broadband
- Sensor location is known

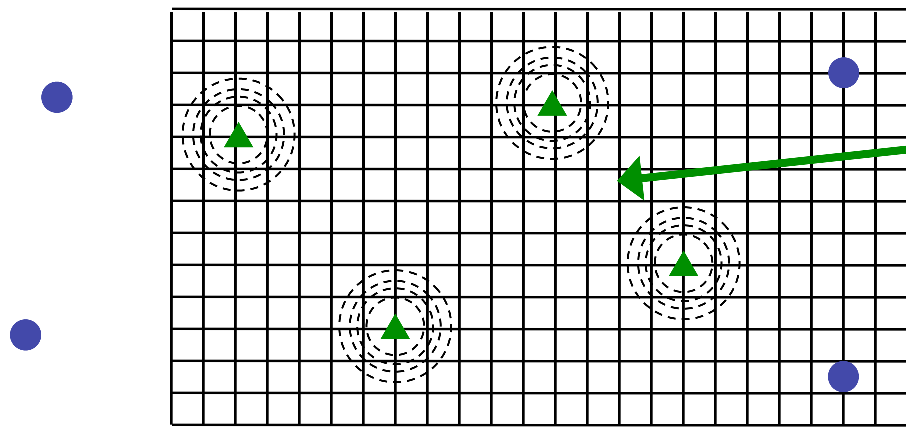
Frequency-Domain Transmission Equation:



Received Signal, Source Signal, Propagation Delay

$$Y_m(\omega) = e^{-j\omega\tau_{s,m}} X_s(\omega)$$

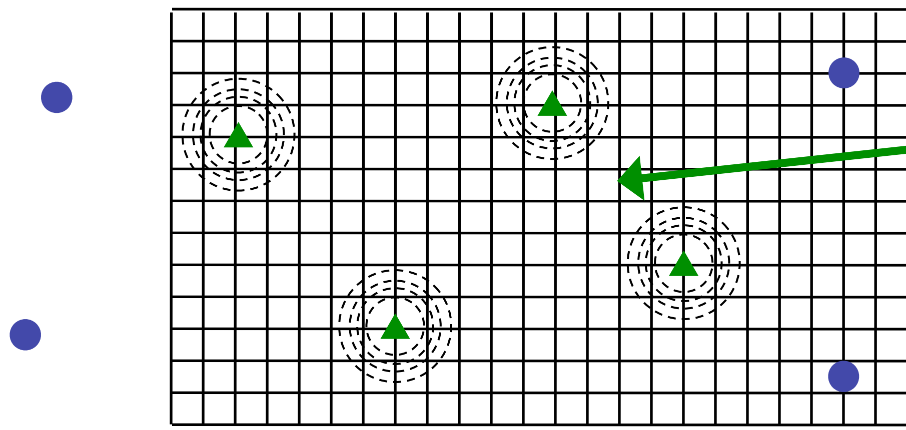
System Model



Discretized Scene:
N Grid points
Each a potential source

- Discrete grid of scene and potential source locations
- N Grid points, any could be a source location
- Scene sparsity: S actual sources
- M microphones (sensors); can be in/out/on/off the grid
- Sensor geometry assumed known
- Distances and delays can be calculated

