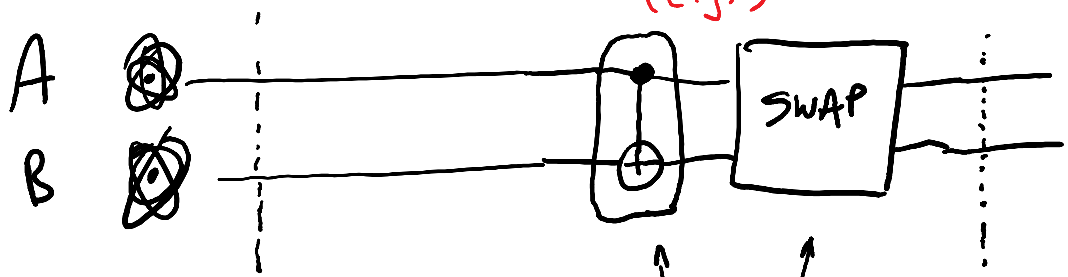
# Lecture 6 - Partial measurements & "Spooky action at a distance"

**Recap:** Alice preps a qubit $|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$

Bob ~~~~~~~~ $|\phi\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

(e.g.)



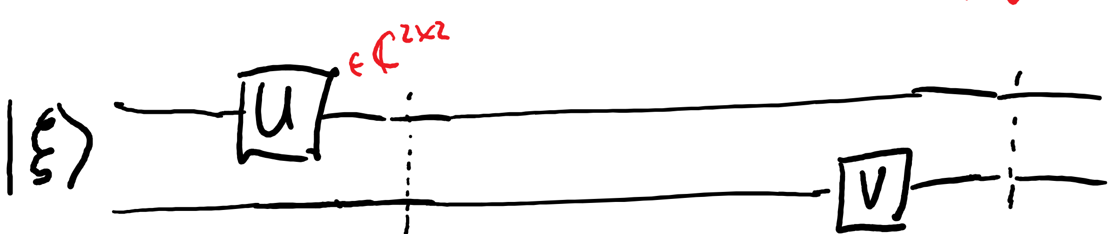

joint state: $|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$

some 4x4 unitaries

state $= |\xi\rangle :=$ SWAP $\cdot$ CNOT $\cdot (|\psi\rangle \otimes |\phi\rangle)$

(note reversed order!)

(So now we have some probably-entangled joint state $|\xi\rangle \in \mathbb{C}^4$. Still consists of 2 physical particles, tho.)

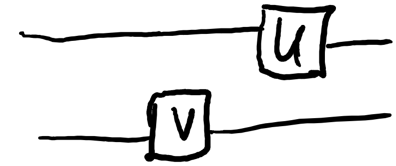



state $= (U \otimes I_{2\times 2})|\xi\rangle$

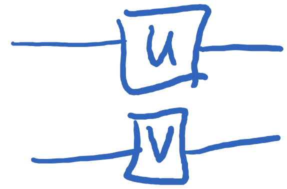
state $=$

$(I_{2\times 2} \otimes V)(U \otimes I_{2\times 2})|\xi\rangle$

$= (U \otimes V)|\xi\rangle$

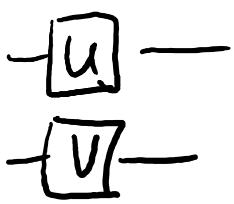
Rem: same as $|\xi\rangle$ 

(Temporally independent; same in either order.



← Most common drawing.)

It's "clear" in the case $|\xi\rangle$ is unentangled:



$$\left.\begin{array}{l} |0\rangle \\ |0\rangle \end{array}\right\} (U|0\rangle)\otimes(V|0\rangle) = (U\otimes V)|00\rangle$$

by def$^n$ of

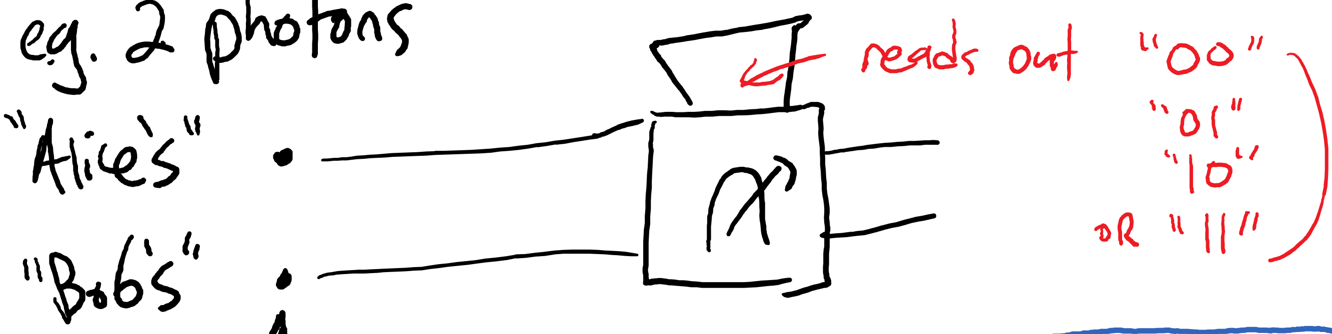
& sim for inputs $|01\rangle, |10\rangle, |11\rangle$.

