Bayesian Approaches to Localization, Mapping, and SLAM

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Quick Probability Review: Bayes Rule

$$p(a|b) = \frac{p(b|a)\ p(a)}{p(b)}$$

$$p(a|b,c) = \frac{p(b|a,c)\ p(a|c)}{p(b|c)}$$

QPR: Law of Total Probability

$$p(a) = \sum_i p(a \wedge b_i)$$

Discrete

$$= \sum_i p(a \mid b_i)\, p(b_i)$$

Continuous

$$p(a) = \int p(a \mid b)\, p(b)\, db$$

it follows that:

$$p(a \mid b) = \int p(a \mid b, c)\, p(c \mid b)\, dc$$

QPR: Markov Assumption

Future is Independent of Past Given Current State

"Assume Static World"

The Problem

- What is the world around me (mapping)
  - sense from various positions
  - integrate measurements to produce map
  - assumes perfect knowledge of position

- Where am I in the world (localization)
  - sense
  - relate sensor readings to a world model
  - compute location relative to model
  - assumes a perfect world model

- Together, these are SLAM (Simultaneous Localization and Mapping)

Localization

Tracking: Known initial position

Global Localization: Unknown initial position

Re-Localization: Incorrect known position

(kidnapped robot problem)

SLAM

Mapping while tracking locally and globally

Challenges

- Sensor processing
- Position estimation
- Control Scheme
- Exploration Scheme
- Cycle Closure
- Autonomy
- Tractability
- Scalability

Representations for Robot Localization

<u>Discrete approaches ('95)</u>
- Topological representation ('95)
  - uncertainty handling (POMDPs)
  - occas. global localization, recovery
- Grid-based, metric representation ('96)
  - global localization, recovery

<u>Kalman filters (late-80s?)</u>
- Gaussians
- approximately linear models
- position tracking

Robotics

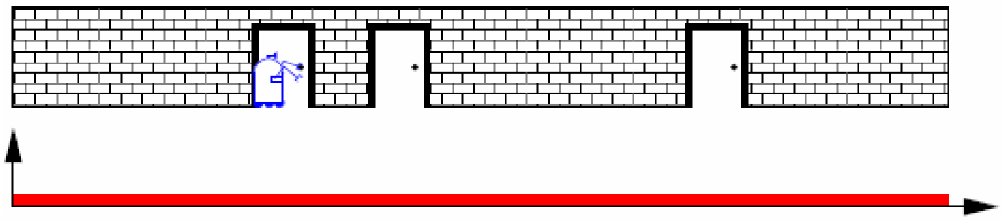
AI

<u>Particle filters ('99)</u>
- sample-based representation
- global localization, recovery

<u>Multi-hypothesis ('00)</u>
- multiple Kalman filters
- global localization, recovery

The Basic Idea

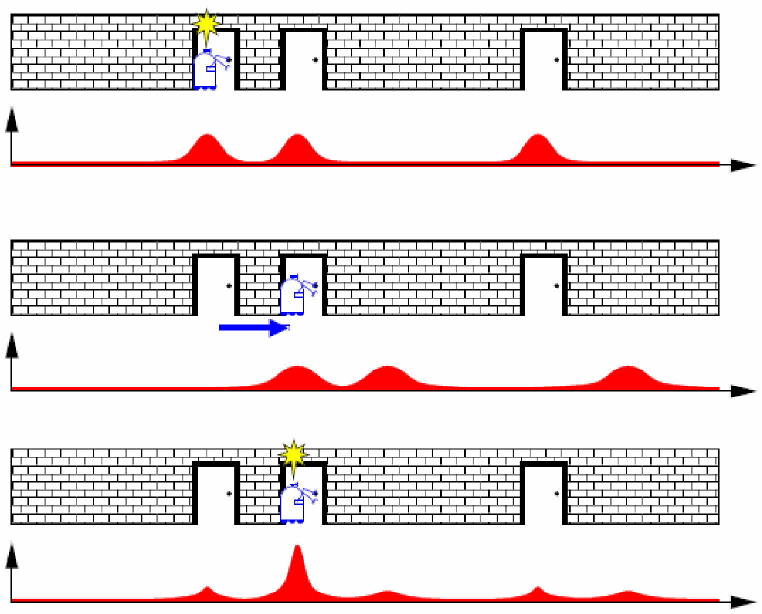
Robot can be anywhere



[Simmons/Koenig 95]
[Kaelbling et al 96]
[Burgard et al 96]

Notes

- Perfect Sensing
  - No false positives/neg.
  - No error
- Data association



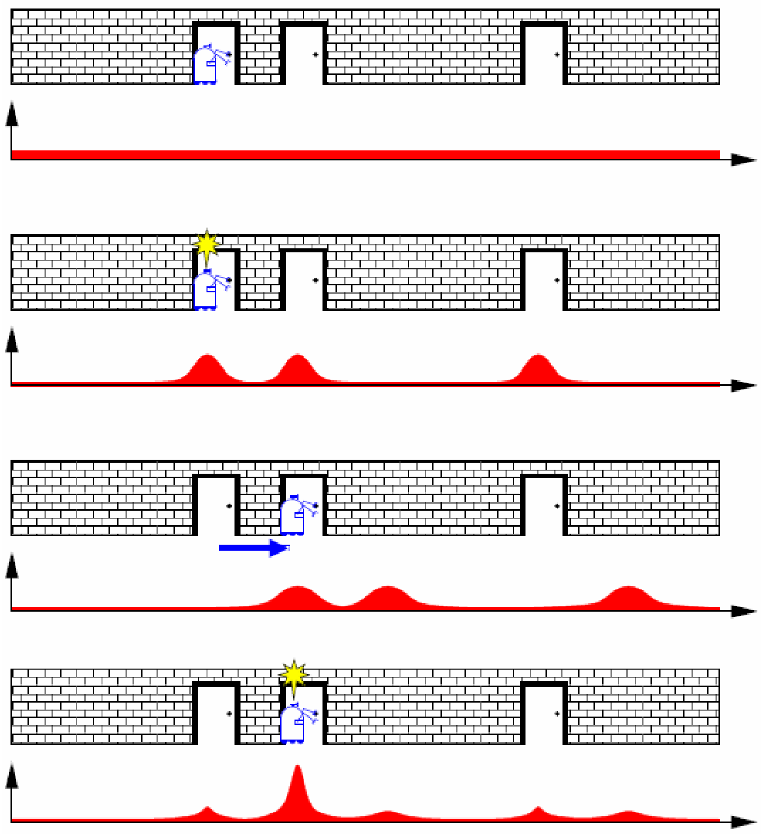
Notation for Localization

The posterior

$$P(x(k) \mid u(0:k-1), y(1:k))$$

- At every step k
- Probability over all configurations
- Given
  - Sensor readings *y* from *1* to *k*
  - Control inputs *u* from *0* to *k-1*
  - Interleaved:

$$u(0), y(1), \ldots, u(k-1), y(k)$$



Map *m*

(should be in condition statements too)

Velocities, force, odometry, something more complicated

Predict and Update, combined

posterior

$$P(x(k) \mid u(0:k-1), y(1:k))$$

prior

$$= \eta(k)\ \underline{P(y(k) \mid x(k))} \sum_{x(k-1)\in X} \left(\underline{P(x(k) \mid u(k-1), x(k-1)} \ \overline{P(x(k-1) \mid u(0:k-2), y(1:k-1))} \right)$$

Motion model: commanded motion moved from robot x(k-1) to x(k)

Sensor model: robot perceives *y(k)* given a map and that it is at *x(k)*

<u>Features</u>

Generalizes beyond Gaussians

Recursive Nature

<u>Issues</u>

Realization of sensor and motion models

Representations of distributions

Prediction Step

- Occurs when an odometry measurement (like a control) or when a control is invoked…. Something with *u(k-1)*
- Suppose *u(0: k-2)* and *y(1: k-1)* known and

  Current belief is $P(x(k-1) \mid u(0:k-2), y(1:k-1))$
- Obtain $P(x(k) \mid u(0:k-1), y(1:k-1))$
  - Integrate/sum over all possible *x(k-1)*
  - Multiply each $P(x(k-1) \mid u(0:k-2), y(1:k-1))$

    by $P(x(k) \mid u(k-1), x(k-1))$

Motion model

Not k

$$P(x(k) \mid u(0:k-1), y(1:k-1))$$
$$= \sum_{x(k-1)\in X} \Big(P(x(k) \mid u(k-1), x(k-1))$$
$$P(x(k-1) \mid u(0:k-2), y(1:k-1)) \Big)$$

Update Step

- Whenever a sensory experience occurs… something with *y(k)*

- Suppose $P(x(k) \mid u(0 : k-1), y(1 : k-1))$ is known

and we just had sensor y(k)

- For each state *x(k)*

Sensor model

Multiply $P(x(k) \mid u(0 : k-1), y(1 : k-1))$ by $P(y(k) \mid x(k))$

$$
\begin{aligned}
& P(x(k) \mid u(0 : k-1), y(1 : k)) \\
& = \quad P(y(k) \mid x(k)) \; P(x(k) \mid u(0 : k-1), y(1 : k-1))
\end{aligned}
$$

That pesky normalization factor

- Bayes rule gives us $\eta(k) \;=\; P(y(k) \mid u(0:k-1), y(1:k-1))^{-1}$

- This is hard to compute:
  - What is the dependency of y(k) on previous controls and sensor readings without knowing your position or map of the world?

- .
$$\eta(k) \;=\; \left[\sum_{x(k) \in X} P(y(k) \mid x(k)) \; P(x(k) \mid u(0:k-1), y(1:k-1)) \right]^{-1}$$

- We know these terms

Summary

$$P(x(k) \mid u(0:k-1), y(1:k))$$

$$= \quad \eta(k)\; P(y(k) \mid x(k)) \sum_{x(k-1)\in X} \Big(P(x(k) \mid u(k-1), x(k-1)\; P(x(k-1) \mid u(0:k-2), y(1:k-1)) \Big)$$

---

prediction:

$$P(x(k) \mid u(0:k-1), y(1:k-1))$$
$$= \sum_{x(k-1)\in X} \Big(P(x(k) \mid u(k-1), x(k-1))$$
$$P(x(k-1) \mid u(0:k-2), y(1:k-1)) \Big)$$

update:

$$\eta(k)$$
$$= \left[\sum_{x(k)\in X} P(y(k) \mid x(k))\; P(x(k) \mid u(0:k-1), y(1:k-1)) \right]^{-1}$$

$$P(x(k) \mid u(0:k-1), y(1:k))$$
$$= \quad \eta(k)\; P(y(k) \mid x(k))\; P(x(k) \mid u(0:k-1), y(1:k-1)).$$

Issues to be resolved

- Initial distribution P(0)
  - Gaussian if you have a good idea
  - Uniform if you have no idea
  - Whatever you want if you have some idea

- How to represent distributions: prior & posterior, sensor & motion models

- How to compute conditional probabilities

$$P(x(k) \mid u(k-1), x(k-1)) \qquad P(y(k) \mid x(k))$$

- Where does this all come from? (we will do that first)

The derivation: $P(x(k) \mid u(0:k-1), y(1:k))$

- Consider odometry and sensor information separately
- Lets start with new sensor reading comes in – a new *y(k)*
  - Assume *y(1:k-1)* and *u(0:k-1)* as known
  - Apply Bayes rule

$$P(x(k) \mid u(0:k-1), y(1:k))$$
$$= \eta \, P(y(k) \mid \quad x(k)) \; P(x(k) \mid u(0:k-1), y(1:k-1))$$

Once state is known, then all previous controls and measurements are independent of current reading

Denominator is a normalizer which is the same for all of *x(k)*

Incorporate motions

- We have

$$P(x(k) \mid u(0:k-1), y(1:k)) \\ = \eta(k)\ P(y(k) \mid x(k)) \boxed{P(x(k) \mid u(0:k-1), y(1:k-1))}$$

- Use law of total probability on right-most term

$$P(x(k) \mid u(0:k-1), y(1:k-1)) \\ = \sum_{x(k-1)\in X} P(x(k) \mid u(k-1), x(k-1)) \\ P(x(k-1) \mid u(0:k-2), y(1:k-1))\big].$$

assume that *x(k)* is independent of sensor readings *y(1:k-1)* and controls *u(1:k-2)* that got the robot to state *x(k-1)* given we know the robot is at state *x(k-1)*

assume controls at k-1 take robot from x(k-1) to x(k), which we don't know x(k) x(k-1) is independent of u(k-1)

Incorporate motions

- We have

$$
\begin{aligned}
&P(x(k) \mid u(0:k-1), y(1:k)) \\
&= \eta(k)\; P(y(k) \mid x(k)) \sum_{x(k-1)\in X} \Big(P(x(k) \mid u(k-1), x(k-1)) \\
&\qquad P(x(k-1) \mid u(0:k-2), y(1:k-1)) \Big)
\end{aligned}
$$

Representations of Distributions

- Kalman Filters
- Discrete Approximations
- Particle Filters

Extended Kalman Filters

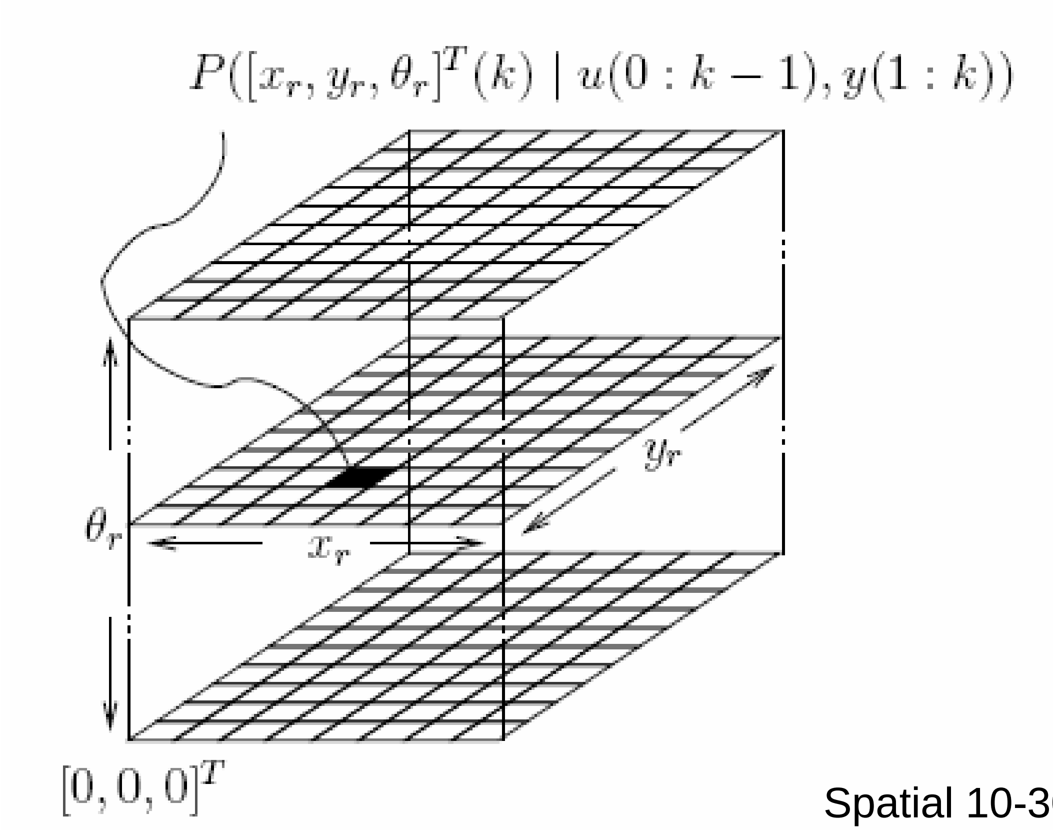
$P(x(k) \mid u(0 : k-1), y(1 : k))$ as a Gaussian

