Stability of ℓ_∞-Ball Slicing Inequalities in Real and Complex Spaces

Nathaniel Scott Glover

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Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

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Thesis Committee:
Giovanni Leoni
Robin Neumayer
Tomasz Tkocz, Chair

For my dog Cypress

Abstract

We give a brief overview of earlier slicing results for the real cube and complex polydisc, and stability results for the cube slicing inequalities. Then, we recount a dimension-free stability result for polydisc slicing, originally proven in the work of Glover, Tkocz, and Wyczesany. Interestingly, unlike in the case of the cube, there is an additional asymptotic maximizer. We utilize Fourier-analytic bounds, probabilistic tools, and a self-improving property of the polydisc slicing inequalities.

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Contents

1 Introduction

As a direct generalization of the hypercube $[0,1]^n$, a Cartesian product of 1-dimensional real unit discs, we consider the polydisc $\mathbb{D}^n$ as the Cartesian product of 1-dimensional complex unit discs, namely $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Note that $\mathbb{D}^n$ takes the following form:

$$\mathbb{D}^n = \prod_{j=1}^{n} \mathbb{D} = \left\{ z \in \mathbb{C} \ : \ \max_{1 \leq j \leq n} |z_j| \leq 1 \right\}.$$

Throughout this paper, we define the function $|\cdot|$ to be the standard Euclidean norm on $\mathbb{R}^n$, and the standard Euclidean norm on $\mathbb{R}^{2n}$ when given an element of $\mathbb{C}^n$. Similarly, we consider the volume of measurable sets $S \subseteq \mathbb{C}^n$ as the real $2n$-dimensional volume $\mathrm{vol}_{2n}(S)$.

Remark 1. Clearly, $\mathbb{D}^n$ is preserved under the operation of permuting vector coordinates since the n-fold maximum of absolute values is preserved under this operation. Additionally, consider an n-fold rotation of the form $z \mapsto (e^{it_1} z_1, \ldots, e^{it_n} z_n)$. As $|e^{it}| = 1$ for all t, this operation maps $\mathbb{D}^n$ to itself. Furthermore, it is clearly invertible (replace each t_j with $-t_j$), so $\mathbb{D}^n$ is preserved under this operation as well.

Lastly, we consider complex hyperplanes of the form $a^\perp = \{z \in \mathbb{C}^n : \langle z, a \rangle = 0\}$, where $a \in \mathbb{C}^n$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^n$. We have studied the $(2n-2)$-dimensional volume of the intersection of $\mathbb{D}^n$ with such hyperplanes, $\mathbb{D}^n \cap a^\perp$.

Theorem 1. *For $n \geq 2$ and every unit vector $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have:*

$$1 + \frac{1}{8}|a - e_1|^2 \leq \frac{1}{\pi^{n-1}} \,\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \leq 2 - \min\left\{ 10^{-40} \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|, \ \frac{1}{76} \sum_{j=1}^{n} a_j^4 \right\}$$

Remark 2. Our result is an instance of *stability* for an existing result, namely the polydisc slicing inequalities (see Section 1.3). This notion of stability is understood easiest by a concrete example: Bonnesen's inequality. The famous isoperimetric inequality states that $0 \leq L^2 - 4\pi A$, where L is the length of a closed curve in $\mathbb{R}^2$, and A is the area of the region it encloses. Equality holds if and only if the curve is a circle. Bonnesen's inequality is a stability result for the isoperimetric inequality; it says that $\pi^2(R-r)^2 \leq L^2 - 4\pi A$, where R and r are the radii of the circumcircle and incircle of the curve, respectively. This stability result relates the quantity $L^2 - 4\pi A$ to quantitative information on how close the curve is to a circle.

Our result is the fourth piece of a history of cube and polydisc slicing results. The result itself and the techniques we use to prove it are much better understood in the context of these previous results, which we will now discuss before moving on to the proof.

1.1 Cube Slicing

In 1986, Keith Ball found a sharp upper bound on the volume of sections of the unit hypercube.

Theorem 2 (Hadwiger, Hensley, Ball, [1], [2], [3]). *For $n \geq 2$ and every unit vector $a \in \mathbb{R}^n$,*

$$1 \leq \operatorname{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq \sqrt{2}, \quad \textit{where} \quad Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n .$$

Furthermore, these bounds are tight:

- *Equality at the lower bound is acheived if and only if a has exactly one nonzero coordinate.*
- *Equality at the upper bound is achieved if and only if a has exactly two nonzero coordinates both equal to $\pm\frac{1}{\sqrt{2}}$.*

The lower bound is due to Handwiger and, independently, Hensley. The upper bound is Ball's celebrated cube slicing result. Prior to the above result, the best known upper bound for the slice volume was 5 (by Hensley). For the improved upper bound of $\sqrt{2}$, Ball provides an elegant and influential proof. The beauty of the proof lies in the reduction from an abstract higher-dimensional geometric statement to a simple one-dimensional integral inequality, which can then be proven with elementary techniques.

Roughly speaking, Ball's proof of the upper bound in Theorem 2 goes as follows:

1. The symmetries of Q_n allow us to assume that $a_i \geq 0$ for all i.
2. First, we address the case when $a_i > \frac{1}{\sqrt{2}}$ for some i with a geometric argument. By considering a linear transformation that sends $Q_n \cap a^{\perp}$ to Q_{n-1}, we are able to conclude the strict inequality $\operatorname{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) < \sqrt{2}$.
3. The remainder of the proof deals with the case where $a_i \leq \frac{1}{\sqrt{2}}$ for all i. We can also assume $a_i > 0$: If $a_i = 0$ for some i, then we cannot possibly have $n = 2$ (otherwise $|a| \leq \frac{1}{\sqrt{2}}$, contradicting the fact that $|a| = 1$). Therefore, going by induction, we must not lie in the base case of $n = 2$, and are thus able to use the inductive hypothesis for $n-1$ dimensions.
4. It remains to show that $\operatorname{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) < \sqrt{2}$ unless $n = 2$ and $a_1 = a_2 = \frac{1}{\sqrt{2}}$. We begin by deriving the following (using Fourier inversion):

$$\operatorname{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) = \int_{\mathbb{R}^{n-1}} 1_{Q_n \cap a^{\perp}}(x)\, \mathrm{d}x = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{n} \frac{\sin(a_j t)}{a_j t} \right) \mathrm{d}t.$$

5. Let $p_j = a_j^{-2}$ so that $p_j \geq 2$ and $\sum_{j=1}^n p_j^{-1} = 1$. By Hölder's inequality,
$$\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq \prod_{j=1}^{n}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin(a_j t)}{a_j t}\right|^{p_j} \mathrm{d}t\right)^{p_j^{-1}} = \prod_{j=1}^{n}\left(\frac{1}{a_j \pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^{p_j} \mathrm{d}t\right)^{p_j^{-1}}. \quad (1)$$

6. As promised, we've reduced the problem to analyzing a one-dimensional integral. By Ball's integral inequality (Lemma 3 below),
$$\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq \prod_{j=1}^{n}\left(\frac{1}{a_j \pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^{p_j} \mathrm{d}t\right)^{p_j^{-1}} \leq \prod_{j=1}^{n}\left(\frac{\sqrt{2}}{a_j\sqrt{p_j}}\right)^{p_j^{-1}} = \sqrt{2}.$$
Lemma 3 also tells us that the second inequality is an equality if and only if $p_j = 2$ for all j. This forces $n = 2$ and $a_1 = a_2 = \frac{1}{\sqrt{2}}$, which of course implies $\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) = \sqrt{2}$.

The remainder of Ball's proof is a proof of the following inequality.

Lemma 3 (Ball, [3]). *For $p \geq 2$,*
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^{p} \mathrm{d}t \leq \sqrt{\frac{2}{p}}.$$
Furthermore, equality holds if and only if $p = 2$.

The original proof of Lemma 3 from [3] is very computation-heavy and quite tricky to follow, but only elementary results and techniques are used in this analysis. A much less computational proof was later given in [7], making use of more advanced techniques in place of numerical calculation.

The bounds obtained in Theorem 2 are independent of a. This naturally leads us to asking if the inequalities can be refined to depend on a, or more specifically a measure of difference between a and the extremizers e_1 and $\frac{e_1+e_2}{\sqrt{2}}$.

This particular refinement on a tight inequality with known extremizers is commonly referred to as *stability* for that inequality. In the context of cube slicing, this question has only recently been answered, which we will discuss in the next section.