Hence true $\forall$ inputs in $\mathbb{C}^4$, by linearity.

What about (the same question for) measurement?

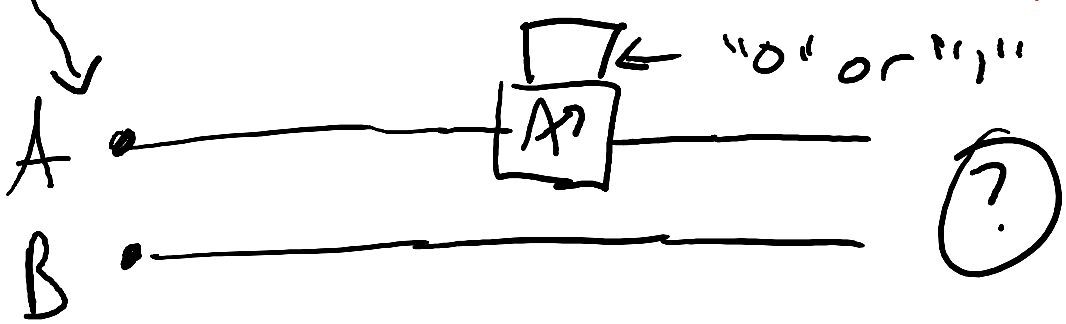
(which is not just a linear transformation on states?)

eg. 2 photons




state $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

(I told you this measuring device can theoretically be built. But we're certainly more used to 1-photon measuring devices. Certainly Alice could get one & put her photon into it. What happens?)




(As I said last time, it's basically the most logical thing you could guess; and it's very analogous to "conditioning" in probability theory.)

**Answer**: Pr[Alice's measuring device reads out "$|0\rangle$"]

$$= |\alpha_{\substack{00\\AB}}|^2 + |\alpha_{\substack{01\\AB}}|^2$$ } take all ampls on basis states "consistent with" readout

$p_0$ : =

→ And if this occurs, joint state "collapses" to

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{p_0}}$$

(Well, that can't be right; not a state, not of "norm 1") →

← ("Normalization")

$$\left( \left|\frac{\alpha_{00}}{\sqrt{p_0}}\right|^2 + \left|\frac{\alpha_{01}}{\sqrt{p_0}}\right|^2 = \frac{p_0}{p_0} = 1. \right)$$ ✓

Pr[Alice's readout is "$|1\rangle$"]

= (well, it must be ...) $p_1 = 1 - p_0$

$= |\alpha_{10}|^2 + |\alpha_{11}|^2$ (← again, the amplitudes on basic states consistent w/ readout)

2) And if this occurs, joint state collapses to $\frac{\alpha_{10}}{\sqrt{p_1}}|10\rangle + \frac{\alpha_{11}}{\sqrt{p_1}}|11\rangle$

**Rem**: In both cases, new state is <u>unentangled</u>. E.g., in former case it's

$$|0\rangle \otimes \left( \frac{\alpha_{00}}{\sqrt{p_0}}|0\rangle + \frac{\alpha_{01}}{\sqrt{p_0}}|1\rangle \right).$$

# Subtlety: "Mixed States"

"State" after Alice's measurement is...

a "classical" prob. distrib. over ("pure") quantum states

"Mixed State"

($p_0$ prob of $|0\rangle \otimes (\sim\sim)$, $p_1$ prob of $|1\rangle \otimes (\sim\sim)$)

(The ultimate actual definition of a "quantum state". We'll study mixed states properly a dozen lectures from now.)

(To keep life simple for now, we mainly try to avoid analyzing 'mixed states': Just to keep the number of things going on a bit less. It's why we almost always consider doing measurements at the end of our Q.M. experiments / Q. Circuits.

But since they've come up, we'll talk about them a bit more in this lecture.)