<u>The Good</u>
Computationally efficient
Easy to implement

<u>The Bad</u>
Linear updates
Unimodal

Discretizations

- Topological structures
- Grids



Spatial 10-30cm
Angular 2-10 degrees

Algorithm to Update Posterior *P(x)*

Start with *u(0: k-1)* and *y(1:k)*

$P(x) \leftarrow P(x(0))$
for $i \leftarrow 1$ **to** k **do**
  for all states $x \in X$ **do**
    $P'(x) \leftarrow \sum_{x' \in X} P(x \mid u(i-1), x') \cdot P(x')$
  end for
  $\eta \leftarrow 0$
  for all states $x \in X$ **do**
    $P(x) \leftarrow P(y(i) \mid x) \cdot P'(x)$
    $\eta \leftarrow \eta + P(x)$
  end for
  for all states $x \in X$ **do**
    $P(x) \leftarrow P(x)/\eta$
  end for

k loops

Integrate *u(i-1)* and *y(i-1)* in each loop

Incorporate motion u(i-1) with motion model

Sensor model

Normalization Constant

Complexity $O(n^2)$

Bypass with convolution details we will skip

Convolution Mumbo Jumbo

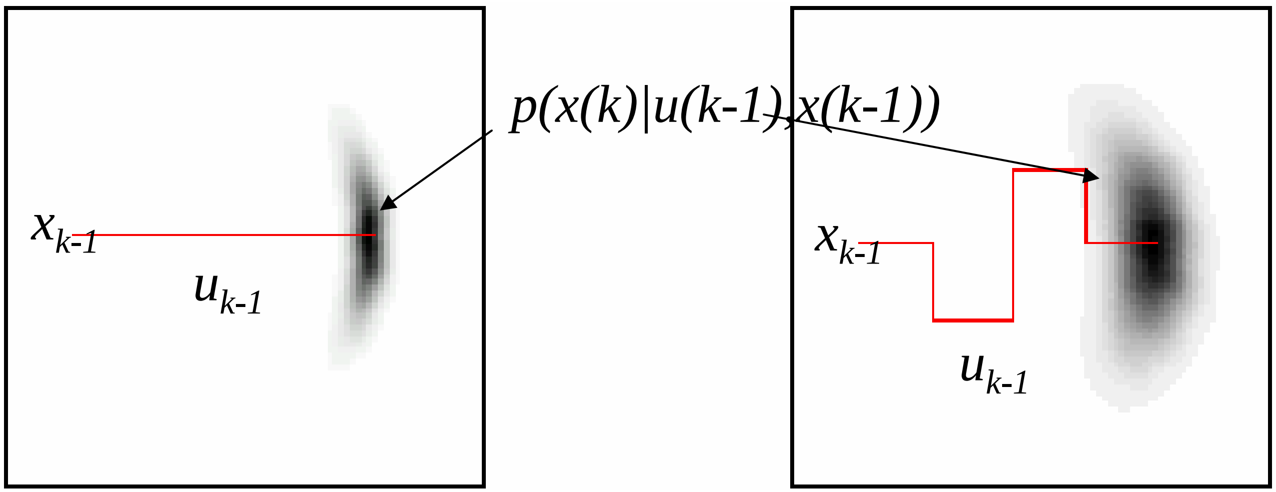
- To efficiently update the belief upon robot motions, one typically assumes a bounded Gaussian model for the motion uncertainty.
- This reduces the update cost from $O(n^2)$ to $O(n)$, where n is the number of states.
- The update can also be realized by shifting the data in the grid according to the measured motion.
- In a second step, the grid is then convolved using a separable Gaussian Kernel.
- Two-dimensional example:

$$\begin{array}{|c|c|c|}\hline 1/16 & 1/8 & 1/16 \\ \hline 1/8 & 1/4 & 1/8 \\ \hline 1/16 & 1/8 & 1/16 \\ \hline \end{array} \cong \begin{array}{|c|}\hline 1/4 \\ \hline 1/2 \\ \hline 1/4 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 1/4 & 1/2 & 1/4 \\ \hline \end{array}$$

- Fewer arithmetic operations
- Easier to implement

Probabilistic Action model

Continuous probability density *Bel(st)* after moving



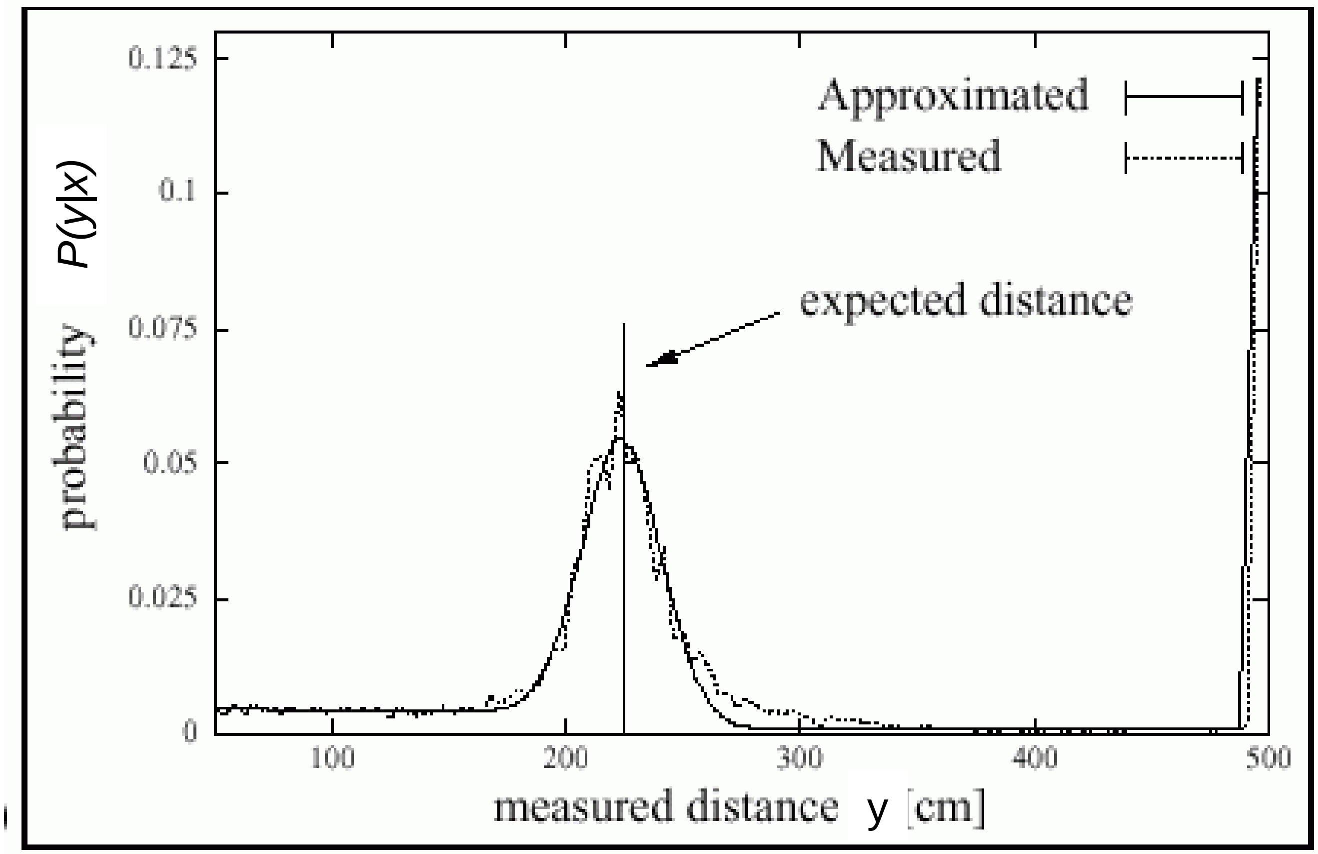
40m

80m.

Darker area has higher probability.

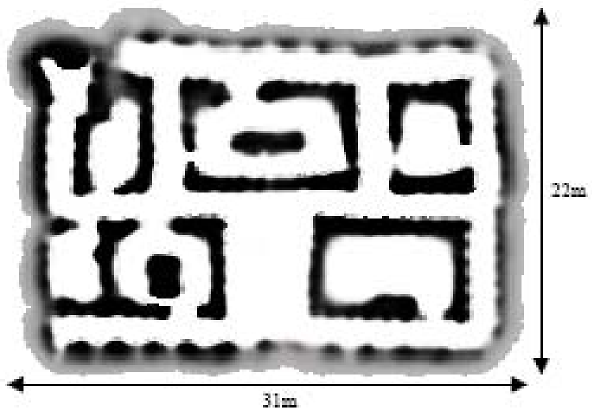
Thrun et. al.

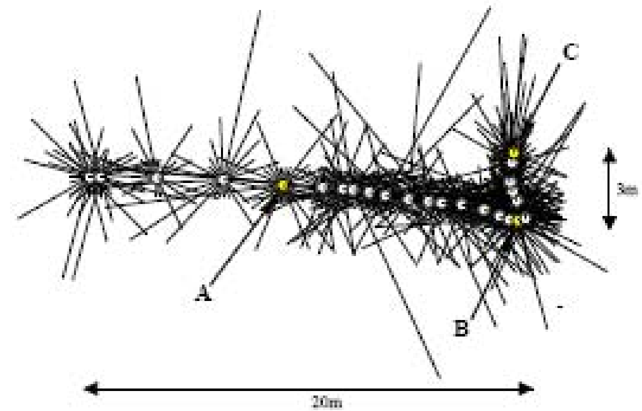
Probabilistic Sensor Model



Probabilistic sensor model for laser range finders

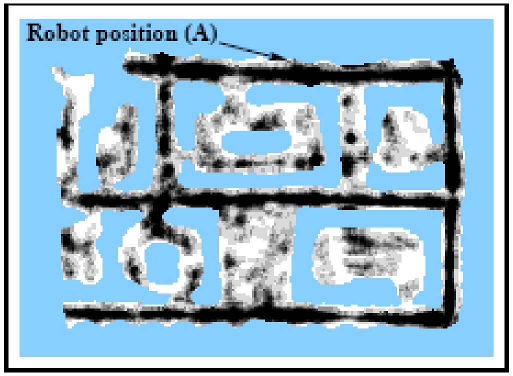
One of Wolfram et al's Experiments



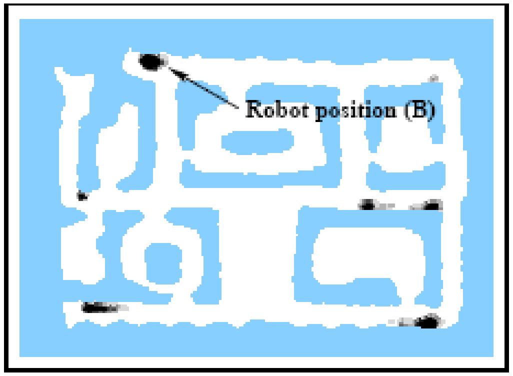


Known map

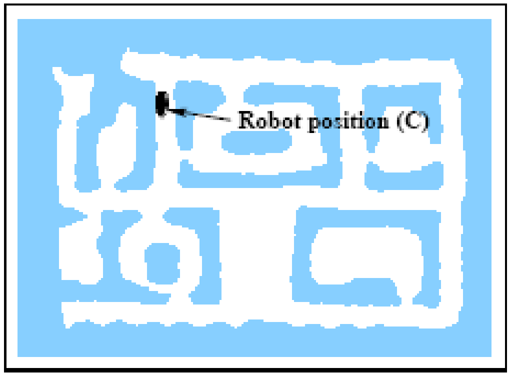
A, after 5 scans;
B, after 18 scans,
C, after 24 scans



5 scans



18 scans



24 scans

What do you do with this info?

- Mean, continuous but may not be meaningful
- Mode, max operator, not continuous but corresponds to a robot position
- Medians of x and y, may not correspond to a robot position too but robust to outliers

Particle Filters

- Represent belief by random samples
- Estimation of non-Gaussian, nonlinear processes

- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter, Particle filter

- Filtering: [Rubin, 88], [Gordon et al., 93], [Kitagawa 96]
- Computer vision: [Isard and Blake 96, 98]
- Dynamic Bayesian Networks: [Kanazawa et al., 95]d

Basic Idea

- Maintain a set of *N* samples of states, *x*, and weights, *w*, in a set called *M*.
- When a new measurement, y(k) comes in, the weight of particle (x,w) is computed as *p(y(k)|x) – observation given a state*
- Resample *N* samples (with replacement) from *M* according to weights *w*

