Problem Set 5

Due: Monday, Oct. 15, beginning of class

Homework policy: Please try to work on the homework by yourself; it isn't intended to be too difficult. Questions about the homework or other course material can be asked on Piazza.

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1. Suppose $f(x) = \mathrm{sgn}(a_0 + a_1x_1 + \cdots a_nx_n)$ is an LTF with $|a_1| \geq |a_2| \geq \cdots \geq |a_n|$. Show that $\mathbf{Inf}_1[f] \geq \mathbf{Inf}_2[f] \geq \cdots \geq \mathbf{Inf}_n[f]$.

2. In class we will discuss the FKN Theorem and the proof of the following: If $f : \{-1,1\}^n \to \{-1,1\}$ has $\mathbf{E}[f] = 0$ and $\mathbf{W}^1[f] \geq 1-\delta$ then f is $O(\delta)$-close to $\pm\chi_i$ for some $i \in [n]$. *Assuming this*, show the following: If $f : \{-1,1\}^n \to \{-1,1\}$ has $\mathbf{W}^{\leq 1}[f] \geq 1-\delta$ then f is $O(\delta)$-close to a 1-junta. (Hint: define $g(x_0, x) = x_0 f(x_0x)$.)

3. (a) Suppose $\ell : \{-1,1\}^n \to \mathbb{R}$ is defined by $\ell(x) = a_0 + a_1x_1 + \cdots a_nx_n$. Define $\widetilde{\ell} : \{-1,1\}^{n+1} \to \mathbb{R}$ by $\widetilde{\ell}(x_0, \ldots, x_n) = a_0x_0 + a_1x_1 + \cdots a_nx_n$. Show that $\|\widetilde{\ell}\|_1 = \|\ell\|_1$ and $\|\widetilde{\ell}\|_2^2 = \|\ell\|_2^2$.

   (b) Show that if $f = \mathrm{sgn}(\ell)$ is any LTF then $\mathbf{W}^{\leq 1}[f] \geq 1/2$. (Recall that we proved this in class assuming $a_0 = 0$ using Homework #2, Problem #6.)

4. Consider the sequence of LTFs $f_n : \{-1,1\}^n \to \{0,1\}$ defined by $f_n(x) = 1$ if and only if $\sum_{i=1}^n \frac{1}{\sqrt{n}} x_i > t$. (I.e., f_n is the indicator of the Hamming ball of radius $\frac{n}{2} - \frac{t}{2}\sqrt{n}$ centered at $(1, 1, \ldots, 1)$.) Show that
$$\lim_{n\to\infty} \mathbf{E}[f_n] = \overline{\Phi}(t), \qquad \lim_{n\to\infty} \mathbf{W}^1[f_n] = \phi(t)^2,$$
where ϕ is the pdf of a standard Gaussian and $\overline{\Phi}$ is the complementary cdf (i.e., $\overline{\Phi}(u) = \int_u^\infty \phi$). You may use the Central Limit Theorem without worrying about error bounds.

5. For integer $0 \leq k \leq n$, define $\mathcal{K}_k : \{-1,1\}^n \to \mathbb{R}$ by $\mathcal{K}_k(x) = \sum_{|S|=k} x^S$. Since $\mathcal{K}_k$ is symmetric, the value $\mathcal{K}_k(x)$ depends only on the number z of -1's in x; or equivalently, on $\sum_{i=1}^n x_i$. Thus we may define $\mathcal{K}_k : \{0, 1, \ldots, n\} \to \mathbb{R}$ by $\mathcal{K}_k(z) = \mathcal{K}_k(x)$ for any x with $\sum_i x_i = n - 2z$.

   (a) Show that $\mathcal{K}_k(z)$ can be expressed as a degree-k polynomial in z. It is called the *Kravchuk (*or *Krawtchouk) polynomial* of degree k. (The dependence on n is usually implicit.)

   (b) Show that $\sum_{k=0}^n \mathcal{K}_k(x) = \begin{cases} 2^n & \text{if } x = (1,\ldots,1), \\ 0 & \text{else.} \end{cases}$.

