Problem Set 1

Due: Monday, Sept. 17, beginning of class

Homework policy: Please work on the homework by yourself; it isn't intended to be too difficult. Questions about the homework or other course material can be asked on Piazza.

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1. Compute the Fourier expansions of the following functions.

   (a) The *selection function* $\mathrm{Sel} : \{-1,1\}^3 \to \{-1,1\}$ which outputs x_2 if $x_1 = -1$ and outputs x_3 if $x_1 = 1$.

   (b) The indicator function $1_{\{a\}} : \{-1,1\}^n \to \{0,1\}$, where $a \in \{-1,1\}^n$.

   (c) The density function corresponding to the product probability distribution on $\{-1,1\}^n$ in which each coordinate has mean $\rho \in [-1,1]$;

   (d) The *inner product mod* 2 *function*, $\mathrm{IP}_{2n} : \mathbb{F}_2^{2n} \to \{-1,1\}$ defined by $\mathrm{IP}_{2n}(x_1,\ldots,x_n,y_1,\ldots,y_n) = (-1)^{x\cdot y}$. (Here $x \cdot y$ denotes the dot-product in the vector space $\mathbb{F}_2^n$.)

   (e) The *hemi-icosahedron function* $\mathrm{HI} : \{-1,1\}^6 \to \{-1,1\}$, defined as follows: $\mathrm{HI}(x)$ is 1 if the number of 1's in x is 1, 2, or 6. $\mathrm{HI}(x)$ is -1 if the number of -1's in x is 1, 2, or 6. Otherwise, $\mathrm{HI}(x)$ is 1 if and only if one of the ten facets in the following diagram has all three of its vertices 1:

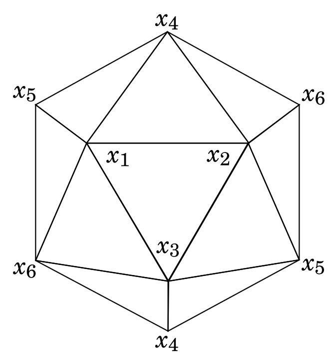

Figure 1: The hemi-icosahedron

   (Please give some indication of how you arrived at the expansion; a bare formula does not suffice.)

2. Let $f : \{-1,1\}^n \to \{-1,1\}$. Let $\boldsymbol{x}, \boldsymbol{x}' \sim \{-1,1\}^n$ be independent uniformly random strings and let $\mu = \mathbf{E}[f(\boldsymbol{x})]$. Show that

$$\begin{aligned}\mathbf{Var}[f] &= \tfrac{1}{2}\,\mathbf{E}[(f(\boldsymbol{x}) - f(\boldsymbol{x}'))^2] = 4\,\mathbf{Pr}[f(\boldsymbol{x}) = 1]\,\mathbf{Pr}[f(\boldsymbol{x}) = -1] \\ &= 2\,\mathbf{Pr}[f(\boldsymbol{x}) \neq f(\boldsymbol{x}')] \quad = \mathbf{E}[|f(\boldsymbol{x}) - \mu|].\end{aligned}$$

3. The *(boolean) dual* of $f : \{-1,1\}^n \to \mathbb{R}$ is the function $f^\dagger$ defined by $f^\dagger(x) = -f(-x)$. The function f is said to be *odd* if it equals its dual; equivalently, if $f(-x) = -f(x)$ for all x. The function f is said to be *even* if $f(-x) = f(x)$ for all x. Given any function $f : \{-1,1\}^n \to \mathbb{R}$, its *odd part* is the function $f^{\mathrm{odd}} : \{-1,1\}^n \to \mathbb{R}$ defined by $f^{\mathrm{odd}}(x) = (f(x) - f(-x))/2$, and its *even part* is the function $f^{\mathrm{even}} : \{-1,1\}^n \to \mathbb{R}$ defined by $f^{\mathrm{even}}(x) = (f(x) + f(-x))/2$.

(a) Express $\widehat{f^\dagger}(S)$ in terms of $\widehat{f}(S)$.