Current state: $\left\{ \begin{array}{l} p_0 \text{ chance of: } |0\rangle \otimes \left( \frac{\alpha_{00}}{\sqrt{p_0}}|0\rangle + \frac{\alpha_{01}}{\sqrt{p_0}}|1\rangle \right) \\ p_1 \text{ chance of: } |1\rangle \otimes \left( \frac{\alpha_{10}}{\sqrt{p_1}}|0\rangle + \frac{\alpha_{11}}{\sqrt{p_1}}|1\rangle \right) \end{array} \right.$

(mixed)

(and Alice read out "0")

(read out "1")

Say now Bob's photon goes into 1-qubit measurer.

<u>In Alice read out "0" case:</u>

$$\Pr[\text{Bob's readout "0"}] = \left| \frac{\alpha_{00}}{\sqrt{p_0}} \right|^2 = \frac{|\alpha_{00}|^2}{p_0}$$

→ and if so, state collapses to... $\frac{\alpha_{00}}{|\alpha_{00}|}|00\rangle$

A "global phase" - i.e. complex # of magnitude 1.

Indistinguishable from $|00\rangle$ (see Lecture 5.5)

$$\Pr[\text{Bob's readout "1"}] = \left| \frac{\alpha_{01}}{\sqrt{p_0}} \right|^2 = \frac{|\alpha_{01}|^2}{p_0}$$

→ and if so, state collapses to $\frac{\alpha_{01}}{|\alpha_{01}|} \cdot |01\rangle$ } indisting from just $|01\rangle$

"global phase"

Note: so far, Pr[combined readout of "00"]

$$= p_0 \cdot \frac{|\alpha_{00}|^2}{p_0} = |\alpha_{00}|^2 \rightarrow \text{and collapse to } |00\rangle.$$

$$\Pr[\text{combined "01"}] = p_0 \cdot \frac{|\alpha_{01}|^2}{p_0} = |\alpha_{01}|^2 \rightarrow \text{collapse to } |01\rangle.$$

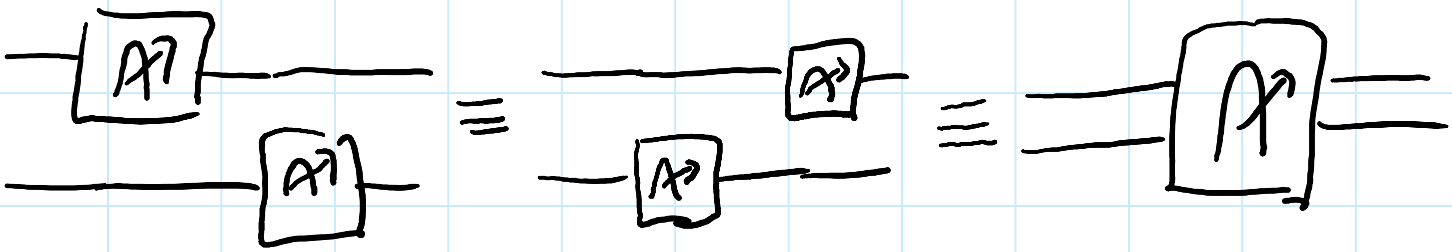
<u>In Alice reads out "1" case:</u>

- - - - - - -

<u>ex</u>: Finish analysis

Conclusion: Same as if you do "overall 4-dim. measurement".

(Same "temporal independence" as when Alice & Bob did $2\times2$ unitaries to their separate qubits.)



(Let me mention a subtlety. Suppose Alice and Bob have a joint state, Alice takes her photon to the moon, never to be seen again. Bob left with his 1 photon — promised it'll never interact with Alice's again. Feels like we should be able to describe Bob's "1-qubit state" for the purposes of future experiments.

Well, you can, but it's a <u>mixed state</u>.
Specifically, it's that (classical) prob. distrib. on 1-qubit states you'd get
<u>if Alice had measured her qubit.</u>
Mathematically, this will correctly determine all experiments Bob might subsequently do (not involving Alice).
Because — in fact — perhaps Alice <u>did</u> measure her qubit on the moon some time. Bob'll never know.....)

(Let's illustrate the partial measurement rule in a more general setting. Instead of stuffily stating "QM Law 6", I'll just give some examples.)