1.2 Stability of Cube Slicing

The stability of an inequality is a measure of how varying the inequality's parameter away from the extremizers will affect the proximity to the bounds. In the setting of cube slicing, a stability result would look something like

$$1 + m_1(a, e_1) \leq \mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq 2 - m_2\left(a, \frac{e_1 + e_2}{\sqrt{2}}\right).$$

In the above, m_1 and m_2 are some notions of closeness between vectors in $\mathbb{R}^n$ (very frequently some function of the standard Euclidean metric). Also, in order for this inequality to not become blatantly false, we make the assumption that the coordinates of a are sorted in nonincreasing order. The symmetries of Q_n ensure that this assumption does not reduce the generality of the problem.

In the case of the lower bound, this inequality says that if a is ε-close to e_1, then the closest the slice volume can get to the lower bound 1 is within a distance ϵ. The upper bound is interpreted similarly.

To my knowledge, the two earliest results on the stability of cube slicing were both obtained in 2021, many years after Ball's original work. The first result, due to Melbourne and Roberto, essentially derived a stability result for Ball's integral inequality (Lemma 3) and used it to deduce a stability result for cube slicing.

The second, due to Chasapis, Nayar, and Tkocz, used an entirely different approach, where the cube slicing inequality (Theorem 2) can be used at a lower dimension to "self-improve" when near the extremizer. Our proof uses ideas from both of these approaches, so it is worth discussing them in more detail.

1.2.1 Proof by Stability of Ball's Integral Inequality

In 2021, Melbourne and Roberto proved the following stability result for cube slicing:

Theorem 4 (Melbourne, Roberto, [4]). *Fix* $\varepsilon \in (0, \frac{1}{75})$. *Let* $a \in \mathbb{R}^n$ *be a unit vector such that* $\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \geq (1-\varepsilon)\sqrt{2}$. *Then, there exist two indices* j_0 *and* j_1 *such that*

$$\frac{1}{\sqrt{2}}\left(1 - \frac{75\varepsilon}{2}\right) \leq |a_{j_0}|, |a_{j_1}| \leq \frac{1}{\sqrt{2}}(1 + 2\varepsilon) \quad \text{and} \quad \sum_{j \neq j_0, j_1} a_j^2 \leq 50\varepsilon.$$

In particular, $|a_j| \leq \sqrt{50\varepsilon}$ *for all* $j \neq j_0, j_1$.

Their proof closely follows Ball's proof of Theorem 2, but instead of applying Ball's integral inequality (Lemma 3) to complete the proof, they use a stability result for that inequality:

Lemma 5 (Melbourne, Roberto, [4]). *If $s \geq 2$ such that*

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi t)}{\pi t} \right|^s \mathrm{d}t \leq (1-\delta)\sqrt{\frac{2}{s}}$$

holds for small $\delta > 0$, then we must have $s \leq 2 + 50\delta$.

This integral inequality is proven by refining Ball's original proof of Lemma 3. Melbourne and Roberto use this lemma twice in an iterated argument; once to deduce bounds on $|a_{j_0}|$, and then on $|a_{j_1}|$. We will need something similar to this lemma in our proof.

1.2.2 Proof by Self-Improvement

Also in 2021, Chasapis, Nayar, and Tkocz proved the following stability result for cube slicing:

Theorem 6 (Chasapis, Nayar, Tkocz, [5]). *There exists a constant $c > 0$ such that, for each $n \geq 1$ and unit vector $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$,*

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq \sqrt{2} - c\left|a - \frac{e_1 + e_2}{\sqrt{2}}\right|.$$

There are a number of tools used in the proof of this theorem that have corresponding analogues in the proof of our result. Firstly, there's the probabilistic formula for the slice volume in terms of independent uniform random variables on the sphere S^2 in $\mathbb{R}^3$:

Lemma 7 (König, Koldobsky, [9]). *For $n \geq 2$ and a unit vector $a \in \mathbb{R}^n$, we have:*

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) = \mathbb{E}\left|\sum_{j=1}^{n} a_j \xi_j\right|^{-1} \qquad \textit{where} \quad \xi_j \sim \mathrm{Unif}(S^2) \textit{ i.i.d. (indep., identically distributed)}$$

To make use of independence, the authors also prove the following lemma:

Lemma 8 (Chasapis, Nayar, Tkocz, [5]). *Let X and Y be two independent rotationally invariant random vectors in $\mathbb{R}^3$. Then,*

$$\mathbb{E}|X+Y|^{-1} = \mathbb{E}\min\left\{|X|^{-1}, |Y|^{-1}\right\} \leq \min\left\{\mathbb{E}|X|^{-1}, \mathbb{E}|Y|^{-1}\right\}.$$

In particular, $\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq \min_{1 \leq j \leq n}\left\{|a_j|^{-1}\right\}$.

Secondly, we have Busemann's theorem, and (as a corollary) Lipschitz continuity of the slice volume.

Theorem 9 (Busemann, [8]). *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$a \mapsto \frac{|a|}{\mathrm{vol}_{n-1}\left(K \cap a^{\perp}\right)}$$

(extended by 0 at 0) defines a norm on $\mathbb{R}^n$.

Lemma 10 (Chasapis, Nayar, Tkocz, [5]). *For unit vectors $a, b \in \mathbb{R}^n$,*

$$\left|\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) - \mathrm{vol}_{n-1}\left(Q_n \cap b^{\perp}\right)\right| \leq 2|a-b|.$$

Interestingly, the proof of this lemma actually invokes the original cube slicing inequality (Theorem 2). At the expense of a constant factor of 2, we can use a much more general result (due to Hensley, see [10]) about sections of arbitrary convex bodies. Although this is not the "self-improvement" I referred to earlier, it is nonetheless an instance of the original inequality being used to prove an improved version of itself.

With these results established, the proof of Theorem 6 is split into two cases: One where a is close to the extremizer, and one where a is far from the extremizer.

There is a lot that goes on in the first case, but I want to highlight the self-improvement argument because it will show up in the proof of our result as well. After dealing with the case of $n = 2$ directly, we assume $n \geq 3$. Let ξ_j be random variables as in Lemma 7, $X = a_1\xi_1 + a_2\xi_2$, and $Y = \sum_{j=3} a_j\xi_j$. From Lemma 7 and Lemma 8,

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) = \mathbb{E}\min\left\{|X|^{-1}, |Y|^{-1}\right\}.$$

Now for the self-improvement step: Applying the probabilistic formulation of the cube slicing inequality (i.e. Theorem 2 applied to Lemma 7, with an additional case for $n = 1$) to Y yields

$$\mathbb{E}|Y|^{-1} \leq \sqrt{2}\left(1 - a_1^2 - a_2^2\right)^{-1/2}.$$

Combining these two establishes

$$\mathrm{vol}_{n-1}\left(Q_n \cap a^{\perp}\right) \leq \mathbb{E}\min\left\{|X|^{-1}, \sqrt{2}\left(1 - a_1^2 - a_2^2\right)^{-1/2}\right\}.$$

So, we no longer need to think about the rest of the coordinates $a_3, \ldots, a_n$, but we didn't sacrifice too much by ignoring them. The proof does not end here, but it is an important starting point.

The second case makes use of Ball's integral inequality (Lemma 3) and also makes use of Lemma 10; an analogous argument will occur in the proof of our result as well.

1.3 Polydisc Slicing

Recalling the discussion in Section 1, the complex polydisc $\mathbb{D}^n$ is a natural generalization of the real hypercube Q_n. In 2000, Oleszkiewicz and Pełczyński proved the following result about sections of the unit polydisc in $\mathbb{C}^n$:

Theorem 11 (Oleszkiewicz, Pełczyński, [6]). *For $n \geq 2$ and every unit vector $a \in \mathbb{C}^n$,*

$$1 \leq \frac{1}{\pi^{n-1}} \operatorname{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \leq 2.$$

The proof of the upper bound in Theorem 11 is quite similar to Ball's proof of the upper bound in Theorem 2. The key difference is that the integrand in the Fourier-analytic formula for the section volume is a different (and somewhat more complicated) function:

1. By Remark 1, we can assume without loss of generality that $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. This is done by first applying an n-fold rotation to a to orient each coordinate along the positive real line, and then sorting the coordinates in nonincreasing order. After proving the result for this transformed a, we invert the transformation to obtain the original $a^\perp$. The geometric symmetries of $\mathbb{D}^n$ ensure that volume is preserved under this inversion.
2. First, we address the case when $a_1 > \frac{1}{\sqrt{2}}$ with a geometric argument.
3. It remains to handle the case where $a_1 \leq \frac{1}{\sqrt{2}}$. We can assume $a_1 > 0$ by induction on n.
4. Using Fourier inversion and Hölder's inequality with exponents $p_j = a_j^{-2}$, we obtain the following bound on the slice volume:

   $$\frac{2}{\pi^{n-1}} \operatorname{vol}_{2n-2}\left(\mathbb{D}^n \cap a^\perp\right) = \int_0^\infty \left(\prod_{j=1}^n j_1(a_j t)\right) t \,\mathrm{d}t \leq \prod_{j=1}^n a_j^{-2p_j^{-1}} \left(\int_0^\infty |j_1(t)|^{p_j}\, t \,\mathrm{d}t\right)^{p_j^{-1}}. \quad (2)$$

   In the above, $j_1(t)$ is $\frac{2J_1(t)}{t}$, where J_1 is the Bessel function of the first kind:

   $$j_1(t) = \frac{2J_1(t)}{t} \text{ for } t > 0\,,\quad j_1(0) = 1\,, \quad \text{where} \quad J_1(t) = \sum_{k=0}^\infty (-1)^k \left(\frac{t}{2}\right)^{2k+1} \frac{1}{k!(k+1)!}.$$

5. The proof concludes with a final upper bound of 4 on (2) by using an analogue of Ball's integral inequality (Lemma 3) for j_1, presented below in Lemma 12.