   (c) Show for $\rho \in [-1,1]$ that $\sum_{k=0}^n \mathcal{K}_k(x)\rho^k = 2^n\,\mathbf{Pr}[\boldsymbol{y} = (1,\ldots,1)]$, where $\boldsymbol{y} = N_\rho(x)$.

   (d) Deduce the following generating function identity: $\mathcal{K}s_k(z) = [\rho^k]((1-\rho)^z(1+\rho)^{n-z})$; i.e., the coefficient on ρ^k in $(1-\rho)^z(1+\rho)^{n-z}$.

6. Say that $\vec{\boldsymbol{Z}}, \vec{\boldsymbol{Z}}'$ are "ρ-correlated d-dimensional Gaussians" if the pairs $(\vec{\boldsymbol{Z}}_1, \vec{\boldsymbol{Z}}'_1), \ldots, (\vec{\boldsymbol{Z}}_d, \vec{\boldsymbol{Z}}'_d)$ are independent ρ-correlated Gaussians. Now given $f : \mathbb{R}^d \to \{-1,1\}$ and $\epsilon \in \mathbb{R}$, define the *rotation sensitivity* of f at ϵ to be
$$\mathbf{RS}_f(\epsilon) = \mathbf{Pr}[f(\vec{\boldsymbol{Z}}) \neq f(\vec{\boldsymbol{Z}}')],$$
where $\vec{\boldsymbol{Z}}$ and $\vec{\boldsymbol{Z}}'$ are $\cos(\epsilon)$-correlated d-dimensional Gaussians.

(a) Show that $\mathbf{RS}_f(0) = 0$, $\mathbf{RS}_f(\pi/2) = \frac{1}{2}$ if $\mathbf{Pr}[f(\vec{\mathbf{Z}}) = 1] = \frac{1}{2}$, and $\mathbf{RS}_f(\pi) = 1$ if f is an odd function (meaning $f(-x) = -f(x)$).

(b) For $j = 0, 1, \ldots, \ell$, let $\vec{u}_j$ be the unit vector in $\mathbb{R}^2$ making an angle of $j\epsilon$ with the x-axis. Let $\vec{\mathbf{Y}}$ be a standard 2-dimensional Gaussian and define $\mathbf{Z}_j = \langle \vec{u}_j, \vec{\mathbf{Y}} \rangle$. Show that $\mathbf{Z}_j$ and $\mathbf{Z}_{j+k}$ are $\cos(k\epsilon)$-correlated standard Gaussians.

(c) Describe how to generate a sequence of d-dimensional Gaussians $\vec{\mathbf{Z}}^{(0)}, \ldots, \vec{\mathbf{Z}}^{(\ell)}$ such that each pair $\vec{\mathbf{Z}}^{(j)}, \vec{\mathbf{Z}}^{(j+k)}$ is $\cos(k\epsilon)$-correlated (as defined in the first sentence of this problem).

(d) Show that for any $f : \mathbb{R}^d \to \{-1, 1\}$, $\epsilon \in \mathbb{R}$, and $\ell \in \mathbb{N}^+$ we have $\mathbf{RS}_f(\ell\epsilon) \leq \ell \mathbf{RS}_f(\epsilon)$. (Hint: previous part + union bound.)

(e) Let $f : \mathbb{R}^d \to \{-1, 1\}$ satisfy $\mathbf{Pr}[f(\vec{\mathbf{Z}}) = 1] = 1/2$ where $\vec{\mathbf{Z}}$ is a d-dimensional Gaussian. Let $\epsilon = \frac{\pi}{2\ell}$ for some $\ell \in \mathbb{N}^+$. Show that $\mathbf{RS}_f(\epsilon) \geq \epsilon/\pi$. (This last result is a special case of a 1985 theorem of Borell and is also generalization of a special case of the Gaussian Isoperimetric Inequality...)

7. (Bonus problem due to [JOW'12].) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be symmetric random variables (meaning $-\boldsymbol{X}$ has the same distribution as $\boldsymbol{X}$, and similarly for $\boldsymbol{Y}$). Show that $\min\{\mathbf{Var}[\boldsymbol{X}], \mathbf{Var}[\boldsymbol{Y}]\} \leq C \cdot \mathbf{Var}[|\boldsymbol{X} + \boldsymbol{Y}|]$ for some universal constant C.