## Partial measurement more generally

Say Alice has qutrit, Bob has qutrit, 9-dim. system. State is

(Like spins of two deuterium nuclei.)

$$\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{13}|13\rangle + \alpha_{21} \cdots + \alpha_{33}|33\rangle,$$

Say Bob measures his qutrit.

$$\Pr[\text{readout "1"}] = |\alpha_{11}|^2 + |\alpha_{21}|^2 + |\alpha_{31}|^2 =: q_1$$

↳ and state collapses to $\frac{\alpha_{11}}{\sqrt{q_1}}|11\rangle + \frac{\alpha_{21}}{\sqrt{q_1}}|21\rangle + \frac{\alpha_{31}}{\sqrt{q_1}}|31\rangle$

$$\Pr[\text{readout "2"}] = |\alpha_{12}|^2 + |\alpha_{22}|^2 + |\alpha_{32}|^2, \text{ etc.}$$

(Also works for "partial-measuring k out of n particles.)

<u>E.g. 2</u>: 3-qubit system: $\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \cdots$

Say we "measure the first two":

$$\Pr[\text{readout "00"}] = |\alpha_{000}|^2 + |\alpha_{001}|^2 =: p_{00}$$

↳ state collapses to $\frac{\alpha_{000}|000\rangle + \alpha_{001}|001\rangle}{\sqrt{p_{00}}}$ ... Hmm

etc.

(Now you really know all the QM Laws you need to know for Quantum Computing. Time for spookiness!)

Say Alice & Bob prepare EPR pair:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

(Remember, they can in principle now take their particles far apart. This is done in actual "quantum teleportation" experiments, a topic we'll talk about in a couple lectures.
In 2012: one particle in La Palma, one Tenerife (Canary Islands): 143km apart.
In 2017: Jian-Wei Pan's group, one in Ngari, Tibet, one on satellite "Micius": 500-1400km.)

Alice goes to moon, measures her qubit.
Say (50% chance) reads out "0". Now joint state is $|00\rangle$. Bob's state "instantaneously" $|0\rangle$. She knows he'll get "0" if he measures his qubit.

(!!??) (Is this crazy? Einstein wasn't keen...)

(Conversely, now if Bob measures on earth, he'll "instantly" know what Alice's readout was.)

(There's a general causality principle in physics that information should not be able to be communicated faster than the speed of light. But...) Alice doesn't actually <u>convey</u> info (of her choice) to Bob.

<u>Classical comparison:</u>

A machine flips a coin, puts a second coin in same H/T state, covers them, gives one to Alice, one to Bob.

"State": $\frac{1}{2}$: "00", $\frac{1}{2}$: "11".

Alice goes to moon. Then uncovers coin, sees "0". Now "instantly" knows Bob will see "0" when he uncovers

(Turns out this isn't a perfect analogy.
The coins have a certain property:)

coins: have "local hidden variable"
("realism")

(That is, the "50% on 00, 50% on 11" state is just a math. model of our ignorance of the "real" state (either 00 or 11). Before we uncover, there's a "hidden variable" — set when the machine flipped the coins — that determines in advance the outcome of future measurements.

In contrast, the qubits "haven't decided" on anything yet when in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.
That's their true state.

Well... that's what I'm telling you.

In '30s, Albert "God does not play dice w/ universe" Einstein didn't believe it, felt Q.M. was incomplete, and there must be "local hidden vbls".

Next lecture, we'll talk about an experiment proving him wrong. But first, let's up the ante....)

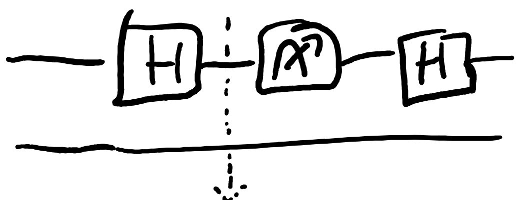
What if moon-Alice measures in another basis?
E.g. $\{|+\rangle, |-\rangle\}$ basis?

(2 equally valid ways to analyze...)

1. Rewrite state in that basis, use "usual" rule.
2. Simulate via unitary & standard meas

(Let's do both. ② first.)

EPR



$(H^{-1} = H)$

$$\frac{1}{\sqrt{2}}(H|0\rangle)\otimes|0\rangle + \frac{1}{\sqrt{2}}(H|1\rangle)\otimes|1\rangle$$
$$= \frac{1}{\sqrt{2}}|+\rangle\otimes|0\rangle + \frac{1}{\sqrt{2}}|-\rangle\otimes|1\rangle$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\otimes|0\rangle + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\otimes|1\rangle$$
$$= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Alice Measures:

- w/ prob. $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$: sees "0" (interp'd as $|+\rangle$), state collapses to

$$\frac{\frac{1}{2}}{\sqrt{1/2}}|00\rangle + \frac{\frac{1}{2}}{\sqrt{1/2}}|01\rangle = |0\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$
$$= |0\rangle\otimes|+\rangle.$$

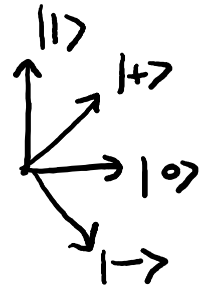
Applies $H^{-1} = H$: $\rightsquigarrow$ $|+\rangle\otimes|+\rangle$.