Lemma 12 (Oleszkiewicz, Pełczyński, [6]). *For $p \geq 2$,*

$$\int_0^\infty |j_1(t)|^p\, t \,\mathrm{d}t \leq \frac{4}{p}.$$

Furthermore, equality holds if and only if $p = 2$.

2 Our Approach and Its Differences From the Real Case

For brevity, we define the normalized section volume function $A_n(a)$ via

$$A_n(a) = \frac{1}{\pi^{n-1}} \operatorname{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp).$$

Generally, our proof is analogous to the work done on the stability of cube slicing; for each lemma in Section 1.2, we will have a corresponding lemma in our proof. We will use those lemmas in a case analysis similar to that discussed in Section 1.2.2, where one case uses a self-improvement argument and other cases make use of these lemmas.

However, there is one major difference between the real and complex cases that complicates our case analysis: The polydisc slicing inequalities have another extremizer for the upper bound! It does not make itself shown in the proof of the polydisc slicing inequalities (Theorem 11) because the dimension is finite, but when trying to prove stability we discovered there is an asymptotic extremizer as $n \to \infty$, namely

$$a^* = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots \frac{1}{\sqrt{n}}\right).$$

This is why our theorem statement in Section 1 contains a minimum with the ℓ_4-norm of a in the upper bound; we need to account for the situation where all coordinates of a are small. This additional obstacle significantly complicates the case analysis; there are now five cases to consider instead of the two discussed in Section 1.2.2. Here is a brief summary of these cases:

1. Case 1 (a_1 and a_2 are close to $\frac{1}{\sqrt{2}}$): We use a self-improvement argument, like the one discussed in Section 1.2.2, applying the polydisc slicing inequalities in a lower dimension.analogous to the one presented in Section 1.2.1.

2. Case 2 (a_1 is below $\frac{1}{\sqrt{2}}$): We use a multidimensional analogue of the Berry-Esseen theorem (see [15]) and a generalization of Lemma 8.

3. Case 3 (a_1 is slightly above $\frac{1}{\sqrt{2}}$): We use an analogue to the Lipschitz continuity result in Lemma 10 to reduce this case to previous cases.

4. Case 4 (a_1 is well above $\frac{1}{\sqrt{2}}$): We use a geometric argument, much like how the case $a_1 \geq \frac{1}{\sqrt{2}}$ handled in the proofs of the cube and polydisc slicing inequalities (Theorem 2, Theorem 11).

5. Case 5 (all coordinates of a are close to 0): We make use of Fourier-analytic tools (like (2) from the proof of the polydisc slicing inequalities) and an integral inequality.

3 Tools and Background Results

3.1 Fourier-Analytic Volume Formula and Integral Inequality

From (2) in the proof of the polydisc slicing inequalities (Theorem 11), we have the following Fourier-analytic formula for the section volume:

$$A_n(a) = \frac{1}{2}\int_0^\infty \left(\prod_{j=1}^n j_1(a_j t)\right) t \,\mathrm{d}t. \tag{3}$$

Also by (2), setting $p_j = a_j^{-2}$ and using the fact that $\sum_{j=1}^n p_j^{-1} = |a|^2 = 1$, we have:

$$\begin{aligned} A_n(a) &\leq \frac{1}{2}\prod_{j=1}^n a_j^{-2p_j^{-1}} \left(\int_0^\infty |j_1(t)|^{p_j} t \,\mathrm{d}t\right)^{p_j^{-1}} \\ &= \frac{2}{4}\prod_{j=1}^n \left(p_j \int_0^\infty |j_1(t)|^{p_j} t \,\mathrm{d}t\right)^{p_j^{-1}} = 2\prod_{j=1}^n \left(\frac{p_j}{4}\int_0^\infty |j_1(t)|^{p_j} t \,\mathrm{d}t\right)^{p_j^{-1}}. \end{aligned}$$

For $s > 0$, we define a normalized integral of j_1 via

$$\Psi(s) = \frac{s}{4}\int_0^\infty |j_1(t)|^s t \,\mathrm{d}t.$$

From 9.2.1 in [12], we can see that the integrand is $O(t^{1-3s/2})$ as $t \to \infty$, which means $\Psi(s)$ is finite whenever $s > 4/3$. For ease, we define $\Phi(s) = \infty$ whenever $s \leq 4/3$. Rewriting the bound on $A_n(a)$ in terms of Ψ yields

$$A_n(a) \leq 2\prod_{j=1}^n \Psi\left(a_j^{-2}\right)^{a_j^2}. \tag{4}$$

The proof of the polydisc slicing inequality (Theorem 11) was heavily dependent on the fact that $\Psi(s) \leq 1$ for $s \geq 2$ with equality if and only if $s = 2$. Now, as discussed in Section 2, our proof requires handling an infinite-dimensional extremizer. Here, this manifests in the fact that $\Psi(s)$ approaches its supremum of 1 as $s \to \infty$. We can quantify its behavior at $s = 2$ and as $s \to \infty$ with the following refinement on the proof of Ball's integral inequality (Lemma 12):

Lemma 13. *We have the following bounds on* $\Psi(s)$*:*

$$\Psi(s) \leq \begin{cases} 1 - \frac{1}{12}(s-2)^2, & \text{if } s \in \left[2, \frac{8}{3}\right], \\ 1 - \frac{1}{151s}, & \text{if } s \in \left[\frac{8}{3}, \infty\right). \end{cases}$$

Proof. First, consider the case when $s \in \left[2, \frac{8}{3}\right]$. As shown in [6] (*Proof of Proposition 1.1 in Case*

(II), p. 289-291),

$$\Psi(s) \le \frac{s}{2} e^{-\frac{s-2}{2}} = (v+1)e^{-v} \;, \quad \text{where} \quad v = \frac{s}{2} - 1 \in \left[0, \frac{1}{3}\right].$$

By an elementary inequality, we get

$$(v+1)e^{-v} \le (v+1)\left(1 - v + \frac{v^2}{2}\right) = 1 - \frac{v^2}{2} - + \frac{v^3}{2} \le 1 - \frac{v^2}{3}.$$

Converting back from v to s yields

$$\Psi(s) \le 1 - \frac{\left(\frac{s}{2} - 1\right)^2}{3} = 1 - \frac{\frac{1}{4}(s-2)^2}{3} = 1 - \frac{(s-2)^2}{12}.$$

This completes the proof of the first case.

Next, suppose $s \in (8/3, \infty)$. As shown in [6] (*Proof of Proposition 1.1 in Case (I)*, p. 288), we have

$$\int_0^\infty |j_1(t)|^s \, t \, \mathrm{d}t \le \frac{32}{3s-4}(60\pi^2)^{-s/4} + \frac{4}{s} - \frac{4}{3s^2} + \frac{4}{3s^3}.$$

Multiplying through by $\frac{s}{4}$, we obtain

$$\Phi(s) \le 1 - \frac{1}{3s} + \frac{2}{3s^2} + \frac{8s}{3s-4}(60\pi^2)^{-s/4}.$$

To match what we're trying to prove, we can pull out a factor of $\frac{1}{s}$ to get

$$\Phi(s) \le 1 - \frac{1}{s}\left(\frac{1}{3} - \frac{1}{3s} - \frac{8s^2}{3s-4}(60\pi^2)^{-s/4}\right).$$

One can check by differentiation that the function in the parentheses is increasing on $[8/3, \infty)$, so it is bounded below by its value at $s = \frac{8}{3}$. It can be checked by calculator that this is slightly greater than $\frac{1}{151}$, completing the proof of the second case. $\square$

3.2 Probabilistic Volume Formula and Independence Lemma

Analogous to Lemma 7 for the cube, we will make use of the following probabilistic formula for the section volume of the polydisc. We also need a generalization of Lemma 8.

