PROBLEM SET 5

Due: Tuesday, April 24, beginning of class

Homework policy: I encourage you to try to solve the problems by yourself. However, you may collaborate as long as you do the writeup yourself and list the people you talked with.

Do at least 4 out of 7.

1. Poincaré Inequality III. For this problem, please assume the setup and results of Problem 1 on Homework #4. Given $f : X^n \to \mathbb{R}$, we will write "$\hat{f}(S)^2$" for $\mathbf{E}_{\boldsymbol{x}}[(f^S(\boldsymbol{x}))^2]$.

(a) Show that $\mathbf{Var}[f] = \sum_{S \neq \emptyset} \hat{f}(S)^2$.

(b) Given $I \subseteq [n]$, let $\boldsymbol{y}$ denote a random draw from $X^{\bar{I}}$, where $\bar{I} = [n] \setminus I$, as usual. Show that

$$\mathop{\mathbf{E}}_{\boldsymbol{y}}[\mathbf{E}[f_{\boldsymbol{y}\to\bar{I}}]^2] = \sum_{S \subseteq \bar{I}} \hat{f}(S)^2.$$

(c) For $i \in [n]$, define

$$\mathrm{Inf}_i(f) = \mathop{\mathbf{E}}_{\boldsymbol{y}}[\mathbf{Var}[f_{\boldsymbol{y}\to\bar{I}}]],$$

where we are writing $I = \{i\}$. Show that $\mathrm{Inf}_i(f) = \sum_{S \ni i} \hat{f}(S)^2$, and conclude that

$$\mathbf{Var}[f] \leq \sum_{i=1}^{n} \mathrm{Inf}_i(f).$$

Remark: This fact (actually, a slightly different equivalent fact) is sometimes known as the *Efron-Stein Inequality*.

2. Talagrand's Lemma. Let $f : \{-1,1\}^n \to \{0,1\}$ and write $p = \mathbf{E}[f]$. (Think of p as small.) Talagrand's Lemma states that $W_1(f) = \sum_{|S|=1} \hat{f}(S)^2 \leq O(p^2 \log(1/p))$. In this problem we will show a slight generalization: the result holds for any $f : \{-1,1\}^n \to [-1,1]$, when $p = \mathbf{E}[|f|]$.

(a) Let $\tilde{\ell} : \{-1,1\}^n \to \mathbb{R}$ be given by $\tilde{\ell}(x) = \sum_{i=1}^{n} \frac{\hat{f}(i)}{\sigma} x_i$, where $\sigma = \sqrt{W_1(f)}$.[1] Show that for any $t \geq 0$,

$$\sigma = \mathbf{E}[\mathbf{1}_{\{|\tilde{\ell}| \leq t\}} \cdot f \cdot \tilde{\ell}] + \mathbf{E}[\mathbf{1}_{\{|\tilde{\ell}| > t\}} \cdot f \cdot \tilde{\ell}].$$

(b) Upper-bound the above by $pt + O(\exp(-t^2/2))$. (Hint: Use Hoeffding.)

(c) Deduce $W_1(f) \leq O(p^2 \log(1/p))$.

[1] If $\sigma = 0$ then we're already done.

3. Degree-1 versus influence. Let $f : \{-1,1\}^n \to \{-1,1\}$. Show that $|\hat{f}(i)| \le \mathrm{Inf}_i(f)$. Show that this is false in general for $f : \{-1,1\}^n \to [-1,1]$.

4. Majority Is Stablest for small ρ. Suppose $f : \{-1,1\}^n \to [-1,1]$ is "$(\epsilon, 1)$-quasirandom"; i.e., $\hat{f}(i)^2 \le \epsilon$ for all i.

(a) Show that $W_1(f) \le \frac{2}{\pi} + O(\sqrt{\epsilon})$. (Hint: Study $\mathbf{E}[f \cdot \tilde{\ell}]$ as in Problem 2 and use Berry-Esseen.)

(b) Show that if in addition $\mathbf{E}[f] = 0$, then $\mathbb{S}_\rho(f) \le \frac{2}{\pi} \arcsin \rho + O(\rho^2 + \sqrt{\epsilon})$ for $\rho \ge 0$.

(Thus the Majority Is Stablest Theorem holds for "small" ρ.)

5. Reverse Majority Is Stablest. Recall that the Majority Is Stablest Theorem is the following:

Fix $0 \le \rho \le 1$. Then if $f : \{-1,1\}^n \to [-1,1]$ is $(\epsilon, 1/\log(1/\epsilon))$-quasirandom and satisfies $\mathbf{E}[f] = 0$, then $\mathbb{S}_\rho(f) \le \frac{2}{\pi} \arcsin \rho + O(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)})$.

Use this to deduce the following "reversed" version:

Fix $-1 \le \rho \le 0$. Then if $f : \{-1,1\}^n \to [-1,1]$ is $(\epsilon, 1/\log(1/\epsilon))$-quasirandom (we don't assume $\mathbf{E}[f] = 0$), then $\mathbb{S}_\rho(f) \ge \frac{2}{\pi} \arcsin \rho - O(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)})$.

(Hint: Odd-ize.)

6. Attenuated influences vs. influences, and the noise sensitivity of Tribes.

(a) Let $f : \{-1,1\}^n \to \{-1,1\}$. Show that $\mathrm{Inf}_i^{(\rho)}(f) \le (\mathrm{Inf}_i(f))^{2/(1+\rho)}$.

(b) Let T denote the Tribes function on n bits. Show that for $0 \le \gamma \le 1/2$,

$$\sum_{|S| \ge 1} |S| \gamma^{|S|-1} \hat{T}(S)^2 \le \frac{O(\log^2 n)}{n^{1-2\gamma}}.$$

(c) Assume $1/n \le \gamma \le 1/\log n$. Show that $\mathbb{NS}_{\frac{1}{2}-\gamma}(T) \ge \frac{1}{2} - \gamma \cdot O(\frac{\log^2 n}{n})$.

(This implies that Tribes is an excellent combining function for hardness amplification — if the hardness is already near $\frac{1}{2}$ to start.)

7. Minimum balanced k-cuts in Kneser-like graphs.

(a) Suppose $f : \{-1,1\}^n \to \{0,1\}$ has $\mathbf{E}[f] = p$. Show that $\mathbb{S}_\rho(f) \le p^{2/(1+\rho)}$.

(b) Let $\epsilon > 0$ and let $k \in \mathbb{N}$ be a power of 2. Consider the weighted complete graph on the vertex set $\{-1,1\}^n$ in which the weight on the edge (u, v) is equal to

$$\Pr_{\boldsymbol{x}, \boldsymbol{y} \sim_{1-\epsilon} \boldsymbol{x}} [(\boldsymbol{x}, \boldsymbol{y}) = (u, v)].$$

A *balanced k-cut* in this graph is a partition of the vertices into k equal-sized parts. The value of the cut is equal to the total weight of edges that have endpoints in different parts. Since the total weight in the graph is 1, the value of a cut is in the range $[0, 1]$. Show that in fact the minimum value among balanced k-cuts in this graph is at least $1 - o_{k \to \infty}(1)$.