- (ex) w/ prob. $\frac{1}{2}$: sees "1" (interp as $|-\rangle$), final state $|-\rangle\otimes|-\rangle$.

(Surprising? Alice reads out $|\pm\rangle$, Bob's state snaps to same. Let's try method ①.)

$$|00\rangle = |0\rangle \otimes |0\rangle = \left(\tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle\right) \otimes |0\rangle$$

$$= \tfrac{1}{\sqrt{2}}|+,0\rangle + \tfrac{1}{\sqrt{2}}|-,0\rangle$$




$$|11\rangle = |1\rangle \otimes |1\rangle = \left(\tfrac{1}{\sqrt{2}}|+\rangle - \tfrac{1}{\sqrt{2}}|-\rangle\right) \otimes |1\rangle$$

$$= \tfrac{1}{\sqrt{2}}|+,1\rangle - \tfrac{1}{\sqrt{2}}|-,1\rangle$$

$$\therefore \text{EPR pair} = \tfrac{1}{2}|+,0\rangle + \tfrac{1}{2}|-,0\rangle + \tfrac{1}{2}|+,1\rangle - \tfrac{1}{2}|-,1\rangle$$

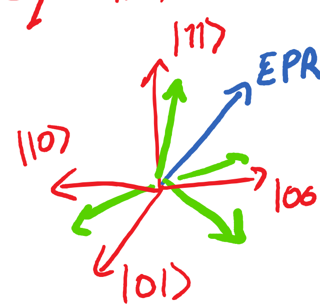
$$= \tfrac{1}{2}|+\rangle \otimes (|0\rangle + |1\rangle) + \tfrac{1}{2}|-\rangle \otimes (|0\rangle - |1\rangle)$$

$$= \tfrac{1}{\sqrt{2}}|+\rangle \otimes |+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle \otimes |-\rangle$$

(!! Like EPR pair but with $|\pm\rangle$! $\tfrac{1}{\sqrt{2}}|++\rangle + \tfrac{1}{\sqrt{2}}|--\rangle$. Strange but true, these are literally equal.)

(So indeed: once Alice measures in $\{|+\rangle, |-\rangle\}$ basis, Bob's qubit collapses to that same outcome.

Spookier? Einstein really didn't like this...)




(two 4-dim ortho bases; one vector in two spans...)

(Now it seems Alice might (?) be able to communicate one bit of info faster than light, by choosing the basis to measure in? Well... still no.....)

Alice choice 1: Measure in $\{|0\rangle, |1\rangle\}$ basis.

50% chance Bob's qubit snaps to $|0\rangle$
" " " " " " $|1\rangle$

$\Big\}$ a "mixed state". Call it "$\rho_1$"

Alice choice 2: Measure in $\{|+\rangle, |-\rangle\}$ basis

50% chance Bob's qubit snaps to $|+\rangle$
" " " " " " $|-\rangle$

$\Big\}$: "$\rho_2$"

$\rho_1$ & $\rho_2$ seem very different! Alice can pick one to instantly create in Bob's hands!

Instant communication... assuming Bob can discriminate $\rho_1$ & $\rho_2$.

Can he??

Actually, $\rho_1$ & $\rho_2$ are <u>indistinguishable</u> by all physical experiments!

(In fact, when we eventually study mixed states, they'll be represented by identical math objects, namely the matrix $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ .....)

(You'll explore this on homework... For now, one example...)

Can Bob distinguish by measuring in $\{|0\rangle, |1\rangle\}$ basis?

$\rho_1$ / 50% $|0\rangle$, 50% $|1\rangle$

$\to$ measurement gives 50% $|0\rangle$ 50% $|1\rangle$

(Same!)

$\rho_2$ / 50% $|+\rangle$, 50% $|-\rangle$

$|+\rangle \to$ measure: 50% $|0\rangle$ 50% $|1\rangle$

$|-\rangle \to$ measure: 50% $|0\rangle$ 50% $|1\rangle$

overall 50% $|0\rangle$, 50% $|1\rangle$.

(Next time: we'll see something Alice & Bob can do with an EPR pair which is truly amazing— and nonlocal.

Proves quantum can't be explained by any local hidden variable theory.

Perhaps the most important experiment in Q.M.)