PROBLEM SET 2
Due: Monday, Sept. 24, beginning of class
Turn in problems #1–#4, plus either #5 or #6

Homework policy: Please work on the homework by yourself; it isn't intended to be too difficult. Questions about the homework or other course material can be asked on Piazza.

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1. Here are some more linear operators on the vector space of functions $f : \{-1,1\}^n \to \mathbb{R}$:

   - The *ith expectation operator* E_i, defined by $\mathrm{E}_i f(x) = \dfrac{f(x^{(i\mapsto+1)}) + f(x^{(i\mapsto-1)})}{2}$.
   - The *ith directional Laplacian operator* L_i, defined by $\mathrm{L}_i f = f - \mathrm{E}_i f$.
   - The *Laplacian operator* L, defined by $\mathrm{L}f = \mathrm{L}_1 f + \mathrm{L}_2 f + \cdots + \mathrm{L}_n f$.

   Prove the following formulas:

   (a) $\mathrm{E}_i f(x) = \sum_{S \not\ni i} \widehat{f}(S) x^S$.

   (b) $f(x) = \mathrm{E}_i f(x) + x_i \mathrm{D}_i f(x)$.

   (c) $\mathrm{L}_i f(x) = \dfrac{f(x) - f(x^{\oplus i})}{2} = \sum_{S \ni i} \widehat{f}(S) x^S$.

   (d) $\langle f, \mathrm{L}_i f \rangle = \langle \mathrm{L}_i f, \mathrm{L}_i f \rangle = \mathbf{Inf}_i[f]$.

   (e) $\mathrm{L}f(x) = (n/2)\big(f(x) - \operatorname{avg}_{i \in [n]} f(x^{\oplus i})\big) = \sum_{S \subseteq [n]} |S| \widehat{f}(S) x^S$.

   (f) $\langle f, \mathrm{L}f \rangle = \mathbf{I}[f]$.

2. In 1965, the Nassau County (New York) Board used a weighted majority voting system to make its decisions, with the 6 towns getting differing weights based on their population. Specifically, the board used the voting rule $f : \{0,1\}^6 \to \{-1,1\}$ defined by $f(x) = \mathrm{sgn}(-58 + 31x_1 + 31x_2 + 28x_3 + 21x_4 + 2x_5 + 2x_6)$. Compute $\mathbf{Inf}_i[f]$ for all $i \in [6]$. (PS: John Banzhaf invented the notion of $\mathbf{Inf}_i$ while suing on behalf of towns #5 and #6.)

3. Let $f : \{-1,1\}^n \to \{-1,1\}$ be unbiased (i.e., $\mathbf{E}[f] = 0$), and let $\mathbf{MaxInf}[f]$ denote $\max_{i \in [n]}\{\mathbf{Inf}_i[f]\}$. Recall that the KKL Theorem implies $\mathbf{MaxInf}[f] \geq \Omega(\frac{\log n}{n})$. In 1987, this was still a conjecture; all that was known was the following results, independently observed by Alon and by Chor and Geréb-Graus…

   (a) Use the Poincaré Inequality to show $\mathbf{MaxInf}[f] \geq 1/n$.

   (b) Prove $|\widehat{f}(i)| \leq \mathbf{Inf}_i[f]$ for all $i \in [n]$. (Hint: consider $\mathbf{E}[|\mathrm{D}_i f|]$.)

   (c) Prove that $\mathbf{I}[f] \geq 2 - n\mathbf{MaxInf}[f]^2$. (Hint: first prove $\mathbf{I}[f] \geq \mathbf{W}^1[f] + 2(1 - \mathbf{W}^1[f])$ and then use the previous exercise.)

   (d) Deduce that $\mathbf{MaxInf}[f] \geq \frac{2}{n} - \frac{4}{n^2}$.

   (Later in 1987, Chor and Geréb-Graus managed to improve the lower bound to $\frac{3}{n} - o(1/n)$.)

4. (Remark: this is really a problem in combinatorics, not Fourier analysis.)

   The *polarizations* of $f : \{-1,1\}^n \to \mathbb{R}$ (also known as compressions, downshifts, or two-point rearrangements) are defined as follows. For $i \in [n]$, the i-polarization of f is the function $f^{\sigma_i} : \{-1,1\}^n \to \mathbb{R}$ defined by

$$f^{\sigma_i}(x) = \begin{cases} \max\{f(x^{(i\mapsto +1)}), f(x^{(i\mapsto -1)})\} & \text{if } x_i = +1, \\ \min\{f(x^{(i\mapsto +1)}), f(x^{(i\mapsto -1)})\} & \text{if } x_i = -1. \end{cases}$$

   (a) Show that $\mathbf{E}[f^{\sigma_i}] = \mathbf{E}[f]$.

