Learning/prediction in "Mistake-Bounded Model"

- A bit different from the "competitive ratio" model...\[1\]
- still an "online" kind of problem, but...
  
  Versus C.R.: - look at cost difference \[versus \text{ratio}\]
  - don't compare to best poss. decisions,
  but to best fixed policy "in hindsight."

Not so clear yet, but will be. Kind of like situation where you
have several possible algs you could use, want to not
regret just picking one & sticking with it at end of day.\[1\]

Game: Each day \( t \), you must predict 0 or 1,

\( a^t \): your action (prediction) (e.g., "bitcoin down" or "bitcoin up"

After each day, an outcome \( o^t \) occurs

"Mistake" if \( a^t \neq o^t \). (Your goal, of course, is to make few
mistakes over the course of time.\[1\]

Well, so far there's absolutely nothing for you to go on, so
we need to have more... \[1\]

- There are \( N \) quote-unquote "experts."
  
  "Don't assume anything about how much they "know". They're
  just people with predictions."

At start of each day \( t \), before you predict, experts
make predictions \( e^t_1, e^t_2, \ldots, e^t_N \) \( \in \{0, 1\} \)

As I said, these may or may not have anything to do with outcomes
They have to do with your...? \( \text{Regret: After } T \text{ days, how are you vs. Best Expert?} \)
e.g. \( N=4: \begin{array}{llllll} \hline t=1 & e_1 & e_2 & e_3 & e_4 & a \\ \hline 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array} \)

You: 2 mistakes

Best expert: retrospectively

1 mistake

\[ \text{Dream}(?) \text{ #Mistakes} \leq 0.01 (\text{#Best expert's mistakes}) \]

\[ \text{OK: #Mistakes} \leq 3 \left( \begin{array}{c} N \end{array} \right) + O(\log N) \]

\[ \text{"additive term"} \]

\[ \text{unavoidable} \]

To see why additive term is unavoidable, suppose...

Say I promise some mystery expert will make 0 mistakes.

What should you do? Well, if anyone ever makes a mistake, henceforth ignore them. Among the rest... go with majority?

"Majority Alg": Dump any expert making a mistake.

O/w predict via majority.

Analysis: Each time you make a mistake, so did \( \geq \frac{1}{2} \) of (remaining) experts

\( \Rightarrow \) you dump \( \geq \frac{1}{2} \) of remaining experts

\( \leq \log_2 N + 1 \) mistakes (Total, independent of TF).

(vs. 0 of Best Expert). This is optimal, and shows necessity of additive \( O(\log N) \).
1. If some expert makes few mistakes, $\frac{1}{m} \approx \frac{1}{l}$, large.

2. If you make a lot of mistakes, $\frac{1}{m} \approx \frac{1}{l}$, becomes small.

$$\text{If } \sum_{i=1}^{m} w_i > 0, \text{ Let } t = \frac{\sum_{i=1}^{m} w_i + \sum_{j=1}^{l} w_j}{2}$$

After seeing outcome, Balance the weight of each mistake expert.

On day $t$, predict according to weighted majority, with $\frac{1}{m}$ weight.

If $\sum_{i=1}^{m} w_i > 0$, $\text{Let } t = \frac{\sum_{i=1}^{m} w_i + \sum_{j=1}^{l} w_j}{2} = T$

$W_i$ for each expert $i$ then $\text{time } t$.

"Weighted Majority" Agi: You shape a "weight" (trust amount)

"Weighted Majority" mistake, just trust them less.

Better alg: Don't completely discount an expert after they make a mistake.

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1. Your mistakes $\leq \log_2 (N + 1) \times \text{Best Expert Mistakes} + 10 N + 1$
2. You make up to $10 N + 11$ more.
3. And
4. +
5. +
6. +
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What's the worst scenario now?

Could still do "Majority" Agi. "Resistant" if all experts get changed.

Say best expert will make $g (\text{m})$ mistakes.
In more detail:

1. Suppose you make a mistake one day. At least half the total weight \( \Phi^t \) gets halved, that contributing to wrong majority vote. 

\[ \Phi^{t+1} \leq \frac{3}{4} \Phi^t. \]

\[ \Phi^{(T+1)} \leq \left( \frac{3}{4} \right)^* \#_{\text{yours}}. \Phi^{(1)} N. \]

2. If expert \( i^* \) makes only \( \Phi \) ("best mistakes") mistakes, \( W^{i^* \uparrow} > \frac{1}{2} \#_{\text{BM}} \).

\[ \left( \frac{1}{2} \right) \#_{\text{BM}} \leq \left( \frac{3}{4} \right) \#_{\text{Yi}} N. \]

\[ 2 \#_{\text{BM}} \leq \left( \frac{3}{4} \right) \#_{\text{Yi}} N \Rightarrow (\log_{\frac{4}{3}}(\#_{\text{Yi}})) \leq \#_{\text{BM}} + \log_2 N. \]

\[ \Rightarrow \text{Your mistakes} \leq \frac{1}{\log_{\frac{4}{3}} \left( \#_{\text{best mistakes}} \right) + \log_2 N} \]

\[ \approx 2.41 \]

 Pretty good. Better on multiplier, worse on additive. Term 1.

Why be so harsh? Say we multiply wrong expert's weight by 0.98

Now: 1. \( \geq \) half of \( \Phi^t \) gets 0.98-timesed.

\[ \Phi^{t+1} \leq \frac{1}{2} \Phi^t + \frac{1}{2} \times 0.98 \Phi^t = 0.99 \Phi^t. \]

2. \( W^{i^* \uparrow} > (0.98) \#_{\text{BM}} \)

\[ (0.98) \#_{\text{BM}} \leq (0.99) \#_{\text{Yi}} N \]

(?)
What's \( \ln(0.99) \)?

Recall one of the greatest facts ever!!

\[ e^x \approx 1 + x \quad \text{for small } x \]

\[ \Rightarrow x \approx \ln(1 + x) \] \( \text{I took } \ln \)

\[ 5 \times 0.01 \times = -0.01 \rightarrow -0.01 \approx \ln(0.99) \]

\[ \therefore \quad \approx \quad (-0.02)(\#BM) \leq (-0.01)(\#BM) + \ln N \]

\[ (0.01)(\#YM) \leq (0.02)(\#BM) + \ln N \]

\[ \#YM \leq 2(\#BM) + 100 \ln N \]

More precisely, using \( \ln(1-x) \approx -x - \frac{1}{2}x^2 - \ldots \)

\[ \Rightarrow \quad \text{If you penalize weight by factor } 1 - \varepsilon, \]

\[ \quad \text{get} \quad \#YM \leq 2(1+\varepsilon)(\#BM) + \frac{\ln N}{\varepsilon} \]

Fact: No deterministic alg. can beat multiplicative factor 2.

Proof: Say just 2 experts:

\[
\begin{array}{cccc}
\varepsilon_1 & \varepsilon_2 & a_1 & a_2 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
\end{array}
\]

at least one expert must make \( \leq T/2 \) mistakes!

Next time:

Beating factor-2 with a randomized alg!