Particle Filter Algorithm and Recursive Localization

$$Bel(x_t) = \eta\, p(y_t \mid x_t) \int p(x_t \mid x_{t-1}, u_{t-1})\, Bel(x_{t-1})\, dx_{t-1}$$

draw x^i_{t-1} from $Bel(x_{t-1})$

draw x^i_t from $p(x_t \mid x^i_{t-1}, u_{t-1})$

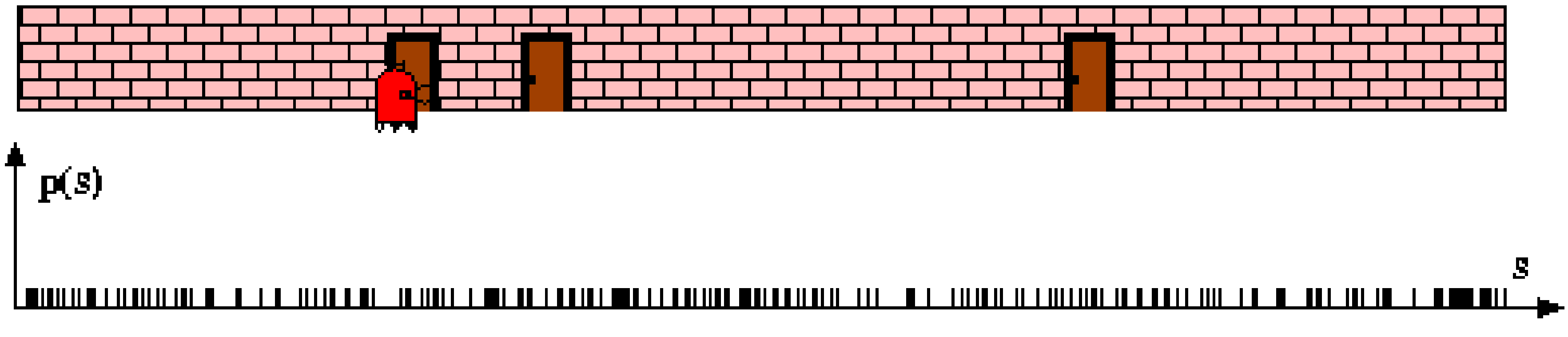
Importance factor for x^i_t:

$$w_t^i = \frac{\text{target distribution}}{\text{proposal distribution}}$$

$$= \frac{\eta\; p(y_t \mid x_t)\; p(x_t \mid x_{t-1}, u_{t-1})\; Bel\;(x_{t-1})}{p(x_t \mid x_{t-1}, u_{t-1})\; Bel\;(x_{t-1})}$$

$$\propto\; p(y_t \mid x_t)$$

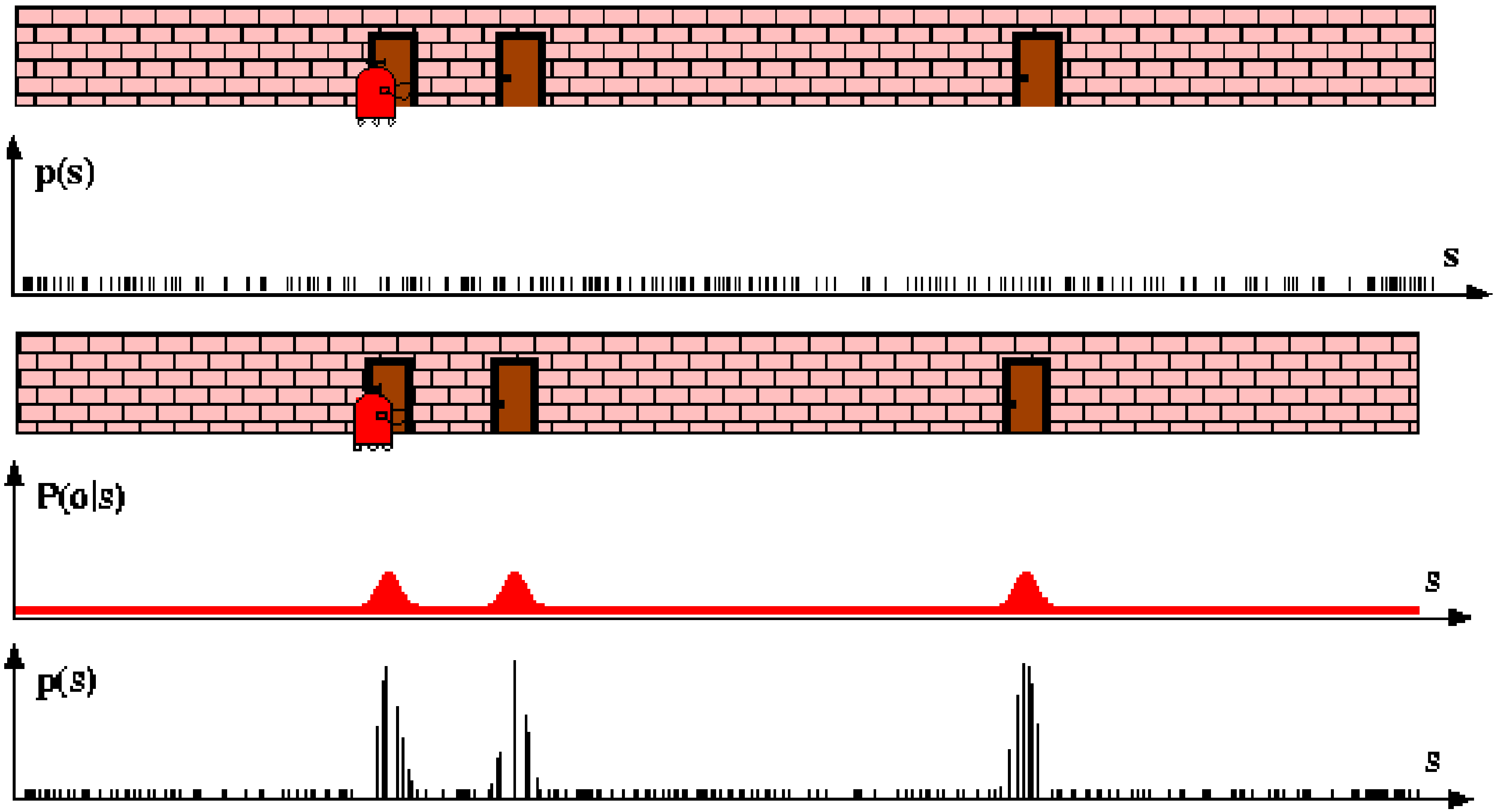
Particle Filters



Sensor Information: Importance Sampling

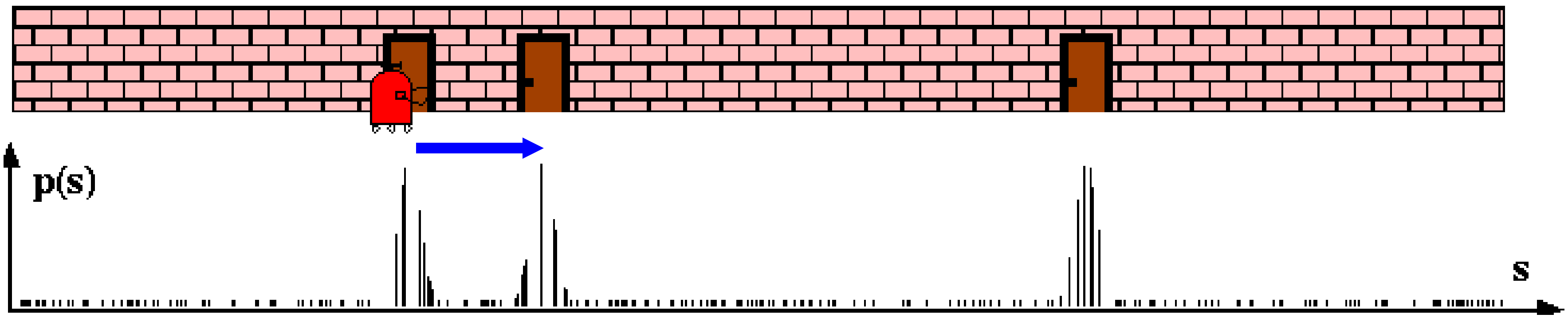
$$Bel(x) \leftarrow \alpha\ p(y \mid x)\ Bel^-(x)$$

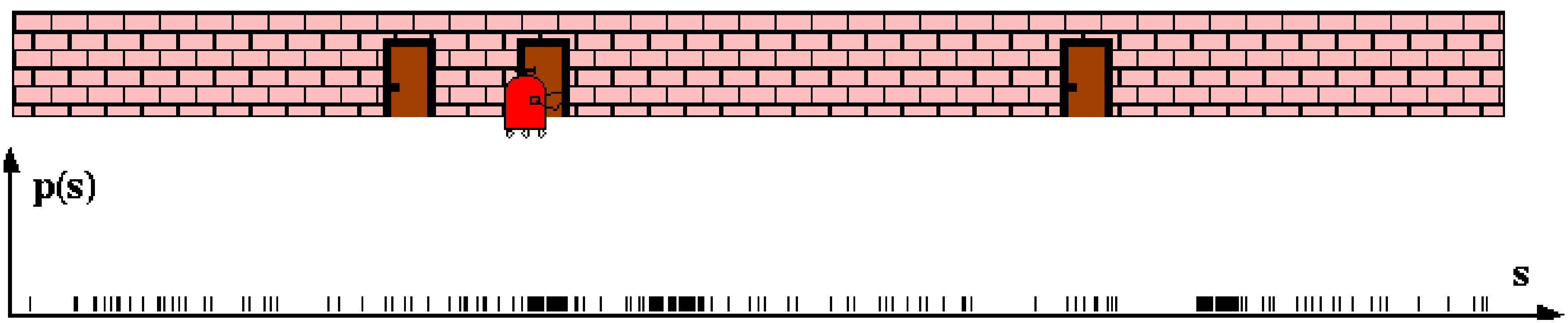
$$w \leftarrow \frac{\alpha\ p(y \mid x)\ Bel^-(x)}{Bel^-(x)} = \alpha\ p(y \mid x)$$



Robot Motion

$$Bel^{-}(x) \;\leftarrow\; \int p(x \mid u, x')\, Bel(x')\; \mathrm{d}x'$$





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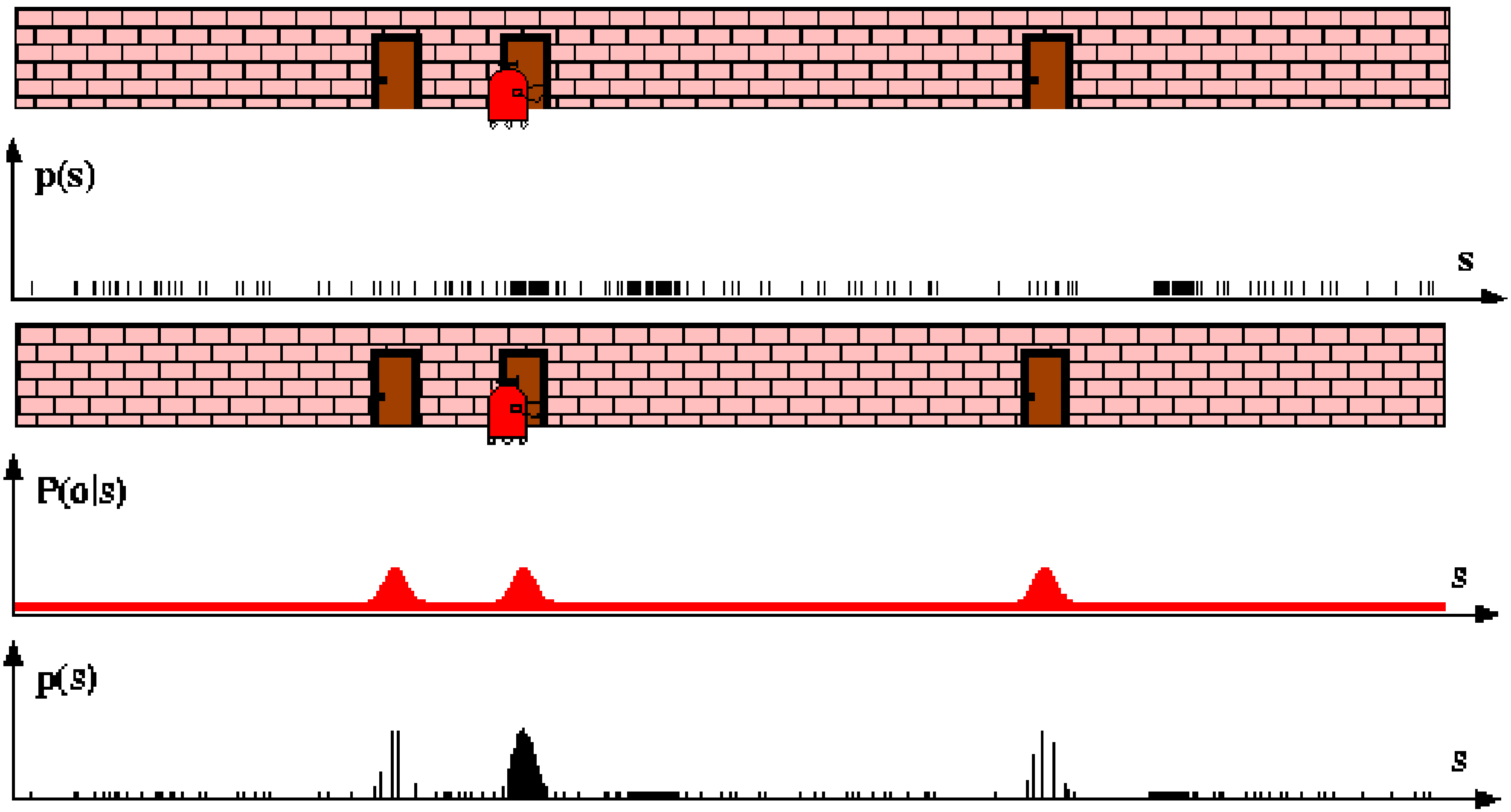
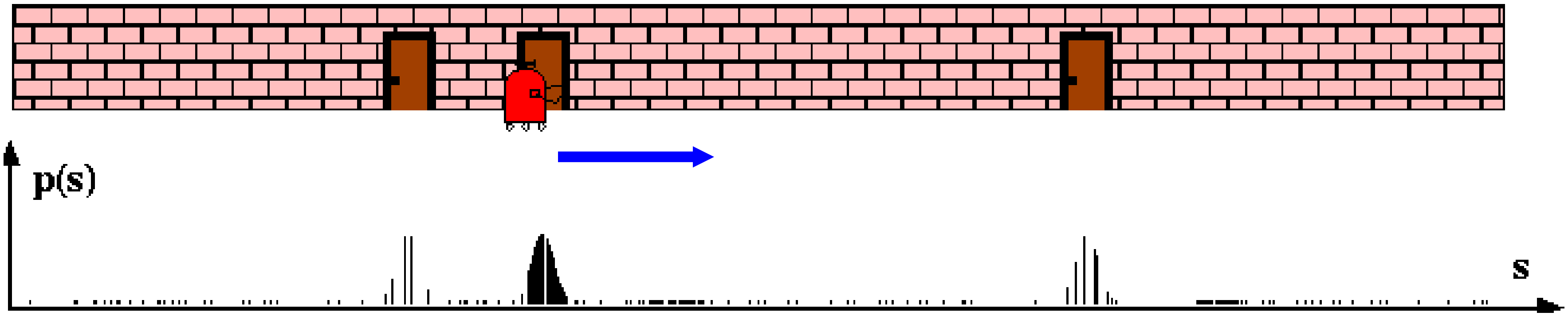
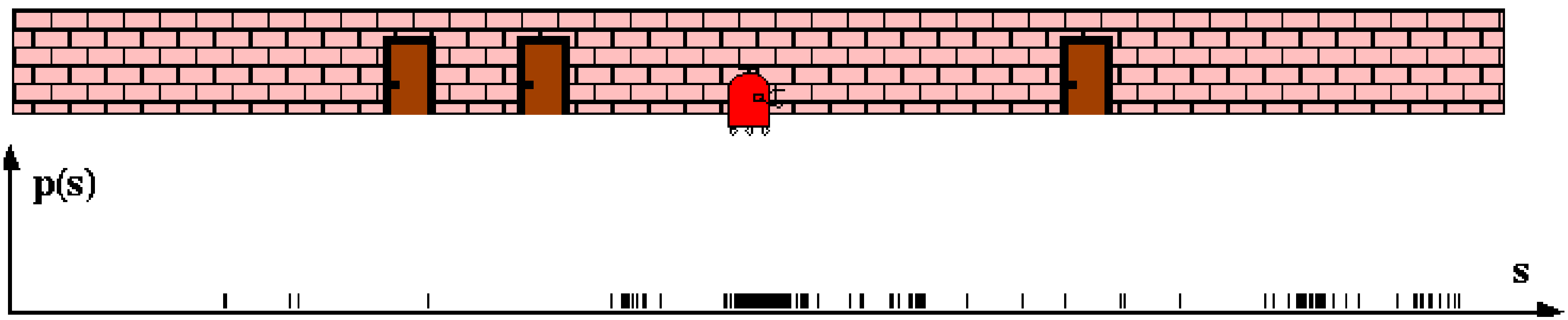