Lemma 14 (Chasapis, Singh, Tkocz, [13]). *For every $n \geq 2$ and unit vector $a \in \mathbb{C}^n$, we have:*

$$A_n(a) = \mathbb{E}\left|\sum_{j=1}^{n} a_j \xi_j\right|^{-2}, \quad \textit{where} \quad \xi_j \sim \mathrm{Unif}(S^3) \ \textit{i.i.d.}$$

Remark 3. Replacing $A_n(a)$ with the probabilistic formulas in Theorem 11 allows us to generalize the polydisc slicing inequalities to $n \geq 1$; the inequalities trivially hold for $\mathbb{E}|a_1\xi_1|^{-2}$.

Lemma 15. *Let $d \geq 3$ and let X, Y be independent, rotationally invariant random vectors in $\mathbb{R}^d$. Then,*

$$\mathbb{E}|X+Y|^{2-d} = \mathbb{E}\min\left\{|X|^{2-d}, |Y|^{2-d}\right\}.$$

Proof. Let ξ_1 and ξ_2 be independent random vectors with uniform distribution on the unit sphere S^{d-1} in $\mathbb{R}^d$. By rotational invariance, $\frac{X}{|X|}$ and $\frac{Y}{|Y|}$ are uniformly distributed. In other words, X and Y have the same distributions as $|X|\xi_1$ and $|Y|\xi_2$, respectively. Therefore, using the law of total expectation to condition on the values of $|X|$ and $|Y|$, it suffices to show for all $a_1, a_2 \geq 0$ that

$$\mathbb{E}\left|a_1\xi_1 + a_2\xi_2\right|^{2-d} = \min\left\{a_1^{2-d}, a_2^{2-d}\right\}.$$

If either a_1 or a_2 is 0, this is trivial. It is also straightforward if $a_1 = a_2$. Furthermore, by commutativity, we can assume $a_1 > a_2$. Therefore, dividing through by a_1, it suffices to show for $t \in (0,1)$ that

$$h(t) = \min\left\{1, t^{2-d}\right\} = 1\,, \quad \text{where} \quad h(t) = \mathbb{E}\left|\xi_1 + t\xi_2\right|^{2-d}.$$

Since ξ_1 has the same distribution as Ue_1 for a random orthogonal matrix (according to the Haar measure), $|\xi_1 + t\xi_2|$ has the same distribution as $|e_1 + t\xi_2|$. Thus,

$$h(t) = \mathbb{E}\left|e_1 + t\xi_2\right|^{2-d} = \frac{1}{\mathrm{vol}_{d-1}(S^{d-1})}\int_{S^{d-1}} |e_1 + t\xi|^{2-d}\,\mathrm{d}\xi.$$

Letting $F(x) = |x|^{2-d}$, and ignoring the constant volume factor, we use the divergence theorem:

$$\begin{aligned} h'(t) &\propto \frac{\mathrm{d}}{\mathrm{d}t}\int_{S^{d-1}} F(e_1 + t\xi)\,\mathrm{d}\xi = \int_{S^{d-1}} \langle \nabla F(e_1 + t\xi), \xi\rangle\,\mathrm{d}\xi \\ &= \int_{B^d} \mathrm{div}_x(\nabla F(e_1 + tx))\,\mathrm{d}x = t\int_{B^d} \Delta F(e_1 + tx)\,\mathrm{d}x = 0. \end{aligned}$$

The last step holds because $e_1 + tx$ is nonvanishing on B^d. Thus, h is constant, and $h(0) = 1$, so $h(t) = 1$ for all $t \in (0,1)$ as desired. □

3.3 Complex Busemann's Theorem and Volume Lipschitz Property

Much like the real case, particularly the work presented in Section 1.2.1, we will need Lipschitz continuity of the section function. We have the following complex analogue of Busemann's theorem:

Theorem 16 (Koldobsky, Paouris, Zymonopoulou, [14]). *Let K be a complex symmetric convex body K in $\mathbb{C}^n$, that is K is a convex body in $\mathbb{R}^{2n}$ with $e^{it}z \in K$ whenever $z \in K$, $t \in \mathbb{R}$. Then*

$$z \mapsto \frac{|z|}{\sqrt{\mathrm{vol}_{2n-2}(K \cap z^{\perp})}}$$

(extended by 0 at 0) defines a norm on $\mathbb{C}^n$.

Lemma 17 (Volume Lipschitz Property). *For unit vectors $a, b \in \mathbb{R}^n$,*

$$|A_n(a) - A_n(b)| \leq 4\sqrt{2} \cdot |a - b|.$$

Proof. Let $K = (\mathbb{D}/\pi)^n$ be the unit-volume polydisc so that $A_n(z) = \mathrm{vol}_{2n-2}(K \cap z^{\perp})$ for all $z \in \mathbb{C}^n$. Clearly K is convex and closed under scalar multiplication by e^{it} for $t \in \mathbb{R}$. So, invoking Theorem 16, we have that the following is a norm on $\mathbb{C}^n$:

$$N(z) = \frac{|z|}{\sqrt{\mathrm{vol}_{2n-2}(K \cap z^{\perp})}} = |z| A_n(z)^{-1/2}.$$

By the polydisc slicing inequalities (Theorem 11),

$$\frac{1}{\sqrt{2}}|z| \leq N(z) \leq |z| \quad \text{for all } z \in \mathbb{C}^n. \tag{5}$$

Therefore, recalling that a and b are unit vectors, we calculate:

$$\begin{aligned}
|A_n(a) - A_n(b)| &= \left|N(a)^{-2} - N(b)^{-2}\right| && \text{definition of } N \\
&= \left|\left(N(a)^{-1} - N(b)^{-1}\right)\left(N(a)^{-1} + N(b)^{-1}\right)\right| && \text{difference of squares} \\
&= \left|\left(\frac{N(b) - N(a)}{N(a)N(b)}\right)\left(\frac{N(b) + N(a)}{N(a)N(b)}\right)\right| && \text{cross-multiplication} \\
&= \frac{N(a) + N(b)}{N(a)^2 N(b)^2}\,|N(a) - N(b)| && \text{homogeneity of } N \\
&\leq \frac{N(a) + N(b)}{N(a)^2 N(b)^2} N(a - b) && \text{triangle inequality for } N \\
&\leq \frac{1 + 1}{(1/\sqrt{2})^2 (1/\sqrt{2})^2} N(a - b) = 8N(a - b) && \text{by (5)} \\
&\leq \frac{8}{\sqrt{2}} \cdot |a - b| = 4\sqrt{2} \cdot |a - b| && \text{by (5)}
\end{aligned}$$

Thanks to the complex analogue of Theorem 9, this proof is very similar to the real case. ☐

4 Main Result: Stability of Polydisc Slicing

For convenience, we restate the polydisc slicing stability inequalities here, this time in terms of some additional definitions and only for $n \geq 3$.

Theorem 18. *For $n \geq 3$ and every unit vector $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have*

$$1 + \frac{1}{8}|a - e_1|^2 \leq A_n(a) \leq 2 - \min\left\{ 10^{-40}\sqrt{\delta(a)},\ \frac{1}{76}\sum_{j=1}^{n} a_j^4 \right\},$$

where we define

$$A_n(a) := \frac{1}{\pi^{n-1}} \operatorname{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \quad \textit{and} \quad \delta(a) := \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|^2 = 2 - \sqrt{2}(a_1 + a_2).$$

Remark 4. Note that we've assumed $n \geq 3$ in the statement of Theorem 18. This assumption will be helpful in the proof, and we can handle the $n = 2$ quite easily; see Appendix B.

Remark 5. The proof of the lower bound (including when $n = 2$) is fairly short, and is not the primary focus of this work. Nonetheless, we include it for completeness; see Appendix A.

It remains to prove the upper bound in Theorem 18. The proof is split into five cases (one of which has two sub-cases) depicted in Figure 1; each region represents a set of possibilities for the point (a_1, a_2). We will refer to these cases by the numbers of the sections in which they are handled.

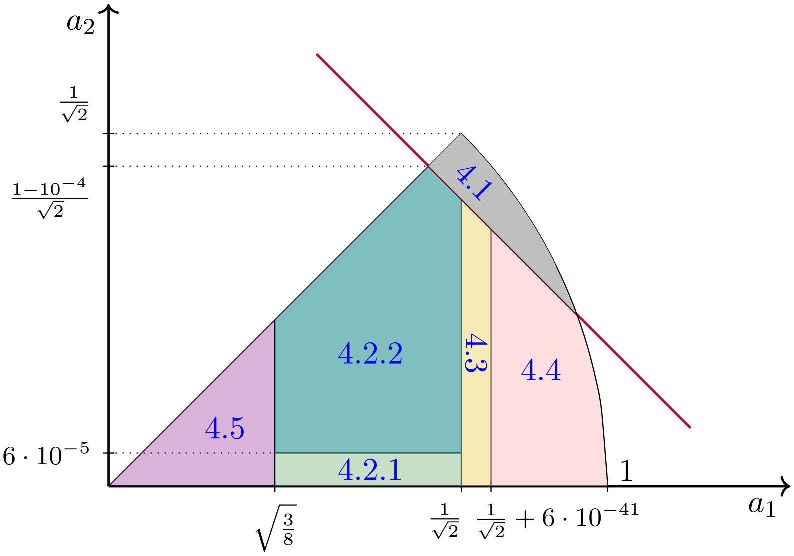

Figure 1: Adapted from [11], this diagram shows the six cases of the proof of the upper bound in Theorem 18. The labels correspond to the section numbers where each case is handled.

