Lecture 17: Online Decision Making Part 1

Traditional algos are "one-shot": you're given the input, you produce the output, the end.

"Online": inputs/data arrives over time - decisions must be made on the fly

We've seen things like this before in the study of data structures.

Input: $I = x_1, x_2, x_3, \ldots$

Alg. must take a decision after each $x_i$.

Generally: Impossible for Alg to achieve the optimal solution, because it doesn't have perfect foresight. So we must be content with "approximation algos" - similar to our study of the NP-hard Min-Weight V.C. problem.

**def:** $\text{OPT}(I) =$ cost of optimal decisions knowing all of $I$

- $\text{Alg}(I)$ = "particular online alg "Alg" on $I$"
- $\text{Alg's "competitive ratio on $I" }: \frac{\text{Alg}(I)}{\text{Opt}(I)}$ (smaller is better)
- $\text{Alg's "competitive ratio"}: \max_I \left\{ \frac{\text{Alg}(I)}{\text{Opt}(I)} \right\}$  
  - Worst factor Alg suffers versus perfect knowledge of $I$.

The most classic of all examples... "Ski rental problem"

- You've never skied, don't know how much you'll like it. You could buy skis now, but maybe you'll hardly use them. Or, could rent every day, but maybe, you'll go zillions of times...
\[ \text{C.R.} = \frac{500}{950} = 0.52 \]

\[ A^g(I) = 950, \quad O^p(I) = 500 \]
\[ A^g: I = 10 \quad (\frac{500}{950}) \]

For this age, which I want:

\[ \text{C.R.} \text{ is 10.} \]

\[ \text{Buy accounts on 1st of Apr.} \]

\[ \text{To buy: $500,} \]
\[ \text{Cost to rent: $60.} \]

\[ \text{Output: ski, ski, ski, ... quiet.} \]
Generally: Rent cost $I$, Buy cost $B$ (say $B$ is an integer $\geq 2$)

BLTN's: C.R. is worst on input $I = B-1$

$$\text{Alg}(B-1) = 2B-1, \quad \text{Opt}(B-1) = B$$

$$\Rightarrow \frac{\text{C.R.}}{B} = \frac{2B-1}{B} = 2 - \frac{1}{B} \quad [\leq 2]$$

Fact: This is best (smallest) possible C.R. for any (deterministic) Alg!

Proof: Any (deterministic) alg. is just "rent $T-1$ times, then buy" for some $1 \leq T \leq \infty$ \( [\text{"\infty" means "always rent"}] \) (on \( \text{Q}^+ \))

(BLTN is $T = \downarrow B \downarrow$)

Consider $I = T$. \( [\text{Intuitively, worst \# of days.}] \)

$$\text{Alg}(T) = T-1 + B$$

$$\text{Opt}(T) = ? \quad [\text{Depends how it compares to } B.]$$

Case 1: $T \geq B$.

$$\Rightarrow \text{Opt}(T) = B \cdot \text{C.R. on } I = T \text{ is } \frac{T-1+B}{B} \geq \frac{B-1+B}{B} = 2 - \frac{1}{B}.$$ 

Case 2: $T \leq B-1$

$$\Rightarrow \text{Opt}(T) = \frac{T-1+B}{T} \cdot \text{C.R. on } I = T \text{ is } \frac{T-1+B}{T} = 1 + \frac{B-1}{T} > 1 + \frac{B-1}{B-1} = 2.$$ 

Either way, $[\text{C.R. of Alg.} \geq 2 - \frac{1}{B}].$

Why did I repeatedly emphasize "deterministic"?

Fact: \([\text{Won't prove.}]\) \exists randomized alg. with competitive ratio

$$\text{C.R.} = \max_I \frac{\mathbb{E}[\text{Alg}(I)]}{\text{Opt}(I)} = \frac{e}{e-1} \approx 1.58$$

[And this is optimal!]
Another e.g.: Hide & Seek

Often studied on geometric graphs but we'll just consider the number line.

You must walk around till finding "\(\ast\)" (i.e., 1) at some \(G \in \mathbb{Z}\).

Opt: \(1G1\).

Alg idea? Go to 1, then -2, then 3, -4, 5, ...

But say \(G\) large, odd, positive (e.g., 99)

\[\text{Alg} = 1 + 3 + 5 + 7 + 9 + ...\]

\[= G^2\]

C.R. of this alg on this inst.: \(1G1\)

: C.R. = \(\infty\) (forget).

Better alg: "Doubling search":

Go to 1, -2, +4, -8, +16, ...

\[G\]
Suppose $G$ is negative. // $G$ positive has basically same analysis; I leave you to check it.

Say $G < -2^i$, $G \geq -2^{i+2}$.

$\therefore$ Alg's cost is $2(1 + 2 + 4 + 8 + \ldots + 2^{i+1}) + 161$.

$= 2(2^{i+2} - 1) + 161$

$\leq 2 \cdot 2^{i+2} + 161 = 8 \cdot 2^i + 161$. 

$(\text{and } \approx)$

$\text{Opt} = 161$.

$\therefore \text{C.R. } \leq \frac{161 + 8 \cdot 2^i}{161} = 1 + \frac{8 \cdot 2^i}{161} \approx 1 + 8 \approx 9$. 

(161 $> 2^i$)

$\therefore \text{C.R. } \leq 9$

(Not so easy): 9 is best possible for deterministic alg.

Fact: 2e $\times$ 5.4 is possible (2 best) for randomized alg.

Generalization:

3 legs:

Fact: Best (det.) strategy:

- 1 on path 1 (from each)
- 3 on path 2
- 1 on path 3

etc.

Ex: C.R. $\leq 14.5$

$L$ legs: exponential incr. search by factor $\frac{L}{L-1}$

C.R. $= 1 + 2L \left(\frac{L}{L-1}\right)^{L-1}$. (??)

$1 + \frac{L}{L-1} \approx e^{\frac{L}{L-1}} \approx 1 + 2L \left(e^{\frac{L}{L-1}}\right) = 1 + 2eL \times 5.4L$

$= O(L)$

Application: tasks where there are several options, linear cost to switch options