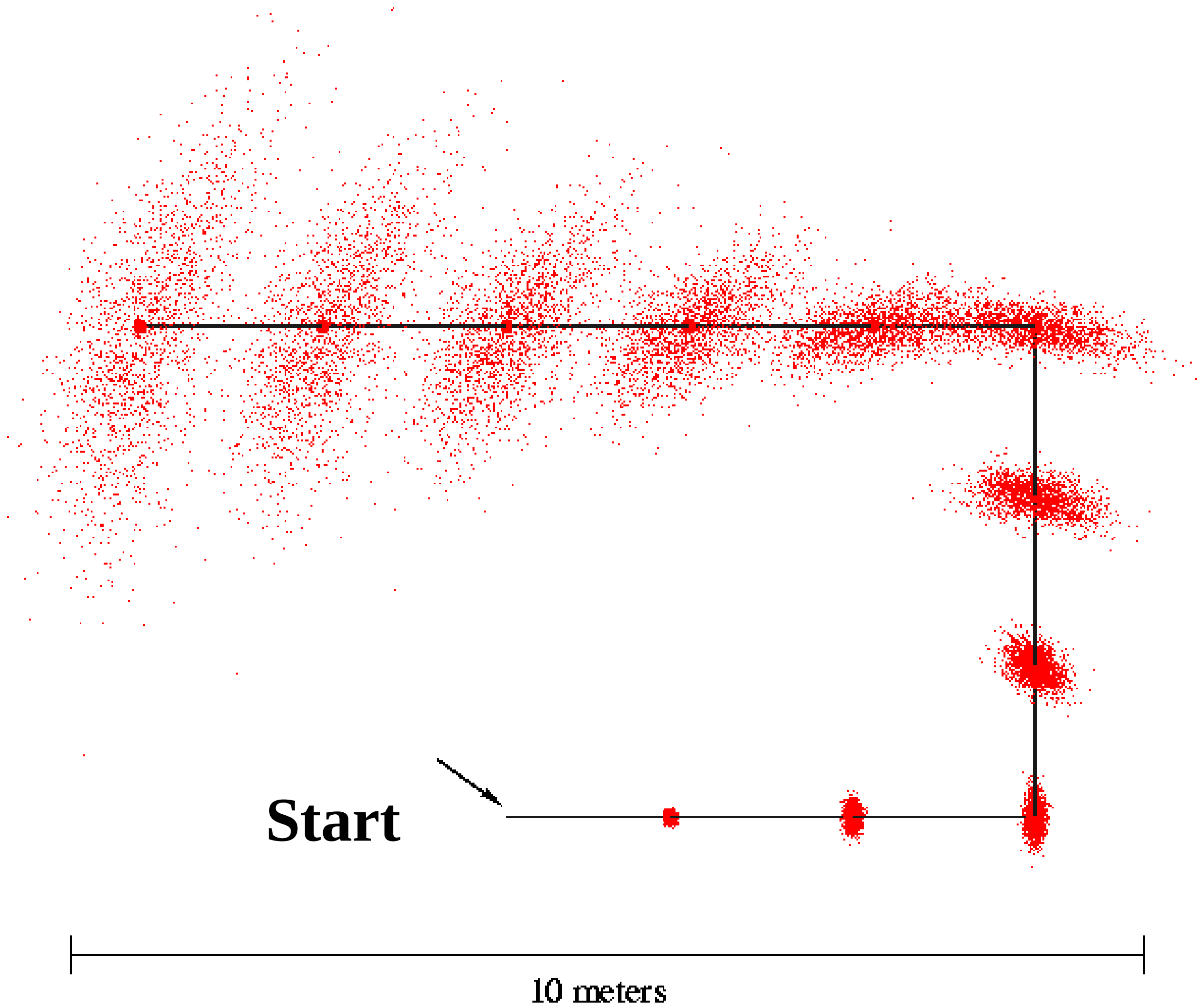
Robot Motion

$$Bel^{-}(x) \quad \leftarrow \quad \int p(x\,|\,u, x')\, Bel(x') \;\, \mathrm{d}\, x'$$



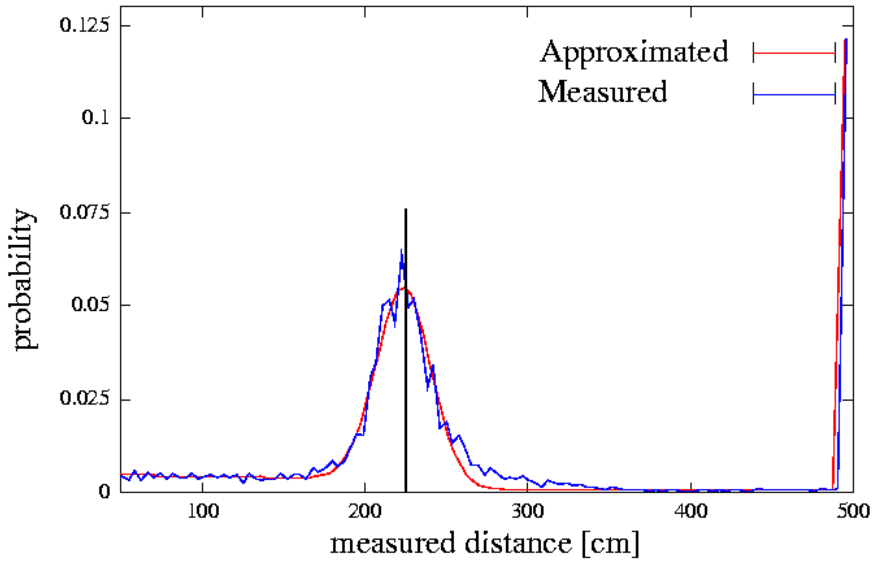


Motion Model Reminder

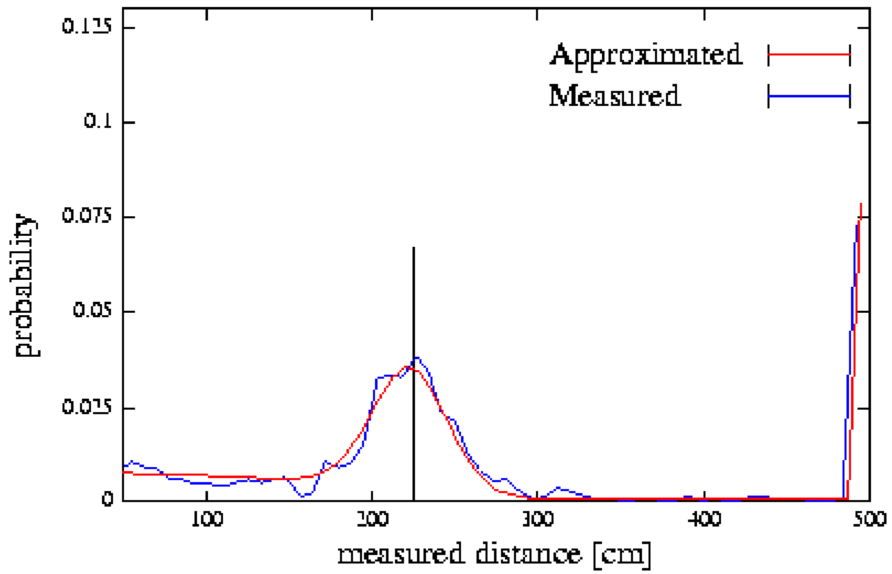


Or what if robot keeps moving and there are no observations

Proximity Sensor Model Reminder



Laser sensor



Sonar sensor

Particle Filter Algorithm

1. Algorithm **particle_filter**(M_{t-1}, u_{t-1} y_t):
2. $M_t = \emptyset, \quad \eta = 0$
3. **For** $i = 1 \ldots n$ ***Generate new samples***
4. Sample index *j(i)* from the discrete distribution given by M_{t-1}
5. Sample x_t^i from $p(x_t \mid x_{t-1}, u_{t-1})$ using $x_{t-1}^{j(i)}$ and u_{t-1}
6. $w_t^i = p(y_t \mid x_t^i)$ ***Compute importance weight***
7. $\eta = \eta + w_t^i$ ***Update normalization factor***
8. $M_t = M_t \cup \{< x_t^i, w_t^i >\}$ ***Insert***
9. **For** $i = 1 \ldots n$
10. $w_t^i = w_t^i / \eta$ ***Normalize weights***
11. RESAMPLE!!!

Resampling

- **Given**: Set M of weighted samples.
- **Wanted** : Random sample, where the probability of drawing x_i is given by w_i.
- Typically done N times with replacement to generate new sample set M'.

Resampling Algorithm

Algorithm **systematic_resampling**(M, n):

$M' = \emptyset, c_1 = w^1$

For $i = 2 \ldots n$ *Generate cdf*

  $c_i = c_{i-1} + w^i$

$u_1 \sim U\,]0, n^{-1}], i = 1$ *Initialize threshold*

For $j = 1 \ldots n$ *Draw samples …*

  While ($u_j > c_i$) *Skip until next threshold reached*

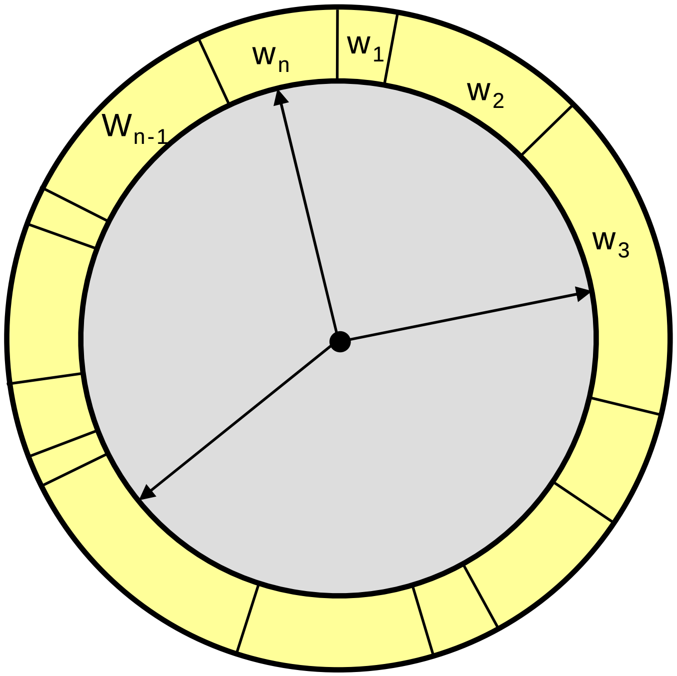
    $i = i + 1$

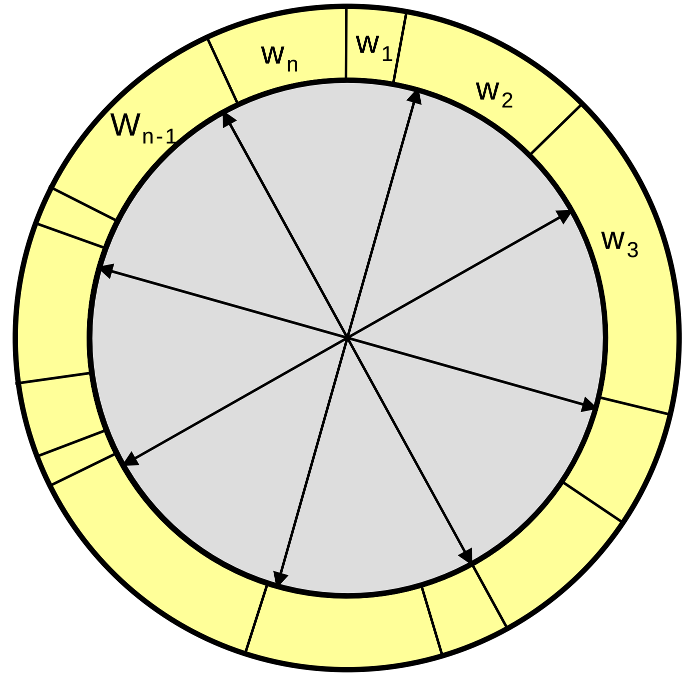
  $M' = M' \cup \left\{ < x^i, n^{-1} > \right\}$ *Insert*

  $u_{j+1} = u_j + n^{-1}$ *Increment threshold*

Return M'