4.1 First two coordinates are close to $\frac{1}{\sqrt{2}}$

In this case, we consider vectors a that are close to the extremizer. We rely on a self-improving property of the polydisc slicing inequalities.

Lemma 19. *Assume $\delta(a) \le \frac{1}{5000}$. Then, we have $A_n(a) \le 2 - \frac{1}{25}\sqrt{\delta(a)}$.*

Proof. Let ξ_j be i.i.d. uniform random vectors on the unit sphere S^3 in $\mathbb{R}^4$. Now, if $a_1^2+a_2^2 = 1$, then the claim trivially reduces to the $n = 2$ case, covered in Appendix B. So, we assume $a_1^2 + a_2^2 < 1$. Since $n \ge 3$, we can let $X = a_1\xi_1 + a_2\xi_2$ and $Y = \sum_{j=3}^n a_j\xi_j$. By Lemma 14, then Lemma 15 with $d = 4$ applied to X and Y, and finally Jensen's inequality applied to the concave function $t \mapsto \min\{\alpha, t\}$ for fixed $\alpha \in \mathbb{R}$, we have

$$A_n(a) = \mathbb{E}\min\{|X|^{-2}, |Y|^{-2}\} \le \mathbb{E}_X \min\{|X|^{-2}, \mathbb{E}_Y|Y|^{-2}\}.$$

By the probabilistic polydisc slicing inequalities applied to Y (generalized for to the 1-dimensional case, see Remark 3), we get $\mathbb{E}_Y|Y|^{-2} \le \frac{2}{1-a_1^2-a_2^2}$. The factor of $1 - a_1^2 - a_2^2$ is to make sure the coefficients in the linear combination inside Y form the coordinates of a unit vector; this is necessary to invoke the polydisc slicing inequalities. Therefore, applying this to the bound on $A_n(a)$, we get

$$A_n(a) \le \mathbb{E}\min\left\{|X|^{-2}, \frac{2}{1-a_1^2-a_2^2}\right\} = \mathbb{E}|X|^{-2} - \mathbb{E}\left(|X|^{-2} - \frac{2}{1-a_1^2-a_2^2}\right)_+ .$$

Once again using Lemma 15 with $d = 4$ applied to X, we have that $\mathbb{E}|X|^{-2} = \min\{a_1^{-2}, a_2^{-2}\} = a_1^{-2}$. Before writing down the final bound on A_n, it will be convenient to introduce the rotated variables

$$u_1 = \frac{a_1 + a_2}{\sqrt{2}} \quad \text{and} \quad u_2 = \frac{a_1 - a_2}{\sqrt{2}},$$

for which $u_1 = 1 - \frac{\delta(a)}{2} \in \left[1 - 10^{-4}, 1\right]$, $u_2 > 0$, and $u_1^2 + u_2^2 = a_1^2 + a_2^2 < 1$. Also, we can see that

$$|X|^2 = \langle a_1\xi_1 + a_2\xi_2, a_1\xi_1 + a_2\xi_2\rangle = a_1^2 + a_2^2 + 2a_1a_2\theta = u_1^2 + u_2^2 + (u_1^2 - u_2^2)\theta, \tag{6}$$

where θ is a random variable with density $x \mapsto \frac{2}{\pi}(1 - x^2)^{1/2}$ on $[-1, 1]$ (the distribution of $\langle\xi_1, \xi_2\rangle$ which is the same as that of $\langle\xi_1, e_1\rangle$). This will be useful later. Anyways, in terms of these variables, our bound on $A_n(a)$ becomes

$$\frac{1}{2}A_n(a) \le \frac{1}{(u_1 + u_2)^2} - \mathbb{E}\left(\frac{1}{2}|X|^{-2} - \frac{1}{1 - u_1^2 - u_2^2}\right)_+ . \tag{7}$$

From here, we case on whether $u_1^2 + 9u_2^2 \ge 1$.

Case 1: $u_1^2 + 9u_2^2 \ge 1$. In this case, we simply ignore the second term in (7) and rearrange the case's assumption to $u_2 \geq \sqrt{\frac{1-u_1^2}{9}}$. This yields the upper bound

$$\frac{1}{2}A_n(a) \leq \frac{1}{(u_1+u_2)^2} \leq \frac{1}{\left(u_1 + \sqrt{\frac{1-u_1^2}{9}}\right)^2}.$$

Letting $\delta = \delta(a) \in \left[0, \frac{1}{5000}\right]$ for brevity, we lower bound the denominator with

$$u_1 + \sqrt{\frac{1-u_1^2}{9}} = 1 - \frac{\delta}{2} + \sqrt{\frac{\delta}{18}\left(2 - \frac{\delta}{2}\right)} \geq 1 - \frac{\delta}{2} + \sqrt{\frac{\delta}{10}} \geq 1 + \frac{1}{2}\sqrt{\frac{\delta}{10}}.$$

These inequalities are straightforward to verify with differentiation and calculation, utilizing the condition that $\delta \leq \frac{1}{5000}$. Applying this to our bound on $A_n(a)$, we obtain

$$A_n(a) \leq 2\left(1 + \frac{1}{2}\sqrt{\frac{\delta}{10}}\right)^{-2} \leq 2\left(1 - \frac{1}{2}\sqrt{\frac{\delta}{10}}\right) = 2 - \frac{1}{\sqrt{10}}\sqrt{\delta(a)} \leq 2 - \frac{1}{25}\sqrt{\delta(a)},$$

where we used the elementary inequality $(1+x)^{-2} \leq 1 - x$ for $x \in \left[0, \frac{1}{2}\right]$. This finishes the case.

Case 2: $u_1^2 + 9u_2^2 < 1$. In this case, we will need to analyze the second term of (7). This is done with the classic probabilistic trick of truncated expectations;

$$\begin{aligned} \mathbb{E}\left(\frac{1}{2}|X|^{-2} - \frac{1}{1-u_1^2-u_2^2}\right)_+ &\geq \mathbb{E}\left[\left(\frac{1}{2}|X|^{-2} - \frac{1}{1-u_1^2-u_2^2}\right)_+ \mathbf{1}_{\left\{\frac{1}{2}|X|^{-2} \geq \frac{2}{1-u_1^2-u_2^2}\right\}}\right] \\ &\geq \frac{1}{1-u_1^2-u_2^2}\mathbb{E}\left[\mathbf{1}_{\left\{|X|^{-2} \geq \frac{2}{1-u_1^2-u_2^2}\right\}}\right] \\ &= \frac{1}{1-u_1^2-u_2^2}\mathbb{P}\left(|X|^2 \leq \frac{1-u_1^2-u_2^2}{4}\right). \end{aligned}$$

Now, recall that $|X|^2 = u_1^2 + u_2^2 + (u_1^2 - u_2^2)\theta$. With this, the condition $|X|^2 \leq \frac{1-u_1^2-u_2^2}{4}$ is equivalent to $\theta \leq \frac{1-5(u_1^2+u_2^2)}{4(u_1^2-u_2^2)} = -1 + \theta_0$ with $\theta_0 = \frac{1-u_1^2-9u_2^2}{4(u_1^2-u_2^2)}$. By the case's assumption, $0 < \theta_0$. Also, $\theta_0 < 1$, since $u_1 > u_2$ and $5(u_1^2+u_2^2) \geq 5u_1^2 = 5(1-\delta(a)/2)^2 \geq 5(1-10^{-4})^2 > 1$ implies $-1+\theta_0 < 0$. Therefore, using this bound $0 < \theta_0 < 1$, we estimate the probability of the event $\theta \leq -1 + \theta_0$ by

$$\mathbb{P}\left(\theta \leq -1 + \theta_0\right) = \frac{2}{\pi}\int_{-1}^{-1+\theta_0} \sqrt{1-x^2}\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\theta_0} \sqrt{x(2-x)}\,\mathrm{d}x \geq \frac{2}{\pi}\int_0^{\theta_0} \sqrt{x}\,\mathrm{d}x = \frac{4}{3\pi}\theta_0^{3/2}.$$