System Model

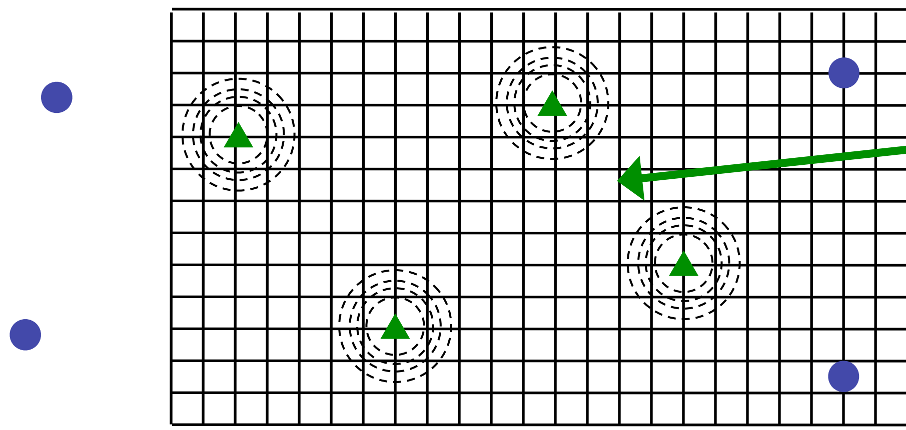


$$\mathbf{Y}(\omega) = \begin{bmatrix} Y_1(\omega) \\ \vdots \\ Y_M(\omega) \end{bmatrix} \quad \mathbf{X}(\omega) = \begin{bmatrix} X_1(\omega) \\ \vdots \\ X_N(\omega) \end{bmatrix}$$

$$\mathbf{A}(\omega) = \begin{bmatrix} e^{-j\omega\tau_{1,1}} & \cdots & e^{-j\omega\tau_{1,N}} \\ \vdots & \ddots & \vdots \\ e^{-j\omega\tau_{M,1}} & \cdots & e^{-j\omega\tau_{M,N}} \end{bmatrix}$$

$$\mathbf{Y}(\omega) = \mathbf{A}(\omega)\mathbf{X}(\omega)$$

System Model



Discretized Scene:
N Grid points
Each a potential source

$\mathbf{X}(\omega)$ is sparse: very few locations contain sources

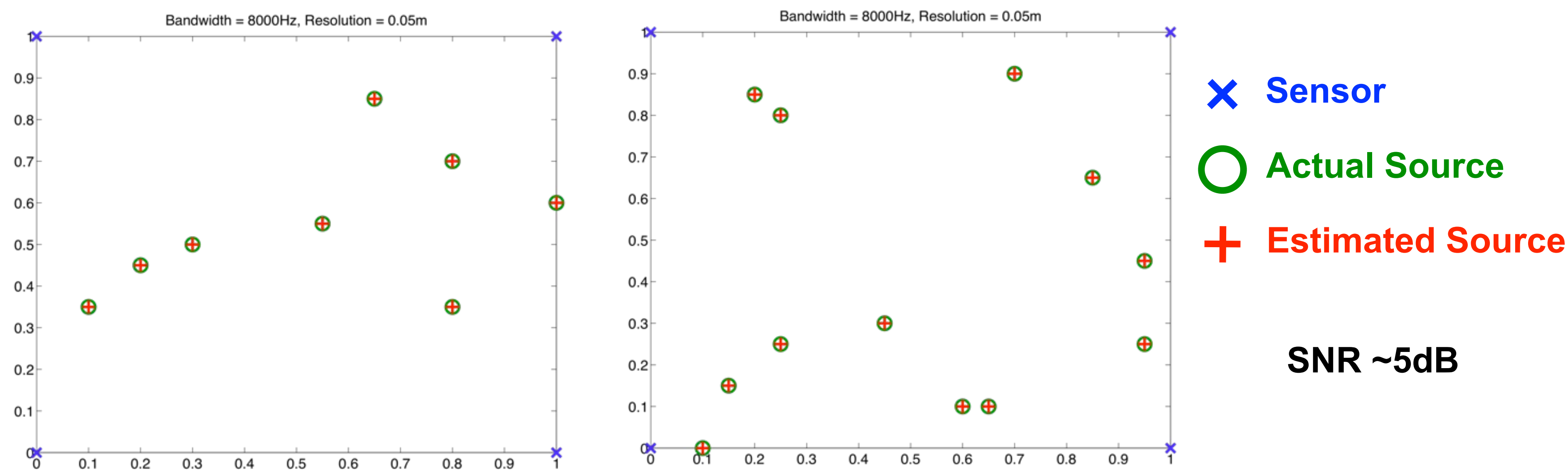
Sparsity pattern depends on source location only