Resampling, an analogy Wolfram likes

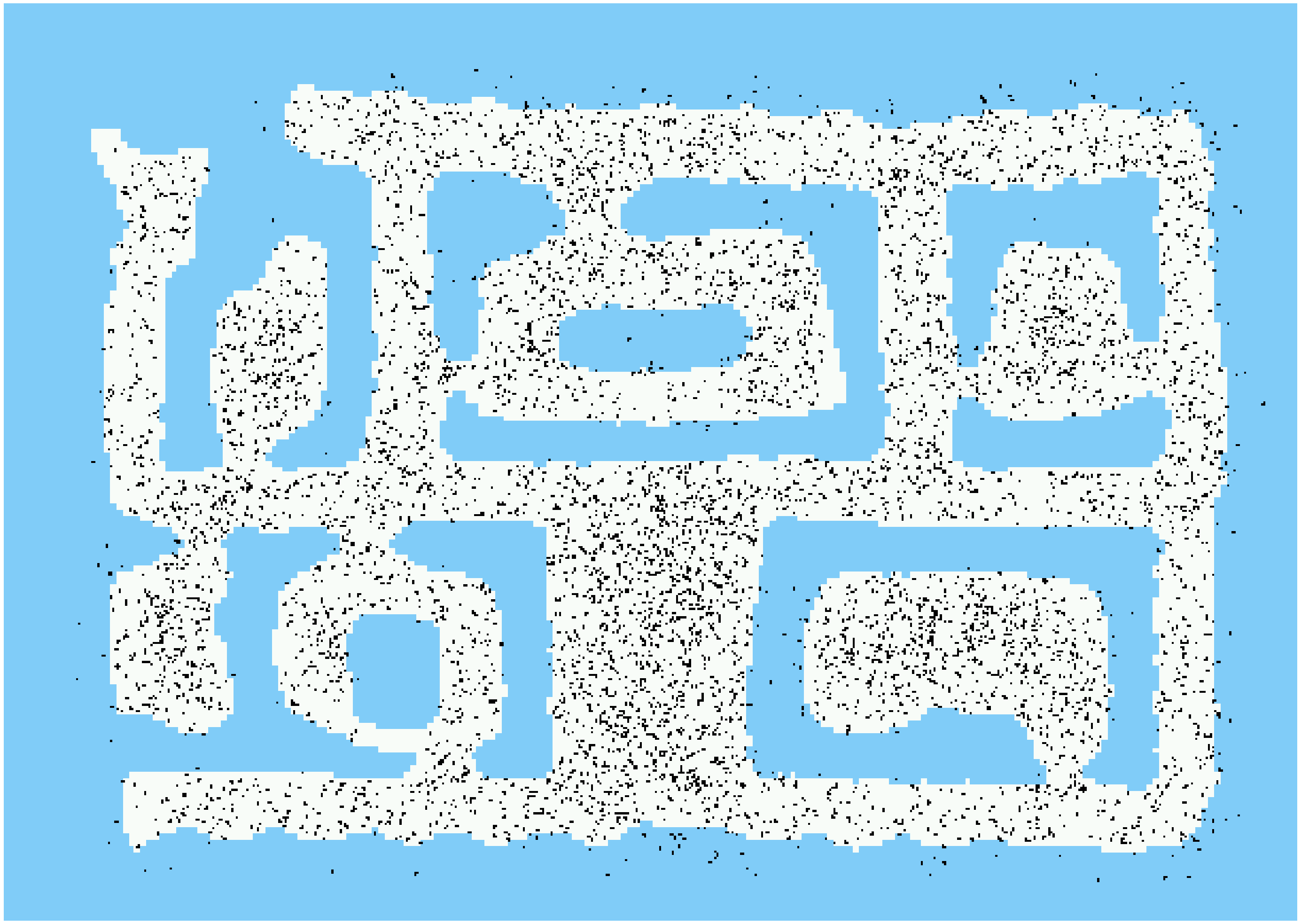




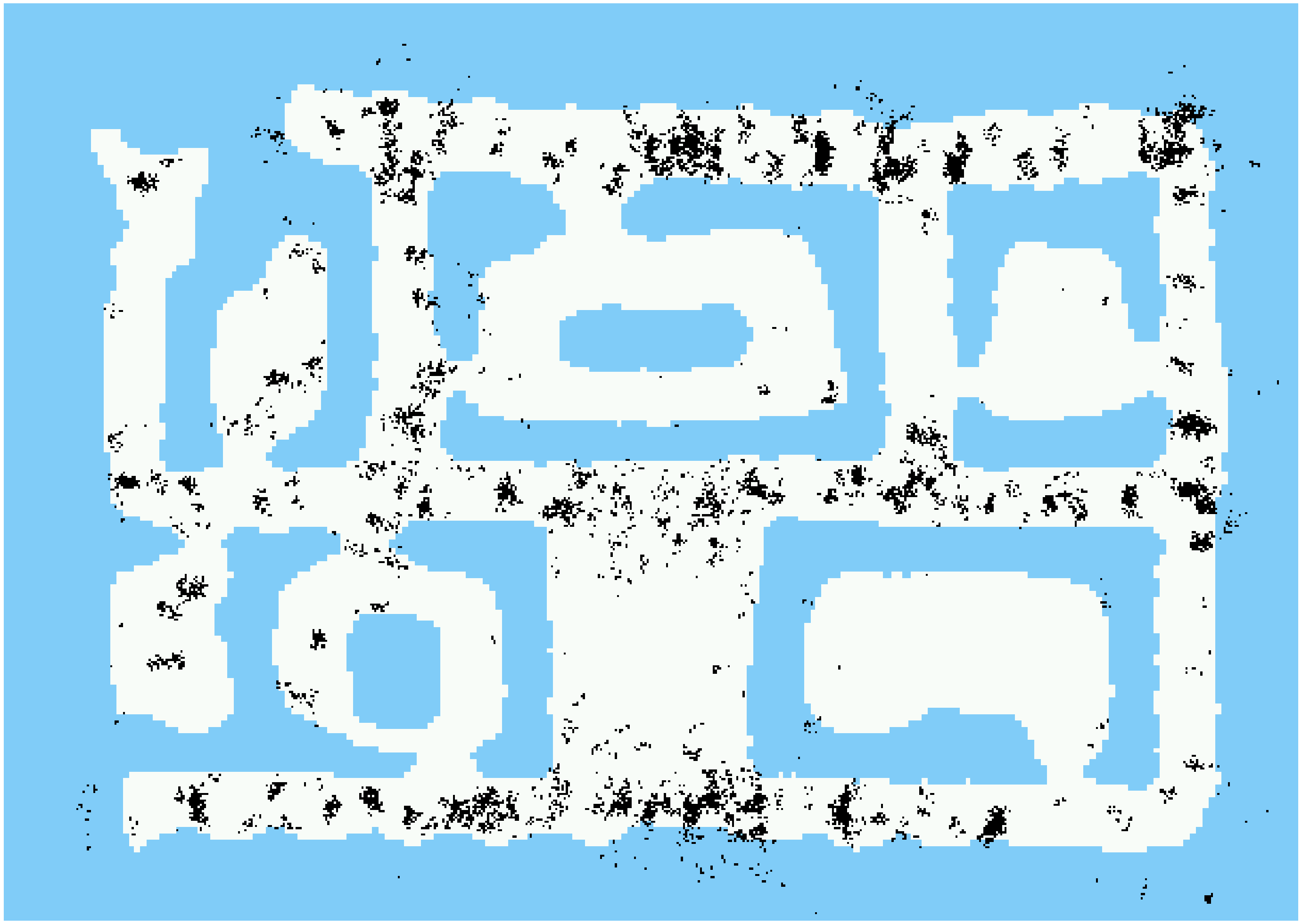
- Roulette wheel
- Binary search, n log n

- Stochastic universal sampling
- Systematic resampling
- Linear time complexity
- Easy to implement, low variance

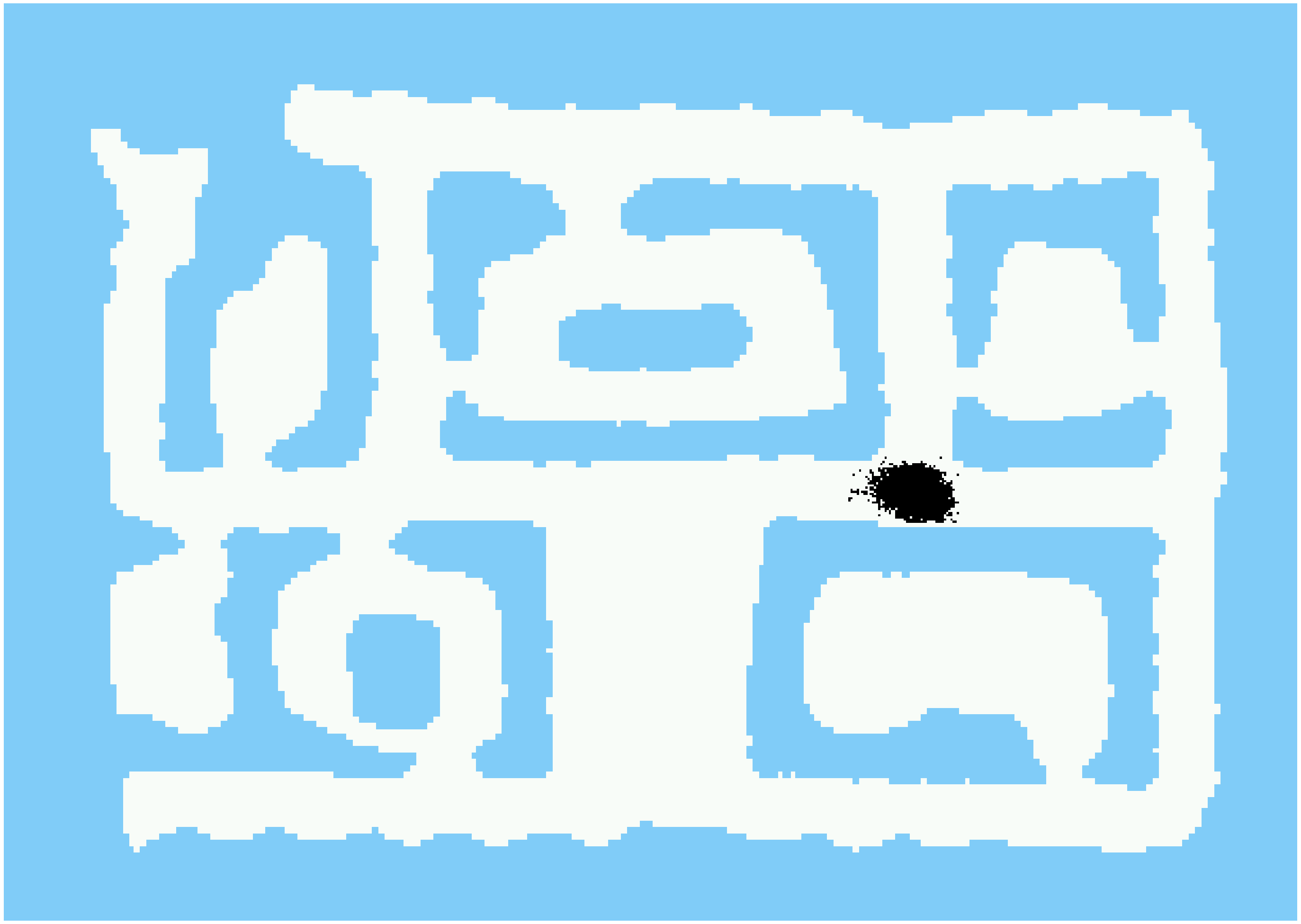
Initial Distribution



After Incorporating Ten Ultrasound Scans



After Incorporating 65 Ultrasound Scans



Limitations

- The approach described so far is able to
  - track the pose of a mobile robot and to
  - globally localize the robot.

- How can we deal with localization errors (i.e., the kidnapped robot problem)?

Approaches

- Randomly insert samples (the robot can be teleported at any point in time).
- Insert random samples proportional to the average likelihood of the particles (the robot has been teleported with higher probability when the likelihood of its observations drops).

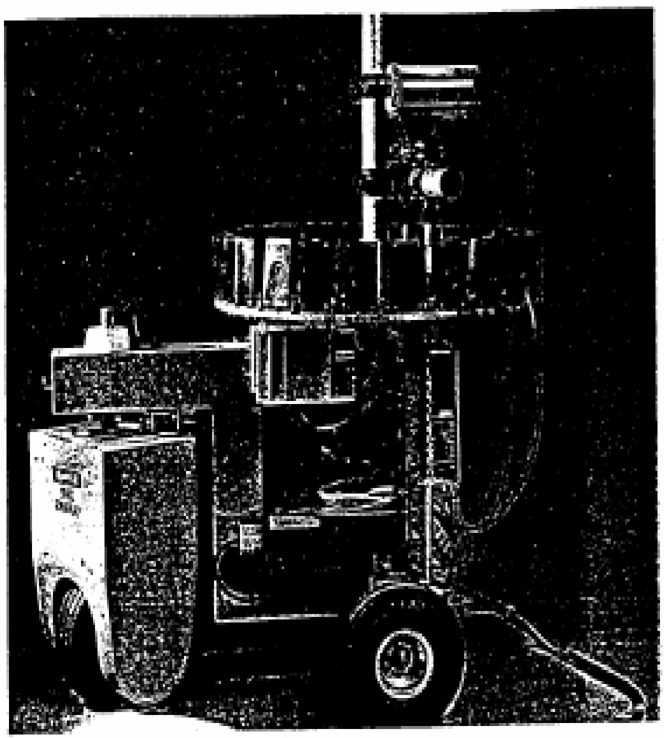
Summary

- Recursive Bayes Filters are a robust tool for estimating the pose of a mobile robot.
- Different implementations have been used such as discrete filters (histograms), particle filters, or Kalman filters.
- Particle filters represent the posterior by a set of weighted samples.

Change gears to

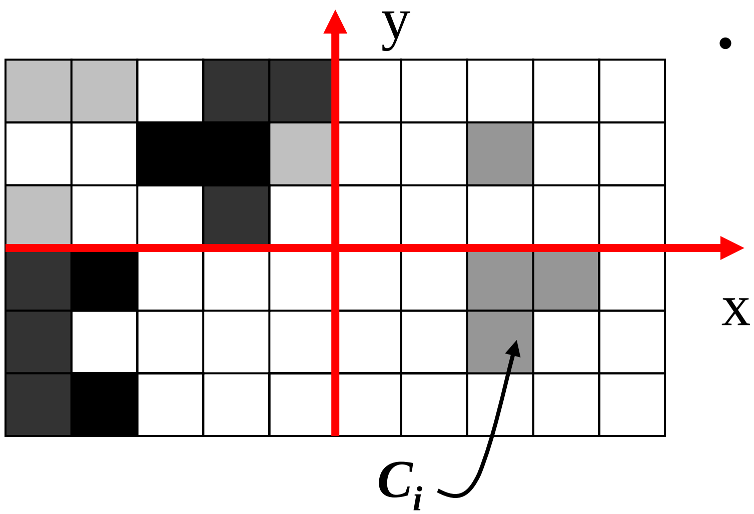
MAPPING

Occupancy Grids [Elfes]



- In the mid 80's Elfes starting implementing cheap ultrasonic transducers on an autonomous robot
- Because of intrinsic limitations in any sonar, it is important to compose a coherent world-model using information gained from multiple reading

Occupancy Grids Defined



- The grid stores the probability that C_i = cell(x,y) is occupied $O(C_i) = P[s(C_i) = OCC](C_i)$
- Phases of Creating a Grid:
  - Collect reading generating $O(C_i)$
  - Update Occ. Grid creating a map
  - Match and Combine maps from multiple locations

Original notation

Binary variable

Cell m_l is occupied

$$P(m_l \mid x(1:k), y(1\ k))$$

Given sensor observations

$$y(1:k) = y(1), \ldots, y(k)$$

Given robot locations

$$x(1:k) = x(1), \ldots, x(k)$$

Bayes Rule Rules!

