21-355 Advanced Calculus I
Exam #1.

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October 14, 2002

Let the non-empty metric spaces (X,d) and (X',d') and the mapping $f : X \to X'$ be given.

1.1 Theorem. *The function $d^* : X \times X' \to [0,\infty[$ defined by $d^*((x,x'),(y,y')) := \max\{d(x,y), d'(x',y')\}$ for all $(x,x'),(y,y') \in X \times X'$ is a metric on $X \times X'$.*

Proof In order to establish that d^* is a metric, we need to show that is adheres to the three properties of a metric.

1. $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$: Because d^* relies solely on d and d', we know that since they are both metrics, they evaluate to positive values only. Therefore, the max of them must be positive. For the case when $p = q$, we know both d and d' will be zero, and thus the max will be zero as well, satisfying the condition.
2. $d(p,q) = d(q,p)$: Again since d^*'s behaviour is completely determined by the established metrics d and d' we know that this will be satisfied. The max value between d and d' will be the same for (p,q) as for (q,p).
3. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$: Without loss of generality, say that d is used as the max for $d^*(p,q)$. We know then from the fact that d is a metric that:

$$d(p,q) \leq d(p,r) + d(r,q) \tag{1}$$

Applying d to p in this case means just take the d distance instead of the max between the d and d' distance. Now supposing that for (p,r) it turns out that d' is the max. The right-hand side of (1) increases if this is the case and the relation still holds. The same argument applies if (r,q) uses the d' distance. The right-hand side goes up again, and the relationship is still intact. Note that this argument is completely symmetric, we could have picked the (p,q) distance to be the distance obtained from d' and the argument flows exactly the same way. So we have $d(p,q) \leq d^*(p,r) + d^*(r,q)$ and $d'(x) \leq d^*(p,r) + d^*(r,q)$ which implies $d^*(x) \leq d^*(p,r) + d^*(r,q)$.

Now since d^* satisfies all three properties, we can say that it is a metric on $X \times X'$. □

1.2 Theorem. *The mapping $p : X \times X' \to X$ defined by*

$$p((x,x')) := x \quad \forall (x,x') \in X \times X' \tag{2}$$

is uniformly continuous. Furthermore, f is continuous if and only if the mapping $g : X \to X \times X'$ defined by

$$g(x) := (x, f(x)) \quad \forall x \in X \tag{3}$$

is continuous.

Proof We first prove the first statement. In order to show uniform continuity, we need to show that for any $\epsilon > 0$,

$$d(p(s), p(t)) < \epsilon \tag{4}$$

Whenever $d^*(s,t) < \delta$ for all $s = (x, x'), t = (y, y') \in X \times X'$. Let $\epsilon > 0$ be given. Letting $\delta = \epsilon/2$ satisfies condition (4). This is seen to be true by noting that for any s and t s.t $d^*(s,t) < \delta$, $d(x,y) < \delta = \epsilon/2 < \epsilon$. Note that the d distance is all that matters because of the nature of p. p evaluates to the element from X, so if the distance between these two elements, s, t, is less than ϵ, then $d(p(s), p(t))$ must be less than ϵ as well. $\square$

To prove the next statement we look first at the "if" part, and then prove the converse.

In order to show $g(x)$ is continuous at a point p, we need to prove the following: For all $\epsilon > 0$, there exists a $\delta > 0$ s.t:

$$d^*(g(p), g(x)) < \epsilon \tag{5}$$

For all $x \in X$ s.t. $d(p,x) < \delta$. Now since f is continuous we know there exists a δ_f s.t. for all $\epsilon > 0$, $d'(f(p), f(x)) < \epsilon$ for all $x \in X$ s.t. $d(p,x) < \delta_f$. Using this, we introduce $\delta_g = \min\{\epsilon/2, \delta_f\}$. The claim now is that δ_g satisfies (5). To see that this δ_g works, we note that $d^*(g(p), g(x))$ uses either d or d' as the metric depending on which is larger. If it uses d, then since $d(p,x) < \delta_g \leq \epsilon/2$, then $d^*(g(x), g(p)) < \epsilon$. If we use d' instead for d^*, then since $d(p,x) < \delta_g \leq \delta_f$, $d^*(g(p), g(x)) < \epsilon$ by the continuity of f. Therefore $g(x)$ is continuous at all points $p \in X$.