$\mathbf{X}(\omega)$ has the same sparsity pattern for all ω

$$\mathbf{Y}(\omega) = \mathbf{A}(\omega)\mathbf{X}(\omega)$$

Solution: Joint Sparsity Models

Simulation Examples



SNR ~5dB

Main features:

Can localize more sources than microphones ($S > M$)

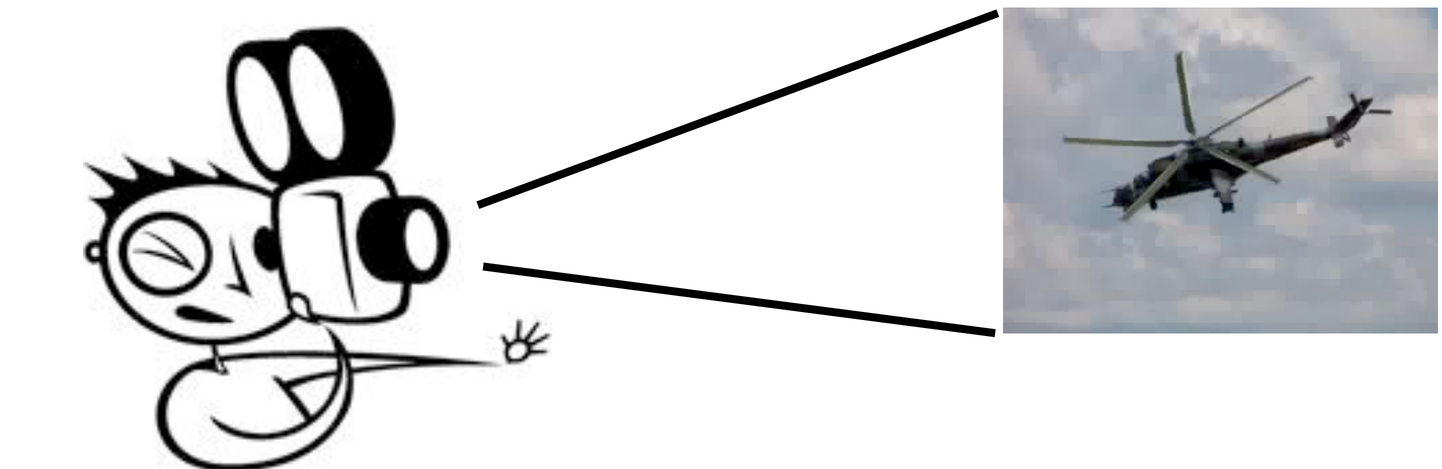
Reconstruction for $S > M$ not as straightforward (working on it)

