Homework 7

Due: 5:00pm, Thursday March 30

Feature: As before, if your homework is typeset (as opposed to handwritten), you will receive 1 bonus point.

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1. **(NP, coNP, PSPACE.)**
   (a) (5 points.) For languages A and B, show that $A \leq_m^P B$ and $B \in \mathsf{PSPACE}$ implies $A \in \mathsf{PSPACE}$.
   (b) (2 points.) Show that $\mathsf{coPSPACE} = \mathsf{PSPACE}$.
   (c) (3 points.) Show that $\mathsf{coNP} \subseteq \mathsf{PSPACE}$. You may use the fact (stated in class) that 3SAT $\in \mathsf{PSPACE}$.

2. **(Median in log-space.)** (10 points.) Consider the problem of finding the median of n integers (n odd). You are to show that this problem can be solved in logarithmic space.

   First, give pseudocode solving this problem, using only a constant number of "integer variables". Explain your code through "comments", or with a short prose description.

   Then, give some more low-level Turing Machine details — such as how things are stored on what tapes, and a little bit about how any computations are done. To be somewhat more precise, you can imagine the following more precise description of the problem: We want a TM of space complexity $O(\log n)$ having the following behavior: The input is supposed to be a string in $\{0, 1, \#\}^*$ of the form $\#y_1\#y_2\# \cdots \#y_m$, where m is odd and each y_i is an integer written in base-2 using exactly $2b$ bits (leading 0's allowed), where b is the number of bits in the base-2 representation of m. The machine rejects if the input is not in the proper form, and otherwise it prints the median value of $y_1, y_2, \ldots, y_n$ on one of its work tape and accepts.

3. **(Verification definition of NL.)** In class we defined NL as the set of languages A for which there is a nondeterministic Turing machine deciding A with space complexity $O(\log n)$. Consider the following "verification-based" definition of a class "VNL". We say language A is in VNL if there is a deterministic TM V with the following properties. First, V has a read-only input tape on which an input $x \in \{0,1\}^n$ is written. Second, V has a special read-*once* input tape on which an input $y \in \{0,1\}^N$ is written, where $N = O(n^c)$ for some constant c. (A read-once tape is one where at each step the head can only stay put or move right; it cannot move left, and it cannot write.) Third, V has one normal (read/write) "work" tape, initially blank. Fourth, V has space complexity $O(\log n)$, in the sense that it accesses at most $O(\log n)$ work tape cells. Finally, V *verifies* A in the sense that for all x it holds that

   $$x \in A \iff \exists y\ V(x, y) \text{ accepts.}$$

   Show that $\mathsf{VNL} = \mathsf{NL}$, as follows:
   (a) (5 points.) Show that $\mathsf{VNL} \subseteq \mathsf{NL}$.
   (b) (5 points.) Show that $\mathsf{NL} \subseteq \mathsf{VNL}$.

4. **(NP vs. LINSPACE.)** (10 points.) Show that $\mathsf{NP} \neq \mathsf{SPACE}(n)$.

   (Remark: it is unknown if $\mathsf{NP} \subseteq \mathsf{SPACE}(n)$ and it is unknown if $\mathsf{SPACE}(n) \subseteq \mathsf{NP}$.

   Hint: you have seen a problem like this before.)