(b) Verify that $f = f^{\mathrm{odd}} + f^{\mathrm{even}}$ and that f is odd (respectively, even) if and only if $f = f^{\mathrm{odd}}$ (respectively, $f = f^{\mathrm{even}}$).

(c) Show that

$$f^{\mathrm{odd}} = \sum_{\substack{S \subseteq [n] \\ |S| \text{ odd}}} \widehat{f}(S)\,\chi_S, \qquad f^{\mathrm{even}} = \sum_{\substack{S \subseteq [n] \\ |S| \text{ even}}} \widehat{f}(S)\,\chi_S.$$

4. Let $f : \{-1,1\}^n \to \{-1,1\}$.

(a) Suppose $\mathbf{W}^1[f] = 1$. Show that $f(x) = \pm\chi_S$ for some $|S| = 1$.

(b) Suppose $\mathbf{W}^{\leq 1}[f] = 1$. Show that f depends on at most 1 input coordinate.

(c) Suppose $\mathbf{W}^{\leq 2}[f] = 1$. Is it true that f depends on at most 2 input coordinates?

5. A *Hadamard matrix* is any $N \times N$ real matrix with ± 1 entries and orthogonal rows. Particular examples are the *Walsh–Hadamard matrices* H_N, inductively defined for $N = 2^n$ as follows: $H_1 = \begin{bmatrix} 1 \end{bmatrix}$, $H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$.

(a) Let's index the rows and columns of H_{2^n} by the integers $\{0, 1, 2, \ldots, 2^n - 1\}$ rather than $[2^n]$. Further, let's identify such an integer i with its binary expansion $(i_0, i_1, \ldots, i_{n-1}) \in \mathbb{F}_2^n$, where i_0 is the least significant bit and i_{n-1} the most. E.g., if $n = 3$, we identify the index $i = 6$ with $(0, 1, 1)$. Now show that the (γ, x) entry of H_{2^n} is $(-1)^{\gamma \cdot x}$.

(b) Show that if $f : \mathbb{F}_2^n \to \mathbb{R}$ is represented as a column vector in $\mathbb{R}^{2^n}$ (according to the indexing scheme from part (a)) then $2^{-n} H_{2^n} f = \widehat{f}$. Here we think of $\widehat{f}$ as also being a function $\mathbb{F}_2^n \to \mathbb{R}$, identifying subsets $S \subseteq \{0, 1, \ldots, n-1\}$ with their indicator vectors.

(c) Show that taking the Fourier transform is essentially an "involution": $\widehat{\widehat{f}} = 2^{-n} f$ (using the notations from part (b)).

(d) (Optional.) Show how to compute $H_{2^n} f$ using just $n2^n$ additions and subtractions (rather than 2^{2n} additions and subtractions as the usual matrix-vector multiplication algorithm would require). This computation is called the *Fast Walsh–Hadamard Transform* and is the method of choice for computing the Fourier expansion of a generic function $f : \mathbb{F}_2^n \to \mathbb{R}$ when n is large.

6. (Sanders '06.) Let $A \subseteq \mathbb{F}_2^n$, let $\alpha = |A|/2^n$, and write $1_A : \mathbb{F}_2^n \to \{0,1\}$ for the indicator function of A.

(a) Show that $\sum_{S \neq \emptyset} \widehat{1_A}(S)^2 = \alpha(1 - \alpha)$.

(b) Define $A + A + A = \{x + y + z : x, y, z \in A\}$, where the addition is in $\mathbb{F}_2^n$. Show that either $A + A + A = \mathbb{F}_2^n$ or else there exists $S^* \neq \emptyset$ such that $|\widehat{1_A}(S^*)| \geq \frac{\alpha}{1-\alpha} \cdot \alpha$. (Hint: if $A + A + A \neq \mathbb{F}_2^n$, show there exists $x \in \mathbb{F}_2^n$ such that $1_A * 1_A * 1_A(x) = 0$.)