Applying this to (7), and using the facts that $1 - u_1^2 - u_2^2 \leq 1 - u_1^2$ and $u_1^2 - u_2^2 \leq 1$, we obtain

$$\begin{aligned} \frac{1}{2}A_n(a) &\leq \frac{1}{(u_1+u_2)^2} - \frac{1}{1-u_1^2-u_2^2}\mathbb{P}\left(4|X|^2 \leq 1 - u_1^2 - u_2^2\right) \\ \leq \frac{1}{(u_1+u_2)^2} - \frac{1}{1-u_1^2-u_2^2} \cdot \frac{4}{3\pi}\left(\frac{1-u_1^2-9u_2^2}{4(u_1^2-u_2^2)}\right)^{3/2} &\leq \frac{1}{(u_1+u_2)^2} - \frac{1}{6\pi}\frac{(1-u_1^2-9u_2^2)^{3/2}}{1-u_1^2}. \quad (8) \end{aligned}$$

We now claim that the quantity in (8) is decreasing as a function of u_2, and therefore bounded from above by its value when $u_2 = 0$. To see this, note that its derivative is

$$-2(u_1+u_2)^{-3} + \frac{9}{2\pi}\frac{u_2(1-u_1^2-9u_2^2)^{1/2}}{1-u_1^2} \le -2(u_1+u_2)^{-3} + \frac{9}{2\pi}\frac{u_2}{\sqrt{1-u_1^2}}.$$

Since $u_1 + u_2 = a_1\sqrt{2} < \sqrt{2}$, the first term is at most $-\frac{1}{\sqrt{2}}$. Since $1 - u_1^2 \ge 9u_2^2$, the second term is at most $\frac{3}{2\pi} < \frac{1}{2}$, and so the derivative is at most $-\frac{1}{\sqrt{2}} + \frac{1}{2} < 0$. Therefore, setting $u_2 = 0$ in (8) gives the upper bound

$$\frac{1}{2}A_n(a) \le \frac{1}{u_1^2} - \frac{1}{6\pi}\sqrt{1-u_1^2} = \left(1-\frac{\delta}{2}\right)^{-2} - \frac{1}{6\pi}\sqrt{\frac{\delta}{2}\left(2-\frac{\delta}{2}\right)} \le 1+2\delta - \frac{1}{6\pi}\sqrt{1-\frac{1}{2}\cdot 10^{-4}}\sqrt{\delta},$$

where we have used the elementary inequality $(1-x/2)^{-2} \le 1+2x$ whenever $0 \le x \le \frac{1}{2}$. Since $\delta(a) \le \sqrt{\frac{1}{5000}}\sqrt{\delta(a)}$, we can further bound the right-hand side with

$$\frac{1}{2}A_n(a) \le 1+2\delta - \frac{1}{6\pi}\sqrt{1-\frac{1}{2}\cdot 10^{-4}}\sqrt{\delta} \le 1 + \left(\frac{2}{\sqrt{5000}} - \frac{1}{6\pi}\sqrt{1-\frac{1}{2}\cdot 10^{-4}}\right)\sqrt{\delta} < 1 - \frac{\sqrt{\delta}}{50}.$$

Multiplying through by 2 finishes this case, and with it the proof. ☐

Since we have just dealt with the case where $\delta(a) \le \frac{1}{5000}$, in some of the remaining cases of the proof we will assume that $\delta(a) \ge \frac{1}{5000}$. This assumption is helpful because it implies a_2 is slightly less than $\frac{1}{\sqrt{2}}$, as shown in the next lemma.

Lemma 20. *Suppose that* $\delta(a) \ge \frac{1}{5000}$. *Then, we have* $a_2 \le \frac{1-10^{-4}}{\sqrt{2}}$.

Proof. As $a_1, a_2 \ge 0$, we have

$$a_2 \le \frac{a_1+a_2}{2} = \frac{1}{2\sqrt{2}}\left(2-(2-\sqrt{2}(a_1+a_2))\right) = \frac{1}{2\sqrt{2}}\left(2-\delta(a)\right) = \frac{1}{\sqrt{2}}\left(1-\frac{\delta(a)}{2}\right).$$

Now, using the assumption on $\delta(a)$,

$$\frac{1}{\sqrt{2}}\left(1-\frac{\delta(a)}{2}\right) \le \frac{1}{\sqrt{2}}\left(1-\frac{5^{-1}\cdot 10^{-3}}{2}\right) = \frac{1}{\sqrt{2}}\left(1-\frac{10^{-3}}{10}\right) = \frac{1-10^{-4}}{\sqrt{2}}.$$

Combining the two inequalities completes the proof. ☐

4.2 First Coordinate is Below $\frac{1}{\sqrt{2}}$

In this case, we assume a_1 is below $\frac{1}{\sqrt{2}}$, but not by very much. We aim to prove the following lemma in this section.

Lemma 21. *Assume that* $\sqrt{\frac{3}{8}} \le a_1 < \frac{1}{\sqrt{2}}$ *and* $a_2 \le \frac{1-10^{-5}}{\sqrt{2}}$. *Then,* $A_n(a) \le 2 - 10^{-19}$.

Proof. Follows immediately from Lemma 23 and Lemma 24. ☐

Note that, instead of assuming $\delta(a) \ge \frac{1}{5000}$ and using Lemma 20 to deduce $a_2 \le \frac{1-10^{-4}}{\sqrt{2}}$, we will directly assume something slightly weaker on a_2. This will be useful in the next case, Section 4.3. Now, as the "proof" suggests, we need to consider two subcases: a_2 is close to zero, and a_2 is bounded away from zero.

The motivation for this casework is that, in the worst case where $a_1 = \frac{1}{\sqrt{2}}$, the Fourier-analytic bound in the proof of Lemma 27 only gives $A_n(a) \le 2\exp\left(-\frac{1}{151}\sum_{j=1}^n a_j^4\right)$. When a_2 is bounded away from 0, this tells us that $A_n(a)$ is bounded away from 2. This argument won't work when a_2 can get arbitrarily close to 0, so we need to consider this case separately.

4.2.1 Second Coordinate is Close to Zero

In this case, we will utilize a Berry-Esseen type inequality with explicit constants to analyze the Gaussian approximation for $\sum_{j=2}^n a_j\xi_j$, where ξ_j are i.i.d. uniform random vectors on the sphere S^3 in $\mathbb{R}^4$. Recently, Raič has obtained such a result for all dimensions.

Theorem 22 (Raič, [15])**.** *Let* $X_1, \dots, X_k$ *be independent mean* 0 *random vectors in* $\mathbb{R}^d$ *such that* $\sum_{j=1}^k X_j$ *has the identity covariance matrix. Let* G *be a standard Gaussian random vector in* $\mathbb{R}^d$. *Then*

$$\sup_A \left| \mathbb{P}\left(\sum_{j=1}^k X_j \in A\right) - \mathbb{P}(G \in A) \right| \le (42d^{1/4} + 16)\sum_{j=1}^k \mathbb{E}|X_j|^3,$$

where the supremum is over all convex Borel sets in $\mathbb{R}^d$.

Lemma 23. *Assume that* $\sqrt{\frac{3}{8}} \le a_1 \le \frac{1}{\sqrt{2}}$ *and* $a_2 \le 6 \cdot 10^{-5}$. *Then,* $A_n(a) \le 2 - 10^{-19}$.

Proof. Let ξ_j be i.i.d. uniform random vectors on the sphere S^3 in $\mathbb{R}^4$ and define $Y = \sum_{j=2}^n a_j\xi_j$. By Lemma 14 and then Lemma 15 applied to $X = a_1\xi_1$ and Y, we have

$$A_n(a) = \mathbb{E}\left|a_1\xi_1 + Y\right|^{-2} = \mathbb{E}\min\left\{a_1^{-2}, |Y|^{-2}\right\} = \int_0^{a_1^{-2}} \mathbb{P}\left(|Y|^{-2} > t\right)\,\mathrm{d}t.$$

We'd like to apply Theorem 22 to Y, but this does require normalizing the X_j to acheive an identity covariance matrix; currently, the covariance matrix of Y is $\frac{1-a_1^2}{4}$ times the identity matrix.