   (b) Show that $\mathbf{Inf}_j[f^{\sigma_i}] \le \mathbf{Inf}_j[f]$ for all $j \in [n]$.

   (c) (Optional.) Show that $\mathbf{Stab}_\rho[f^{\sigma_i}] \ge \mathbf{Stab}_\rho[f]$ for all $0 \le \rho \le 1$.

   (d) Show that f^{σ_i} is monotone in the ith direction. (We say g is "monotone in the ith direction" if $g(x^{(i\mapsto +1)}) \ge g(x^{(i\mapsto -1)})$ for all x.) Further, show that if f is monotone in the jth direction for some $j \in [n]$ then f^{σ_i} is still monotone in the jth direction.

   (e) Let $f^* = f^{\sigma_1\sigma_2\cdots\sigma_n}$. Show that f^* is monotone, $\mathbf{E}[f^*] = \mathbf{E}[f]$, $\mathbf{Inf}_j[f^*] \le \mathbf{Inf}_j[f]$ for all $j \in [n]$, and $\mathbf{Stab}_\rho[f^*] \ge \mathbf{Stab}_\rho[f]$ for all $0 \le \rho \le 1$ (you may use part (c)).

5. (Enflo, 1970.) The Hamming distance $\mathrm{Dist}(x,y) = \#\{i : x_i \neq y_i\}$ on the discrete cube $\{-1,1\}^n$ is an example of an ℓ_1 *metric space*. For $D \ge 1$, we say that the discrete cube can be *embedded into* ℓ_2 *with distortion* D if there is a mapping $F : \{-1,1\}^n \to \mathbb{R}^m$ for some $m \in \mathbb{N}$ such that:

$$\|F(x) - F(y)\|_2 \ge \mathrm{Dist}(x,y) \text{ for all } x,\ y; \qquad \text{("no contraction")}$$

$$\|F(x) - F(y)\|_2 \le D \cdot \mathrm{Dist}(x,y) \text{ for all } x,\ y. \qquad \text{("expansion at most } D\text{")}$$

   In this problem you will show that the least distortion possible is $D = \sqrt{n}$.

   (a) Recalling the definition of f^{odd} from Homework 1, show that for any $f : \{-1,1\}^n \to \mathbb{R}$ we have $\|f^{\mathrm{odd}}\|_2^2 \le \mathbf{I}[f]$ and hence

$$\mathop{\mathbf{E}}_{\boldsymbol{x}}[(f(\boldsymbol{x}) - f(-\boldsymbol{x}))^2] \le \sum_{i=1}^{n} \mathop{\mathbf{E}}_{\boldsymbol{x}}\left[\left(f(\boldsymbol{x}) - f(\boldsymbol{x}^{\oplus i})\right)^2\right].$$

   (b) Suppose $F : \{-1,1\}^n \to \mathbb{R}^m$, and write $F(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ for functions $f_i : \{-1,1\}^n \to \mathbb{R}$. By summing the above inequality over $i \in [m]$, show that any F with no contraction must have expansion at least $\sqrt{n}$.

   (c) Show that there is an embedding F achieving distortion $\sqrt{n}$.

6. (Latała–Oleszkiewicz, 1994.) Let V be a vector space with norm $\|\cdot\|$ and fix $w_1, \ldots, w_n \in V$. Define $g : \{-1,1\}^n \to \mathbb{R}$ by $g(x) = \|\sum_{i=1}^n x_i w_i\|$.

   (a) Recalling the operator L from Problem 1, show that $\mathrm{L}g \le g$ pointwise. (Hint: triangle inequality.)

   (b) Deduce $2\,\mathbf{Var}[g] \le \mathbf{E}[g^2]$ and thus the *Khintchine–Kahane inequality*:

$$\mathop{\mathbf{E}}_{\boldsymbol{x}}\left[\left\|\sum_{i=1}^{n} \boldsymbol{x}_i w_i\right\|\right] \ge \frac{1}{\sqrt{2}} \cdot \mathop{\mathbf{E}}_{\boldsymbol{x}}\left[\left\|\sum_{i=1}^{n} \boldsymbol{x}_i w_i\right\|^2\right]^{1/2}.$$

   (Hint: first, show that the improved Poincaré inequality $\mathbf{Var}[f] \le \frac{1}{2}\mathbf{I}[f]$ holds whenever $f : \{-1,1\}^n \to \mathbb{R}$ is even, as defined in Homework 1.)

   (c) Show that the constant $\frac{1}{\sqrt{2}}$ above is optimal (Hint: take $V = \mathbb{R}$ and $n = 2$.)