- Seek to find *m* to maximize $P(m \mid x(1:k), y(1:k))$

Local map

$$P(m \mid x(1:k), y(1:k)) = \frac{\boxed{P(y(k) \mid m, x(1:k), y(1:k-1))}\, P(m \mid x(1:k), y(1:k-1))}{P(y(k) \mid x(1:k), y(1:k-1))}$$

Assume that current readings is independent of all previous states and readings given we know the map

$$P(m \mid x(1:k), y(1:k)) = \frac{\boxed{P(y(k) \mid m, x(k))}\, P(m \mid x(1:k), y(1:k-1))}{P(y(k) \mid x(1:k), y(1:k-1))}$$

Bayes rule on $P(y(k) \mid m, x(k))$

$$P(m \mid x(1:k), y(1:k)) = \frac{P(m \mid x(k), y(k))\; P(y(k) \mid x(k))\; P(m \mid x(1:k-1), y(1:k-1))}{P(m)\; P(y(k) \mid x(1:k), y(1:k-1))}$$

A cell is occupied or not

- The m

$$P(m \mid x(1:k), y(1:k)) = \frac{P(m \mid x(k), y(k))\ P(y(k) \mid x(k))\ P(m \mid x(1:k-1), y(1:k-1))}{P(m)\ P(y(k) \mid x(1:k), y(1:k-1))}$$

- Or not the m

$$P(\neg m \mid x(1:k), y(1:k)) = \frac{P(\neg m \mid x(k), y(k))\ P(y(k) \mid x(k))\ P(\neg m \mid x(1:k-1), y(1:k-1))}{P(\neg m)\ P(y(k) \mid x(1:k), y(1:k-1))}$$

$$\frac{P(m \mid x(1:k), y(1:k))}{1 - P(m \mid x(1:k), y(1:k))}$$

$$P(\neg A) = 1 - P(A)$$

$$= \frac{P(m \mid x(k), y(k))}{1 - P(m \mid x(k), y(k))} \frac{1 - P(m)}{P(m)} \frac{P(m \mid x(1:k-1), y(1:k-1))}{1 - P(m \mid x(1:k-1), y(1:k-1))}$$

The Odds

$$\mathrm{Odds}(x) \quad = \quad \frac{P(x)}{1 - P(x)}$$

$$\frac{P(m \mid x(k), y(k))}{1 - P(m \mid x(k), y(k))} \frac{1 - P(m)}{P(m)} \frac{P(m \mid x(1:k-1), y(1:k-1))}{1 - P(m \mid x(1:k-1), y(1:k-1))}$$

$$\begin{aligned} &\mathrm{Odds}(m \mid x(1:k), y(1:k)) \\ &\quad = \quad \frac{\mathrm{Odds}(m \mid x(k), y(k)) \;\; \mathrm{Odds}(m \mid x(1:k-1), y(1:k-1))}{\mathrm{Odds}(m)} \end{aligned}$$

RECURSION

$$\begin{aligned} &\log \mathrm{Odds}(m \mid \boxed{x(1:k), y(1:k))} \\ &\quad = \quad \log \mathrm{Odds}(m \mid x(k), y(k)) - \log \mathrm{Odds}(m) \\ &\qquad\quad + \log \mathrm{Odds}(m \mid \boxed{x(1:k-1), y(1:k-1))} \end{aligned}$$

Recover Probability

$$P(x) = \frac{\text{Odds}(x)}{1+\text{Odds}(x)} = \left[1+\frac{1}{\text{Odds}(x)}\right]^{-1}$$

$$\begin{aligned}
&P(m \mid x(1:k), y(1:k)) \\
&= \left[1+\frac{\text{Odds}(m)}{\text{Odds}(m \mid x(k), y(k))\ \text{Odds}(m \mid x(1:k-1), y(1:k-1))}\right]^{-1} \\
&= \left[1+\frac{1-P(m \mid x(k), y(k))}{P(m \mid x(k), y(k))}\ \frac{P(m)}{1-P(m)}\right. \\
&\qquad \left.\frac{1-P(m \mid x(1:k-1), y(1:k-1))}{P(m \mid x(1:k-1), y(1:k-1))}\right]^{-1}.
\end{aligned}$$

Given a sequence of measurements *y(1:k),* known positions x(1:k), and an initial distribution *P_0(m)*

Determine $P_m = P(m \mid x(1:k), y(1:k))$

THE PRIOR

$$\begin{aligned}
&P_m \leftarrow P_0(m) \\
&\textbf{for } i \leftarrow 1 \text{ to } k \textbf{ do} \\
&\quad P_m \leftarrow \left[1+\frac{1-P(m|x(i),y(i))}{P(m|x(i),y(i))}\ \frac{P(m)}{1-P(m)}\ \frac{1-P_m}{P_m}\right]^{-1} \\
&\textbf{end for}
\end{aligned}$$

Actual Computation of $P(m \mid x(k), y(k))$

Local map

- Big Assumption: All Cells are Independent

$$P(m) = \prod_l P(m_l)$$

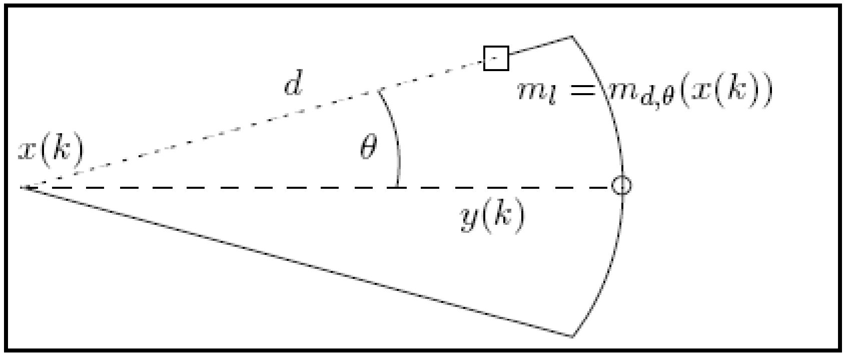
- Now, we can update just a cell $P(m_l \mid x(k), y(k)) = P(m_{d,\theta}(x(k)) \mid y(k), x(k))$

$$P(m_{d,\theta}(x(k)) \mid y(k), x(k)) = P(m_{d,\theta}(x(k)))$$

The prior

$$+ \begin{cases} -s(y(k),\theta) & d < y(k) - d_1 \\ -s(y(k),\theta) + \frac{s(y(k),\theta)}{d_1}\,(d - y(k) + d_1) & d < y(k) + d_1 \\ s(y(k),\theta) & d < y(k) + d_2 \\ s(y(k),\theta) - \frac{s(y(k),\theta)}{d_3 - d_2}\,(d - y(k) - d_2) & d < y(k) + d_3 \\ 0 & \text{otherwise.} \end{cases}$$

Depends on current cell, distance to cell and angle to central axis



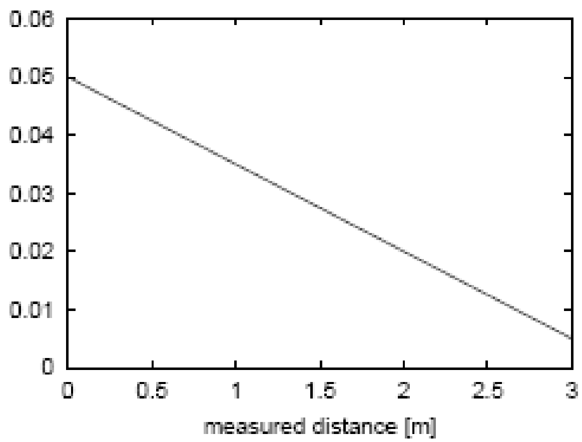
More details on s

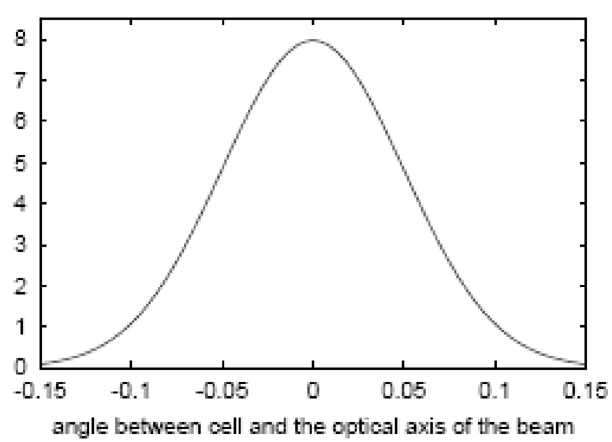
$$P(m_{d,\theta}(x(k)) \mid y(k), x(k)) = P(m_{d,\theta}(x(k))) + \begin{cases} -s(y(k),\theta) & d < y(k) - d_1 \\ -s(y(k),\theta) + \frac{s(y(k),\theta)}{d_1}\,(d - y(k) + d_1) & d < y(k) + d_1 \\ s(y(k),\theta) & d < y(k) + d_2 \\ s(y(k),\theta) - \frac{s(y(k),\theta)}{d_3 - d_2}\,(d - y(k) - d_2) & d < y(k) + d_3 \\ 0 & \text{otherwise.} \end{cases}$$

Else if’s

Deviation from occupancy probability from the prior given a reading and angle

$$s(y(k),\theta) \;=\; g(y(k))\,\mathcal{N}(0,\sigma_\theta)$$





Break it down

- d_1, d_2, d_3 specify the intervals

- Between the arc and current location, lower probability

$d < y(k) - d_1$ $\qquad\qquad$ $P(m_l) - s(y(k), \theta)$

- Cells close to the arc, ie. Whose distances are close to readings

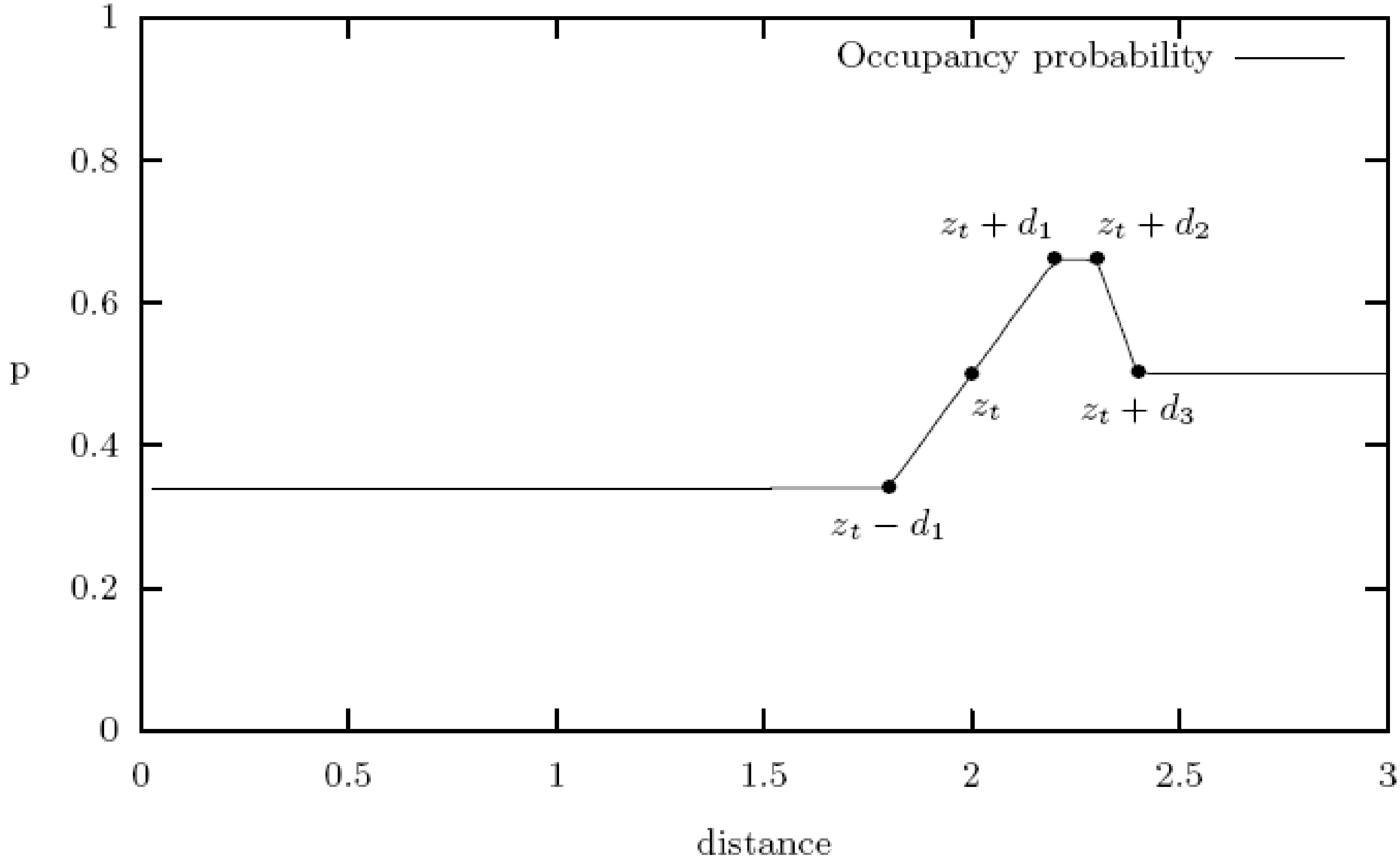
$y(k) - d_1 \leq d < y(k) + d_1$ $\qquad\qquad$ **Some linear function**

- Immediately behind the cell (obstacles have thickness)

$y(k) + d_1$ **< d <** $y(k) + d_2$ $\qquad\qquad$ $P(m_l) + s(y(k), \theta)$

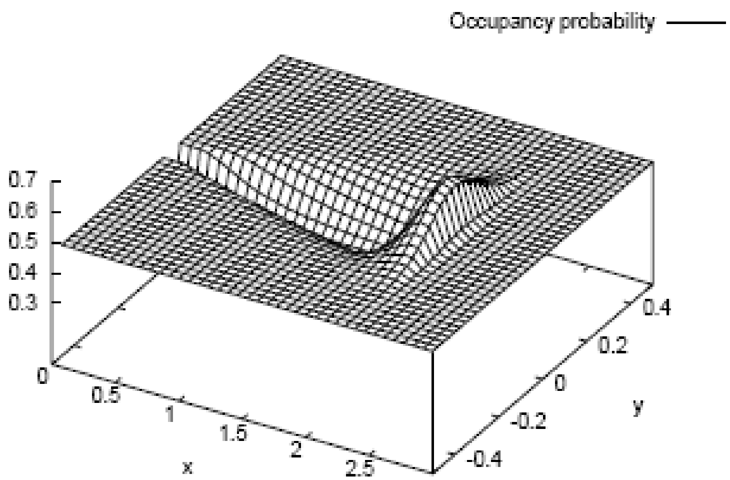
- No news is no news $P(m_{d,\theta}(x(k)) \mid y(k), x(k))$ is prior beyond