So, applying Theorem 22 with $d = 4$ to $X_j = \frac{2}{\sqrt{1-a_2^2}} a_j \xi_j$ for $j \geq 2$, we get

$$\mathbb{P}\left(|Y|^{-2} > t\right) \leq \mathbb{P}\left(\left(\sqrt{\tfrac{1-a_1^2}{4}}|G|\right)^{-2} > t\right) + (42\sqrt{2}+16)\sum_{j=2}^{n} \mathbb{E}\left|\frac{2}{\sqrt{1-a_1^2}} a_j \xi_j\right|^3, \tag{9}$$

where G is a standard Gaussian random vector in $\mathbb{R}^4$. Since $|G|^2$ has density $x \mapsto \frac{x}{4} e^{-x/2}$ on $(0, \infty)$, the first term of (9) is

$$\begin{aligned}
\int_0^{a_1^{-2}} \mathbb{P}\left(\left(\sqrt{\tfrac{1-a_1^2}{4}}|G|\right)^{-2} > t\right) \mathrm{d}t &= \mathbb{E} \min\left\{a_1^{-2}, \left(\sqrt{\tfrac{1-a_1^2}{4}}|G|\right)^{-2}\right\} \\
&= \int_0^\infty \min\left\{a_1^{-2}, \frac{4}{1-a_1^2}\frac{1}{x}\right\} \frac{x}{4} e^{-x/2}\, \mathrm{d}x \\
&= \frac{1}{a_1^2}\left(1 - e^{-\frac{2a_1^2}{1-a_1^2}}\right).
\end{aligned}$$

The integral on the second line is computed by splitting the domain of integration to eliminate the minimum function, and then using elementary techniques. For the second term of (9), we have

$$\sum_{j=2}^{n} \mathbb{E}\left|\frac{2}{\sqrt{1-a_1^2}} a_j \xi_j\right|^3 = \frac{8}{(1-a_1^2)^{3/2}} \sum_{j=2}^{n} a_j^3 \leq \frac{8a_2}{(1-a_1^2)^{3/2}} \sum_{j=2}^{n} a_j^2 = \frac{8a_2}{\sqrt{1-a_1^2}},$$

where we have used the facts that $a_j \leq a_2$ for $j \geq 2$ and $\sum_{j=2}^{n} a_j^2 = 1 - a_1^2$. Applying these to (9) and then to the bound on $A_n(a)$, we get

$$A_n(a) \leq \frac{1}{a_1^2}\left(1 - e^{-\frac{2a_1^2}{1-a_1^2}}\right) + \frac{8(42\sqrt{2}+16)a_2}{a_1^2\sqrt{1-a_1^2}}.$$

Taking the derivative of the first term with respect to a_1^2, we can see that it is a decreasing function of a_1^2, and thus bounded by its value at $a_1^2 = \frac{3}{8}$ (which is our assumed lower bound for a_1^2). Thus, using the bounds on a_1^2 and the upper bound on a_2, we obtain

$$A_n(a) \leq \frac{8}{3}\left(1 - e^{-\frac{6}{5}}\right) + \frac{8(42\sqrt{2}+16)\cdot 6 \cdot 10^{-5}}{\frac{3}{8}\sqrt{\frac{1}{2}}} < 2 - 10^{-5}.$$

The last inequality can be checked by calculator. This is less than $2 - 10^{-19}$, so we are done. $\square$

4.2.2 Second Coordinate is Well Above Zero

Lemma 24. *Assume that* $\sqrt{\frac{3}{8}} \leq a_1 \leq \frac{1}{\sqrt{2}}$ *and* $6 \cdot 10^{-5} \leq a_2 \leq \frac{1-10^{-5}}{\sqrt{2}}$. *Then,* $A_n(a) \leq 2 - 10^{-19}$.

Proof. Note that $\Psi(a_j^{-2}) \leq 1$ for all j by Lemma 13, since $a_1^{-2} \geq 2$ and therefore $a_j^{-2} \geq 2$ for all j. Invoking (4) and applying this bound for every j except $j = 2$ yields

$$A_n(a) \leq A_n(a) \leq 2 \prod_{j=1}^{n} \Psi\left(a_j^{-2}\right)^{a_j^2} \leq 2\Psi(a_2^{-2})^{a_2^2}.$$

Furthermore, again using Lemma 13 and our assumptions on a_2, we obtain

$$\begin{aligned} \Psi(a_2^{-2}) &\leq 1 - \min\left\{\tfrac{1}{151} a_2^2,\ \tfrac{1}{12}(a_2^{-2} - 2)^2\right\} \\ &\leq 1 - \min\left\{\tfrac{36}{151} 10^{-10}, \tfrac{1}{3}((1 - 10^{-5})^{-2} - 1)^2\right\} = 1 - \frac{36}{151} \cdot 10^{-10}. \end{aligned}$$

Applying this to our bound on $A_n(a)$ and using Bernoulli's inequality, we get

$$A_n(a) \leq 2\left(1 - \frac{36}{151} \cdot 10^{-10}\right)^{a_2^2} \leq 2\left(1 - \frac{36}{151} \cdot 10^{-10} a_2^2\right) < 2 - 10^{-19}.$$

The last inequality comes from our assumed lower bound on a_2 and can be checked by calculator. $\square$

4.3 First Coordinate is Slightly Above $\frac{1}{\sqrt{2}}$

In this case, we assume a_1 is strictly above $\frac{1}{\sqrt{2}}$, by no more than a margin of $6 \cdot 10^{-41}$.

Lemma 25. *Assume that* $\frac{1}{\sqrt{2}} < a_1 < \frac{1}{\sqrt{2}} + 6 \cdot 10^{-41}$ *and* $\delta(a) \geq \frac{1}{5000}$. *Then,* $A_n(a) \leq 2 - 10^{-20}$.

Proof. Consider a magnitude-preserving distortion of a, namely

$$b = \left(\frac{1}{\sqrt{2}}, \sqrt{a_1^2 + a_2^2 - \frac{1}{2}}, a_3, \dots, a_n \right).$$

Clearly, we have $b_3 \geq \cdots \geq b_n \geq 0$. Furthermore, by the lower bound in our first assumption, $a_1^2 + a_2^2 > \frac{1}{2} + a_2^2 \geq \frac{1}{2} + a_3^2 = \frac{1}{2} + b_3^2$. Thus, we must have

$$b_2 = \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} \geq \sqrt{\frac{1}{2} + b_3^2 - \frac{1}{2}} = b_3.$$

Additionally, by the upper bound in our first assumption and Lemma 20 applied to our second assumption, we get that

$$b_2^2 \leq \left(\frac{1}{\sqrt{2}} + 6 \cdot 10^{-41} \right)^2 + \left(\frac{1 - 10^{-4}}{\sqrt{2}} \right)^2 - \frac{1}{2} < \left(\frac{1 - 10^{-5}}{\sqrt{2}} \right)^2.$$

The above inequality can be checked via calculator and will be used to invoke Lemma 21. Note the final bound on b_2^2 is of course less than $\frac{1}{2}$, so $b_1 \geq b_2$. Thus, $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ and $|b| = 1$; everything we've proven about a, we can now apply to b as well. In particular, applying the Lipschitz property (Lemma 17) to a and b, and then the previous case of the proof (Lemma 21) to b, we obtain

$$A_n(a) \leq A_n(b) + 4\sqrt{2} \cdot |a - b| \leq 2 - 10^{-19} + 4\sqrt{2} \cdot |a - b|.$$

It remains to estimate $|a - b|$, which intuitively should be quite small given the proximity of a_1 to $\frac{1}{\sqrt{2}}$ and the fact that the last $n - 2$ coordinates of a and b are equal. To make this precise, note

$$(b_2 - a_2)(b_2 + a_2) = \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 \right) \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} + a_2 \right) = a_1^2 - \frac{1}{2}$$
$$\leq \left(\sqrt{a_1^2 - \frac{1}{2}} \right) \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} \right) \leq \left(\sqrt{a_1^2 - \frac{1}{2}} \right) \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} + a_2 \right).$$

Dividing through by $b_2 + a_2$, we obtain a bound on $b_2 - a_2$. We can use this and then our first assumption to bound $|a - b|$:

$$|a - b|^2 = \left(a_1 - \frac{1}{\sqrt{2}} \right)^2 + \left(\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 \right)^2 \leq 2a_1 \left(a_1 - \frac{1}{\sqrt{2}} \right) < 10^{-40}.$$

Therefore, $A_n(a) \leq 2 - 10^{-19} + 4\sqrt{2} \cdot 10^{-20} < 2 - 10^{-20}$ as desired. □

4.4 First Coordinate is Well Above $\frac{1}{\sqrt{2}}$

In this case, we assume a_1 is bounded above $\frac{1}{\sqrt{2}}$ by a strictly positive quantity.

Lemma 26. *Assume that* $a_1 \geq \frac{1}{\sqrt{2}} + 6 \cdot 10^{-41}$. *Then,* $A_n(a) \leq 2 - 12\sqrt{2} \cdot 10^{-41}$.