Working on theoretical guarantees

Very good performance in practice

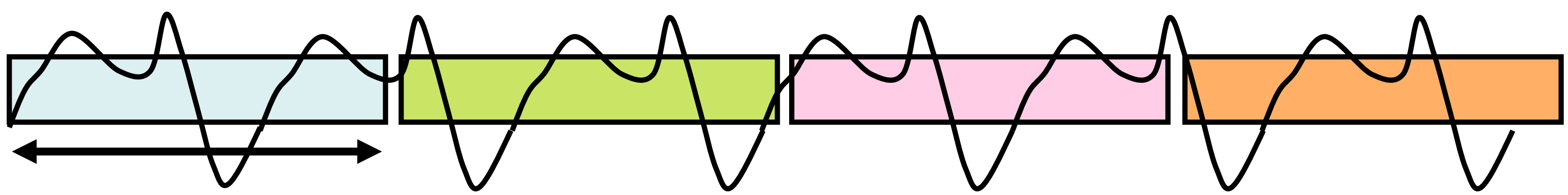
HIGH SPEED VIDEO ACQUISITION

Time Aliasing in Video Acquisition

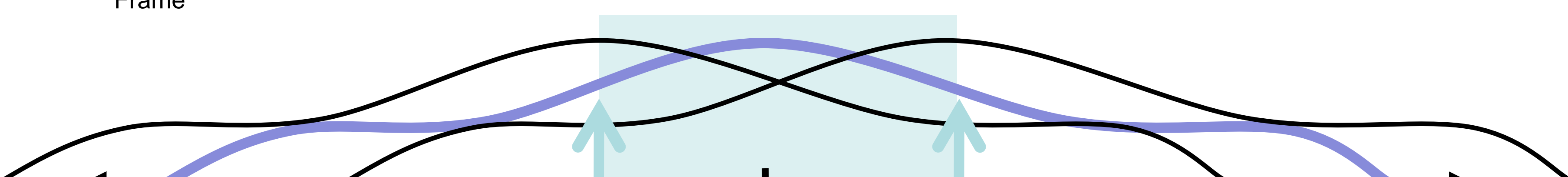


Each frame integrates light (signal) over time and samples

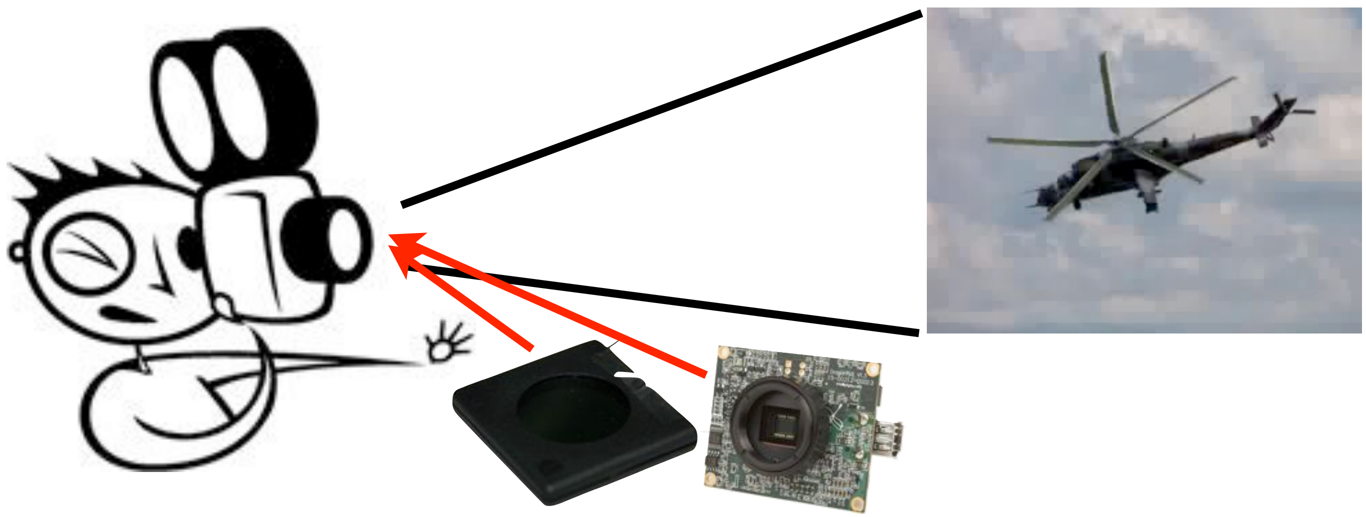
High frequency information is **lost**.



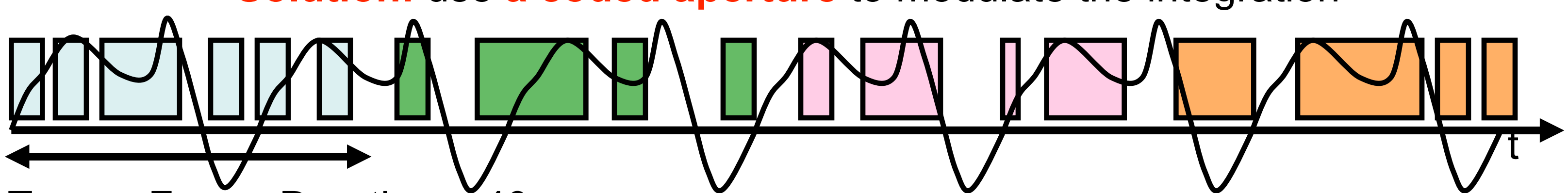
T_{Frame}= Frame Duration = 40ms



Time Domain Coded Strobing [w/ Asif, Reddy, Veeraraghavan]