Example $P(m_{d,\theta}(x(k)) \mid y(k), x(k))$

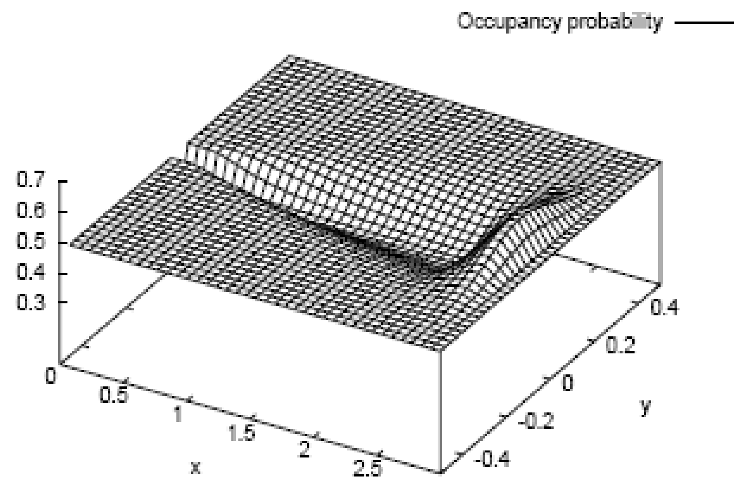


y(k) = 2m, angle = 0, s(2m,0) = .16

Example $P(m_{d,\theta}(x(k)) \mid y(k), x(k))$

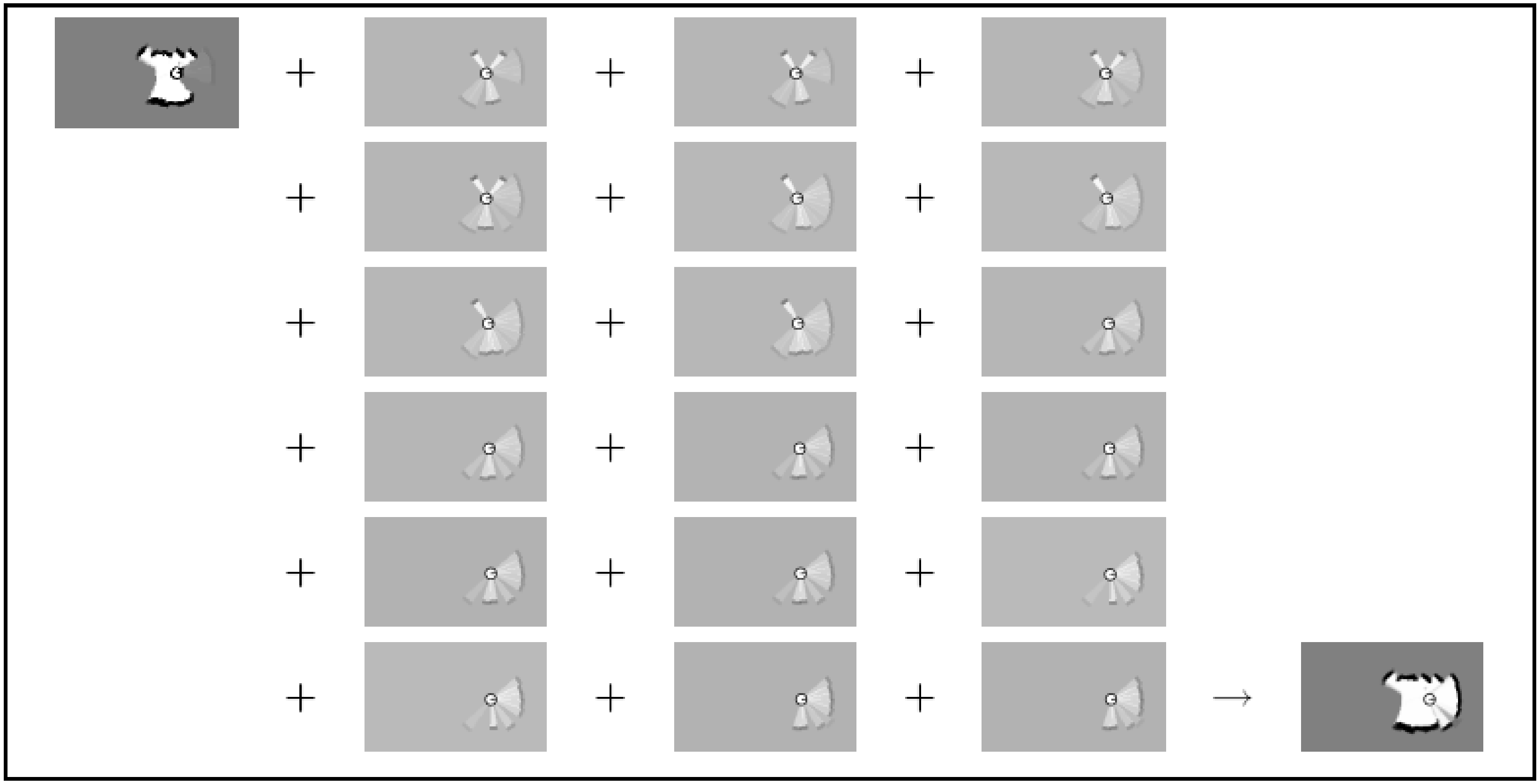


y(k) = 2m



y(k) = 2.5m

A Wolfram Mapping Experiment with a B21r with 24 sonars



$$
P(m_{d,\theta}(x(k)) \mid y(k), x(k)) = P(m_{d,\theta}(x(k))) + \begin{cases} -s(y(k),\theta) & d < y(k) - d_1 \\ -s(y(k),\theta) + \frac{s(y(k),\theta)}{d_1}\,(d - y(k) + d_1) & d < y(k) + d_1 \\ s(y(k),\theta) & d < y(k) + d_2 \\ s(y(k),\theta) - \frac{s(y(k),\theta)}{d_3 - d_2}\,(d - y(k) - d_2) & d < y(k) + d_3 \\ 0 & \text{otherwise.} \end{cases}
$$

18 scans, note each scan looks a bit uncertain but result starts to look like parallel walls

Are we independent?

- Is this a bad assumption?

SLAM!

- A recursive process.

$$\boxed{P(x(1:k), m \mid u(0:k-1), y(1:k))} = \alpha \boxed{P(y(k) \mid x(k), m)}$$
$$\int \left(\boxed{P(x(k) \mid u(k-1), x(k-1))} \right.$$
$$\left. P(x(1:k-1), m \mid u(0:k-2), y(1:k-1)) \right) dx(1:k-1)$$

Motion model

Sensor model

Posterior, hard to calculate

"Scan Matching"

At time $k-1$ **the robot is given**

1. **An estimate** $\hat{x}(k-1)$ **of state**
2. **A map estimate** $\hat{m}(\hat{x}(1:k-1), y(1:k-1))$

The robot then moves and takes measurement *y(k)*

And robot chooses state estimate which maximizes

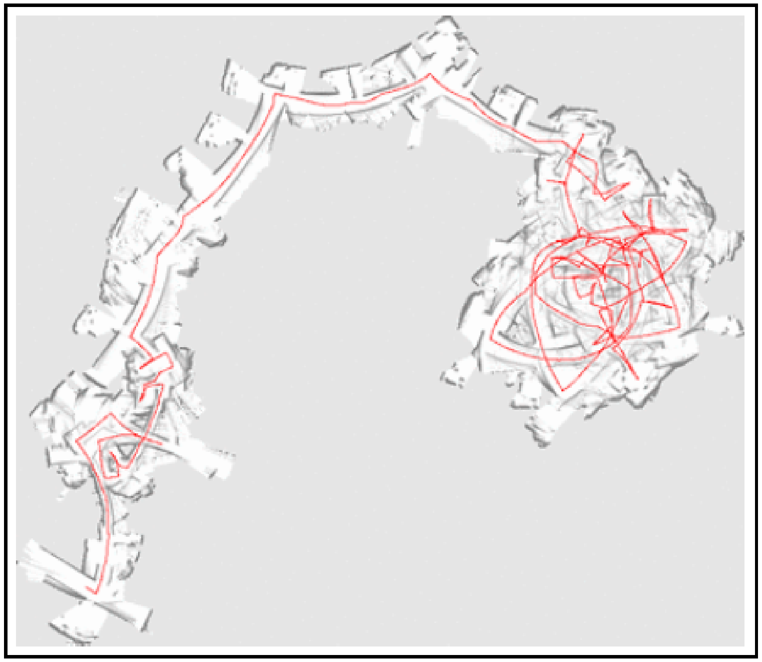
$$\hat{x}(k) = \underset{x(k)}{\operatorname{argmax}} \Big\{ P(y(k) \mid x(k), \hat{m}(\hat{x}(1:k-1), y(1:k-1))) \\ P(x(k) \mid u(k-1), \hat{x}(k-1)) \Big\}.$$

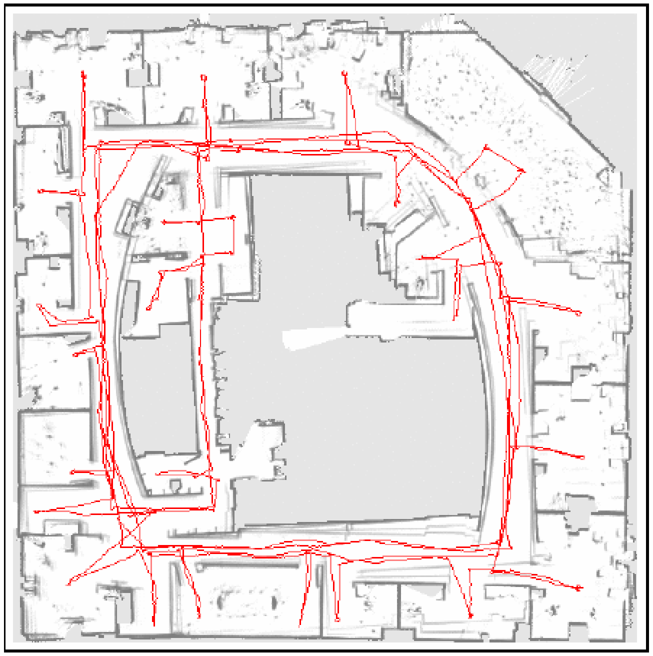
And then the map is updated with the new sensor reading

Another Wolfram Experiment

28m x 28m, .19m/s, 491m

Another Wolfram Experiment

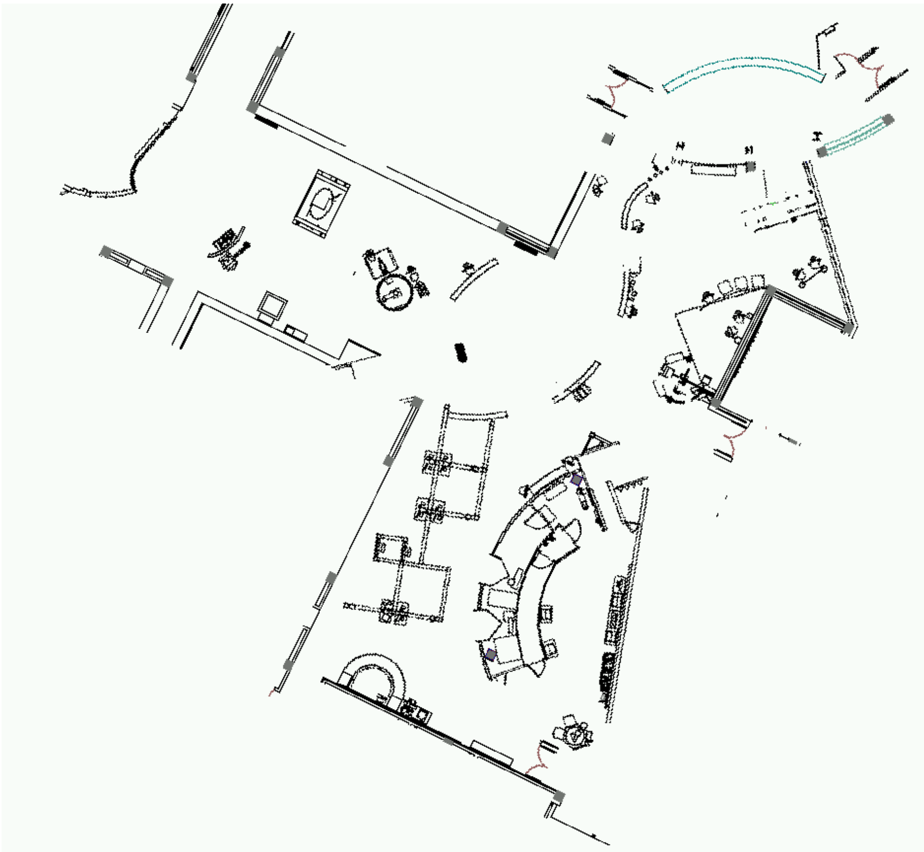




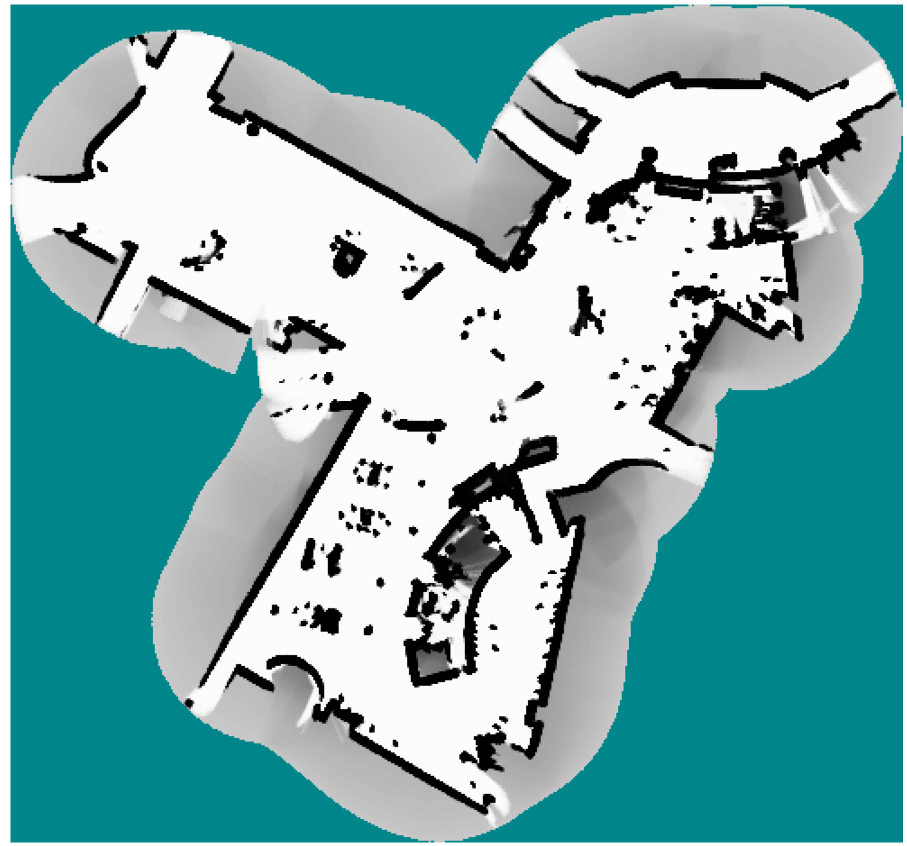
before **after**

28m x 28m, .19m/s, 491m

Tech Museum, San Jose



CAD map



occupancy grid map

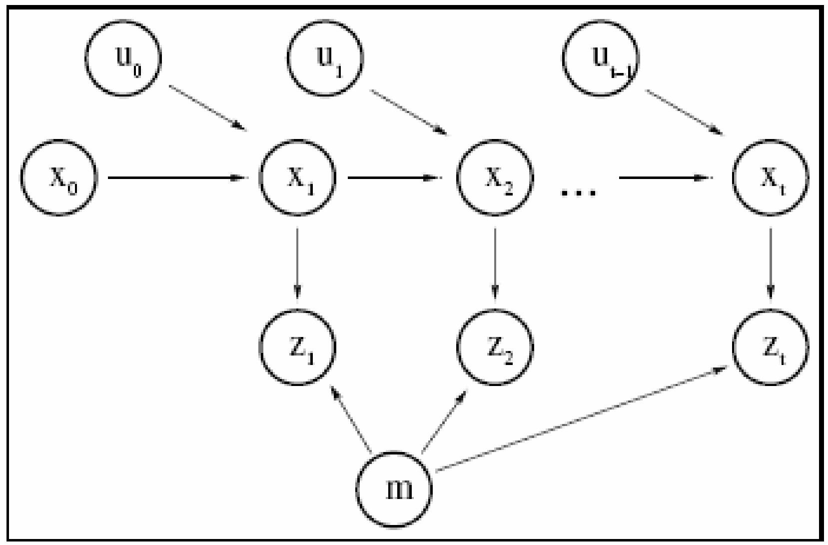
Issues

- Greedy maximization step (unimodal)
- Computational burden (post-processing)
- Inconsistency (closing the loop, global map?)