Proof. Applying Lemma 15 with $d = 4$ to the the probabilistic formula for the section function (Lemma 14), taking $X = a_1\xi_1$ and $Y = \sum_{j=2}^{n} a_j\xi_j$, we have

$$A_n(a) = \mathbb{E}|X + Y|^{-2} = \mathbb{E}\min\left\{|X|^{-2}, |Y|^{-2}\right\} \leq \mathbb{E}|X|^{-2} = a_1^{-2}.$$

Using our assumption on a_1,

$$A_n(a) \leq a_1^{-2} \leq \left(\frac{1}{\sqrt{2}} + 6 \cdot 10^{-41}\right)^{-2} = 2\left(1 + 6\sqrt{2} \cdot 10^{-41}\right)^{-2}.$$

Finally, using the simple inequality $(1 + x)^{-2} \leq 1 - x$ for $x \in (0, \frac{1}{2})$, we immediately obtain the desired bound on $A_n(a)$. □

4.5 All Coordinates are Small

In this case, we assume that a_1 is well below $\frac{1}{\sqrt{2}}$, which of course entails the same for all coordinates. This case deals with the asymptotic extremizer, which is $a^* = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ as $n \to \infty$. The distance to this extremizer is quantified by $\|a\|_4^4$, the 4th power of the ℓ_4 norm.

Lemma 27. *Assume* $a_1 \leq \sqrt{\frac{3}{8}}$. *Then,* $A_n(a) \leq 2 - \frac{1}{76}\|a\|_4^4$.

Proof. By assumption, $a_j^{-2} \geq \frac{8}{3}$ for all k. Using Eq. (4) and then Lemma 13 for each j,

$$A_n(a) \leq 2\prod_{j=1}^{n} \Psi(a_j^{-2})^{a_j^2} \leq 2\prod_{j=1}^{n} \left(1 - \tfrac{1}{151}a_j^2\right)^{a_j^2} \leq 2\prod_{j=1}^{n} \exp\left(-\frac{1}{151}a_j^4\right).$$

In the last step, we have used the elementary inequality $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$, and of course the fact that $(e^x)^y = e^{xy}$. From here, we have

$$A_n(a) \leq 2\prod_{j=1}^{n} \exp\left(-\frac{1}{151}a_j^4\right) = 2\exp\left(-\frac{1}{151}\sum_{j=1}^{n} a_j^4\right) = 2\exp\left(-\frac{1}{151}\|a\|_4^4\right).$$

Now, it is well known that if $1 \leq p \leq r \leq \infty$, then $\|a\|_r \leq \|a\|_p$. Taking $r = 4$ and $p = 2$, we have $\|a\|_4^4 \leq |a|^4 = 1$. We have the elementary inequality $2\exp(-x) \leq 2 - \frac{151}{76}x$ for $x \in \left[0, \frac{1}{151}\right]$ (easily checked by differentiation and direct calculation). Invoking this on the above exponential bound for $A_n(a)$ completes the proof. □

4.6 Proof of Main Result

We can finally combine the results of the previous sections to prove the upper bound in Theorem 18.

Proof. Putting together the central lemmas from each case, namely Lemma 19, Lemma 21, Lemma 25, Lemma 26, and Lemma 27, we have that at least one of the following inequalities holds:

1. $A_n(a) \leq 2 - \frac{1}{25}\sqrt{\delta(a)}$
2. $A_n(a) \leq 2 - 10^{-19}$
3. $A_n(a) \leq 2 - 10^{-20}$
4. $A_n(a) \leq 2 - 12\sqrt{2} \cdot 10^{-41}$
5. $A_n(a) \leq 2 - \frac{1}{76}\|a\|_4^4$

Each inequality is of the form $A_n(a) \leq 2 - x$, so the smallest x of these 5 inequalities yields the most general upper bound on $A_n(a)$. Clearly inequalities 2 and 3 have larger constants than that of inequality 4, so we can ignore them. This yields

$$A_n(a) \leq 2 - \min\left\{\frac{1}{25}\sqrt{\delta(a)}, 12\sqrt{2} \cdot 10^{-41}, \frac{1}{76}\|a\|_4^4\right\}.$$

To remove the constant entirely, we can use the crude bound $\sqrt{\delta(a)} = \sqrt{2 - \sqrt{2}(a_1 + a_2)} \leq \sqrt{2}$, so that $12\sqrt{2} \cdot 10^{-41} \geq 12\sqrt{\delta(a)} \cdot 10^{-41} \geq 10^{-40}\sqrt{\delta(a)}$. Therefore, we have

$$A_n(a) \leq 2 - \min\left\{\frac{1}{25}\sqrt{\delta(a)}, 10^{-40}\sqrt{\delta(a)}, \frac{1}{76}\|a\|_4^4\right\} = 2 - \min\left\{10^{-40}\sqrt{\delta(a)}, \frac{1}{76}\|a\|_4^4\right\}.$$

This concludes the proof the upper bound, and thus the proof of Theorem 18. ☐

A The Lower Bound

Theorem 28. *For $n \geq 2$ and every unit vector $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have*

$$1 + \frac{1}{8}|a - e_1|^2 \leq A_n(a),$$

where we define

$$A_n(a) := \frac{1}{\pi^{n-1}} \operatorname{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp).$$

Proof. Let ξ_j be independent, identically distributed uniform random vectors on the sphere S^3 in R^4. Begin by defining

$$Y = 2\sum_{j<k} a_j a_k \langle \xi_j, \xi_k \rangle, \text{ so that } \left| \sum_{j=1}^{n} a_j \xi_j \right|^2 = 1 + Y.$$

We have the obvious inequality $\frac{1}{4}y^2(y-1)^2 \geq 0$ for all y. Assuming $y > -1$ and rearranging this inequality, we obtain $(1+y)^{-1} \geq 1 - y + \frac{3}{4}y^2 - \frac{1}{4}y^3$. Applying this to Y and using the probabilistic formula for $A_n(a)$ (see Lemma 14), we get

$$A_n(a) \geq 1 - \mathbb{E}Y + \frac{3}{4}\mathbb{E}Y^2 - \frac{1}{4}\mathbb{E}Y^3.$$

By symmetry, $\mathbb{E}Y = 0$. As for $\mathbb{E}Y^2$ and $\mathbb{E}Y^3$, it was shown in Section 6.1 of [5] that $\mathbb{E}Y^3 \leq \mathbb{E}Y^2$ and $\mathbb{E}Y^2 \geq \frac{1}{4}|a - e_1|^2$. Therefore, we have

$$A_n(a) \geq 1 + \frac{3}{4}\mathbb{E}Y^2 - \frac{1}{4}\mathbb{E}Y^3 \geq 1 + \frac{3}{4}\mathbb{E}Y^2 - \frac{1}{4}\mathbb{E}Y^2 = 1 + \frac{1}{2}\mathbb{E}Y^2 \geq 1 + \frac{1}{8}|a - e_1|^2,$$

finishing the proof of the lower bound. Note that we assumed $n \geq 2$ here, so together with Appendix B, we have proven the lower and upper bounds for all $n \geq 2$. □

B The Two-Dimensional Upper Bound

Theorem 29. *For every unit vector $a \in \mathbb{R}^2$ with $a_1 \geq a_2 \geq 0$, we have*

$$A_2(a) \leq 2 - \min\left\{10^{-40}\sqrt{\delta(a)},\ \frac{1}{76}\left(a_1^4 + a_2^4\right)\right\},$$

where we define

$$A_2(a) := \frac{1}{\pi}\operatorname{vol}_2(\mathbb{D}^2 \cap a^\perp) \quad \textit{and} \quad \delta(a) := \left|a - \frac{e_1 + e_2}{\sqrt{2}}\right|^2 = 2 - \sqrt{2}(a_1 + a_2).$$

Proof. Applying Lemma 15 with $d = 4$ to Lemma 14, we get that $A_2(a) = \min\{a_1^{-2}, a_2^{-2}\} = a_1^{-2}$. One can check using trigonometry and trigonometric inequalities that $a_1^{-2} \leq 2 - \sqrt{\delta(a)}$. Therefore,

$$A_2(a) \leq 2 - \sqrt{\delta(a)} \leq 2 - 10^{-40}\sqrt{\delta(a)}.$$

It remains to show that $10^{-40}\sqrt{\delta(a)} \leq \frac{1}{76}\left(a_1^4 + a_2^4\right)$, so that the above inequality becomes the desired result. Since $a_1^2 + a_2^2 = 1$, we have

$$\frac{1}{76}\left(a_1^4 + a_2^4\right) = \frac{1}{76}\left(a_1^4 + (1 - a_1^2)^2\right) = \frac{1}{76}\left(a_1^4 - a_1^2 + 1\right)$$

Analyzing the polynomial $x \mapsto x^4 - x^2 + 1$ using basic calculus techniques, we can see it is minimized by the value $\frac{3}{4}$. Therefore,

$$10^{-40}\sqrt{\delta(a)} \leq 10^{-40}\sqrt{2} \leq \frac{1}{76} \cdot \frac{3}{4} \leq \frac{1}{76}\left(a_1^4 - a_1^2 + 1\right),$$

completing the proof. □

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