Solution: use **a coded aperture** to modulate the integration



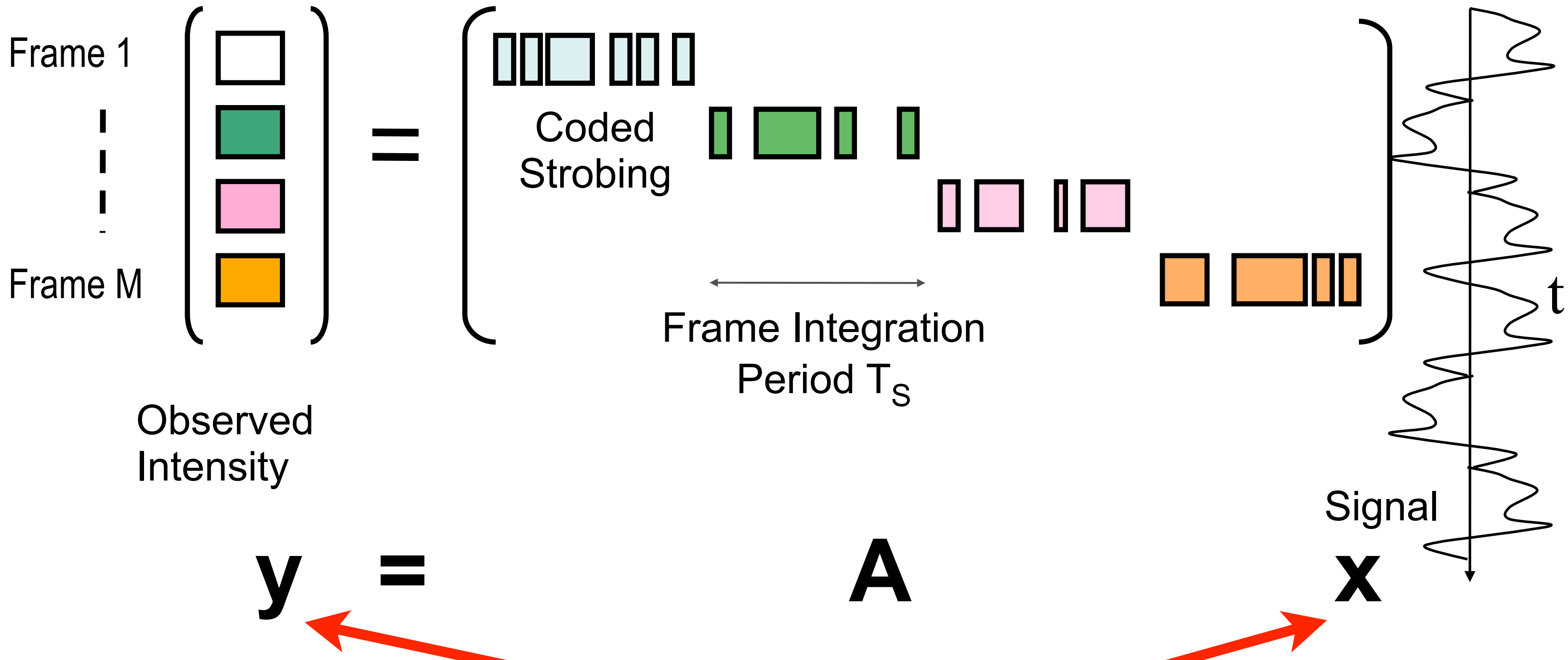
T_{Frame}= Frame Duration = 40ms

Camera **acquisition** (i.e., sampling/recording) still **at slow rate**

Math model similar to **random demodulator** (0/1 instead of +/-1)