Solutions [still maintain one map, but update at loop closing]

- Grid-based technique (Konolodige et. al)
- Particle Filtering (Thrun et. al., Murphy et. al.)
- Topological/Hybrid approaches (Kuipers et. al, Leonard et al, Choset et a.)

Probabilistic SLAM
Rao-Blackwell Particle Filtering



If we know the map, then it is a localization problem
If we know the landmarks, then it is a mapping problem

Some intuition: if we know x(1:k) (not x(0)), then we know the "relative map" but Not its global coordinates

<u>The promise</u>: once path (x(1:k)) is known, then map can be determined analytically

Find the path, then find the map

Mapping with Rao-Blackwellized Particle Filters

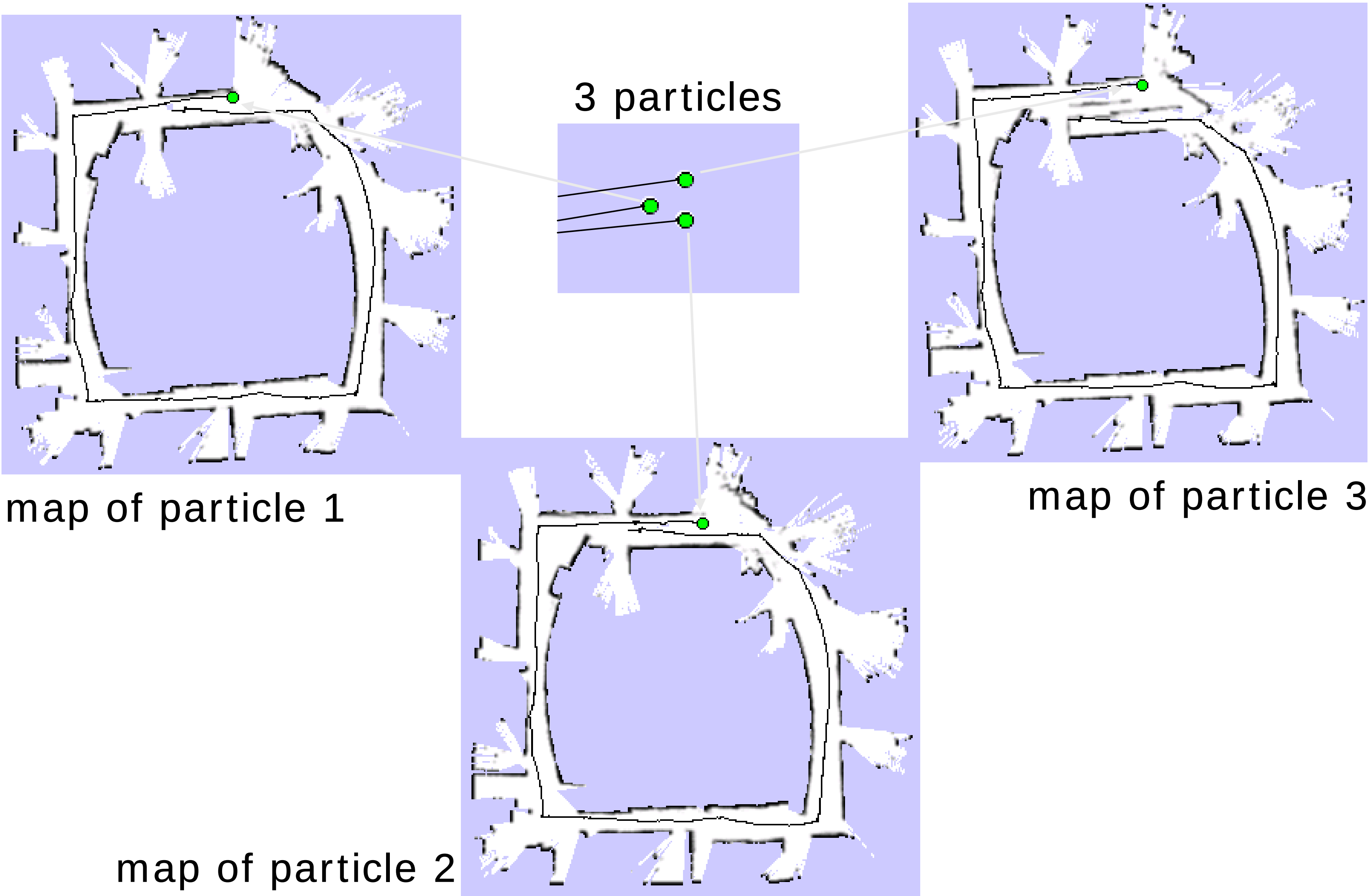
- **Observation**:

  Given the true trajectory of the robot, all measurements are independent.

- **Idea**:
  - Use a particle filter to represent potential trajectories of the robot (multiple hypotheses). Each particle is a path (maintain posterior of paths)
  - For each particle we can compute the map of the environment (mapping with known poses).
  - Each particle survives with a probability that is proportional to the likelihood of the observation given that particle and its map.

[Murphy et al., 99]

RBPF with Grid Maps



Some derivation

$$P(x(1:k), m \mid u(0:k-1), y(1:k))$$

$$\begin{aligned} & P(x(1:k), m \mid u(0:k-1), y(1:k)) \\ = \quad & P(m \mid x(1:k), y(1:k), u(0:k-1)) \\ & P(x(1:k) \mid y(1:k), u(0:k-1)). \end{aligned}$$

P(A,B) = P(A|B)P(B)

$$P(m \mid x(1:k), y(1:k), u(0:k-1)) \quad = \quad P(m \mid x(1:k), y(1:k))$$

m is independent of $u(0:k-1)$ given $x(1:k)$

$$\begin{aligned} & P(x(1:k), m \mid u(0:k-1), y(1:k)) \\ = \quad & \boxed{P(m \mid x(1:k), y(1:k))} \boxed{P(x(1:k) \mid y(1:k), u(0:k-1)).} \end{aligned}$$

We can compute **Use particle filtering**

Computing prob map (local map) given trajectory for each particle

Methodology

- M be a set of particles where each particle starts at $[0,0,0]^T$
- Let $h^{(j)}(1{:}k)$ be the jth path or particle
- Once the path is known, we can compute most likely map

$$m^{(j)}(1:k-1) \quad = \quad \underset{m}{\operatorname{argmax}}\, P(m \mid h^{(j)}(1:k), y(1:k-1))$$

Hands start waving….. Just a threshold here

- Once a new $u(i\text{-}1)$ is received (we move), do same thing as in localization, i.e., sample from $P(x \mid x_j, u(i-1))$.

Not an issue, but in book

  - Note, really sampling from $P(x \mid x_j, u(i-1), m^{(j)}(1:k-1))$
  - Ignore the map for efficiency purposes, so drop the m

- Get our $y(k)$'s to determine weights, and away we go (use same sensor model as in localization)

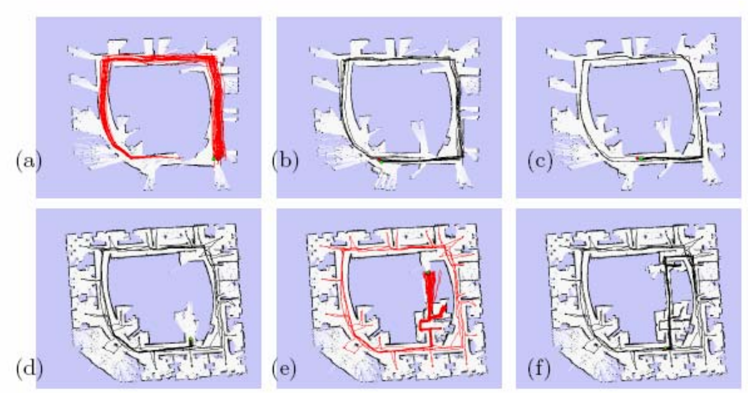
Rao-Blackwell Particle Filtering

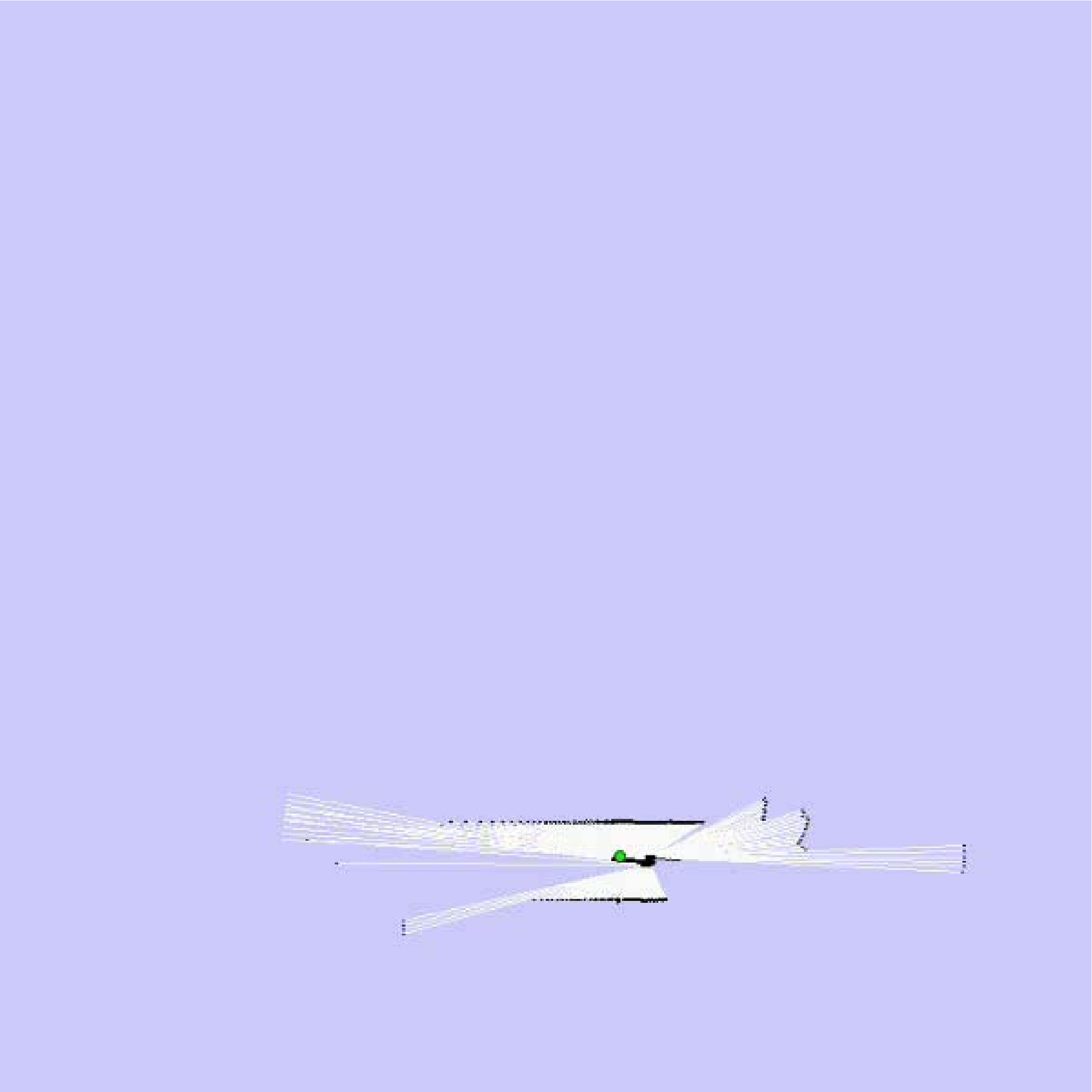
Input: Sequence of measurements $y(1:k)$ and movements $u(0:k-1)$ and set $\mathcal{M}$ of N samples (x_j, ω_j)
Output: Posterior $P(x(1:k), m \mid u(0:k-1), y(1:k))$ represented by $\mathcal{M}$ about the path of the robot at time and the map

for $j \leftarrow 1$ to N **do**
  $x_j \leftarrow (0,0,0)$
end for
for $i \leftarrow 1$ to k **do**
  for $j \leftarrow 1$ to N **do**
    compute a new state x by sampling according to $P(x \mid u(i-1), x_j)$.
    $x_j \leftarrow x$
  end for
  $\eta \leftarrow 0$
  for $j \leftarrow 1$ to N **do**
    $w_j = P(y(i) \mid x_j, m^{(j)}(1:i-1)))$
    $\eta = \eta + w_j$
  end for
  for $j \leftarrow 1$ to N **do**
    $w_j = \eta^{-1} \cdot w_j$
  end for
  $\mathcal{M} = resample(\mathcal{M})$
end for

$$\begin{aligned} &P(x(1:k), m \mid u(0:k-1), y(1:k)) \\ &\quad = \quad P(m \mid x(1:k), y(1:k))\ P(x(1:k) \mid y(1:k), u(0:k-1)). \end{aligned}$$

Wolfram Experiment





Most Recent Implementations

- **15 particles**
- four times faster than real-time P4, 2.8GHz
- 5cm resolution during scan matching
- 1cm resolution in final map

Courtesy by Giorgio Grisetti & Cyrill Stachniss

Maps, space vs. time

Maintain a map for each particle

OR

Compute the map each time from scratch

Subject of research

- Montermerlou and Thrun look for tree-like structures that capture commonality among particles.
- Hahnel, Burgard, and Thrun use recent map and subsample sensory experiences

How many particles?

- What does one mean?
- What does an infinite number mean?