Incoherent with **frequency-sparse signals**

Observation Model



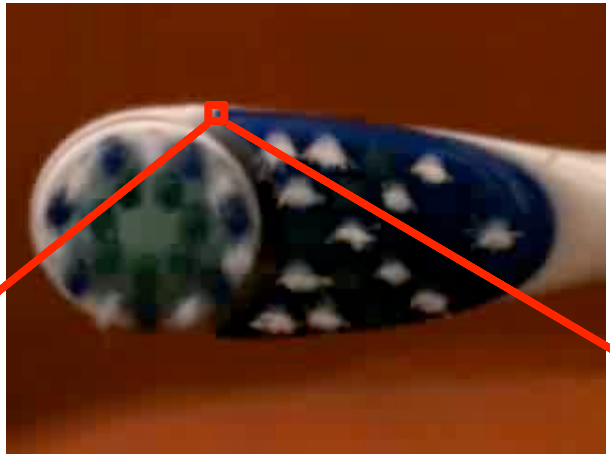
Represents 1-pixel location of the video

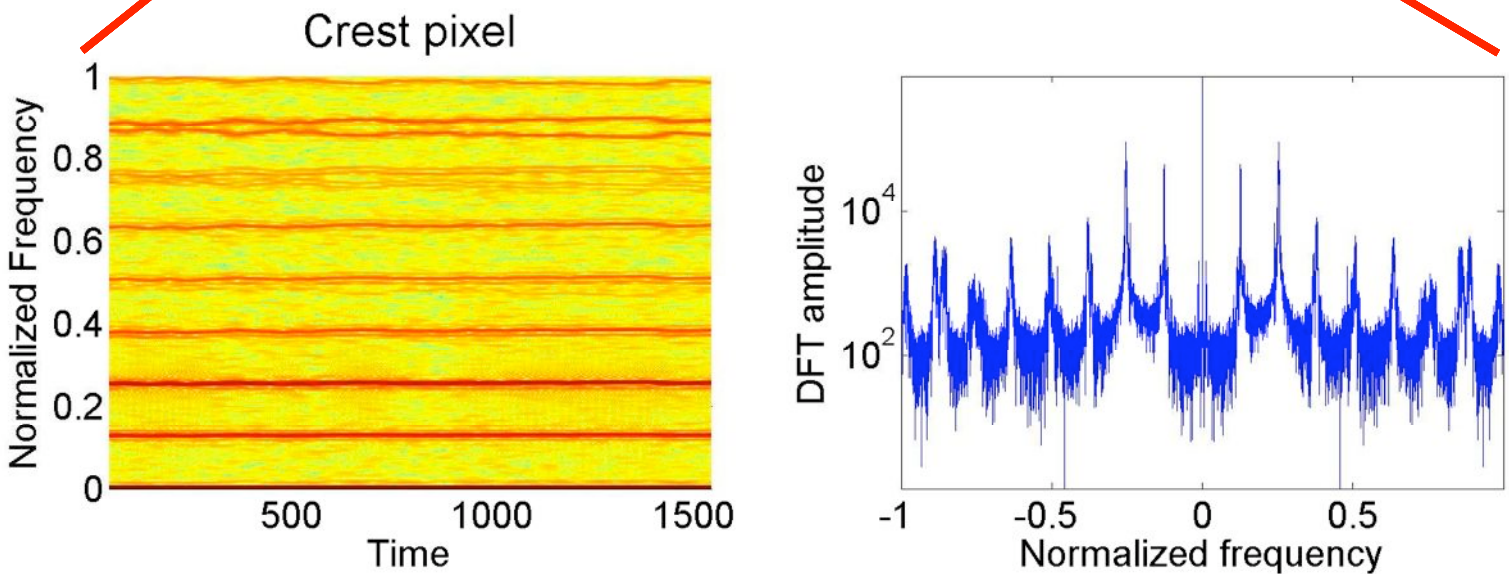
Same **A** for all pixels

Assume video is locally periodic (sparse in frequency)

Nearby pixels have similar sparsity pattern

Example: Crest Toothbrush Video





Reconstructed Video

5x Undersampling

10x Undersampling

15x Undersampling

20x Undersampling

Questions/Comments?

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