Now we show the converse. Since $g(x)$ is continuous at any point p, we know of the existence of a δ_g s.t for all $\epsilon > 0$, $d^*(g(p), g(x)) < \epsilon$ for all $x \in X$ s.t $d(p,x) < \delta_g$. The key observation now is

$$d^*(g(p), g(x)) \geq d'(f(p), f(x)) \tag{6}$$

This is easily seen to be true by noting that d^* takes the max of the two metrics d and d', therefore the value of d^* must be greater than or equal to the d' distance. (6) allows us to say that for any $\epsilon > 0$, $d'(f(p), f(x)) < \epsilon$ for all $x \in X$ s.t $d(p,x) < \delta_g$. Therefore f is continuous at all points $p \in X$. $\square$

1.3 Theorem. *The subset $G := \{(x, f(x)) \mid x \in X\}$ of $X \times X'$ is called the* **graph of** *f. The following statements are equivalent:*
(1): G is a compact subset of $X \times X'$.
(2): X is compact and f is continuous.

Proof

Lemma. *The mapping $h : X \to G$ where $h(x) := (x, f(x))$ is bijective.*

Proof h is clearly 1-to-1. The first element in the pair $(x, f(x))$ guarantees this. h is also "onto" since the range of h is G by definition. So since h is 1-to-1 and "onto" it is a bijection. $\square$

(1⇒2)

Since h is bijective, there exists a function $q(x) := h^{-1}(x) : G \to X$. The exact definition of q is $q(x) := x$. By Theorem 1.2 we know that q is uniformly continuous and therefore continuous. Since h is bijective, we know that X and G are the same cardinality. Therefore, $q(G) = X$. By Rudin 4.14, since G is compact, $q(G) = X$ is compact. Since q is continuous and a 1-1 mapping (since in G there is at most 1 pair with x in it) from a compact space G to X, by Rudin 4.17 we know that the inverse of q, which is h, is continuous. Now by Theorem 1.2, since h is continuous, so is f, and the implication is proven.

(2⇒1)

This implication follows directly from Rudin 4.14. By Theorem 1.2 we know that if f is continuous, then $h(x) := (x, f(x))$ is continuous as well. Again note that our h maps all of X to all of G, so $h(X) = G$ since h is a bijection. So since h is continuous and X is compact, $h(X) = G$ implies that G is compact by Rudin 4.14.

Since (1) ⇒ (2) and (2) ⇒ (1) then, (1) ⇔ (2) and we are done. $\square$

2.1 Theorem. *Let the intervals D and D' of $\mathbb{R}$ and the bijection $f : D \to D'$ be given. If f is continuous, then the inverse mapping $g = f^{-1} : D' \to D$ is continuous.*

Proof

Lemma 1. *Given an invertible mapping f, then f is either isotone or antitone.*

Proof If f is neither isotone nor antitone, that means that there exists at least two points r and s and a point p such that $f(r) = p$ and $f(s) = p$. Because of this, the inverse of f, $g = f^{-1}$, is not a function since $g(p) = r$ and $g(p) = s$. Therefore the inverse does not exist, contradicting the fact that f is invertible. So we conclude that f is either isotone or antitone. $\square$

Lemma 2. *Given an isotone, invertible mapping $f : D \to D'$, $g = f^{-1} : D' \to D$ is isotone as well.*

Proof Pick two points x_1',$x_2' \in D'$ s.t $x_1' < x_2'$. Let $x_1 = g(x_1')$, $x_2 = g(x_2')$. Clearly, $x_1' = f(x_1)$ and $x_2' = f(x_2)$. Since f is isotone, $x_1' < x_2' \Rightarrow x_1 < x_2$. Furthermore, this relation implies that g is isotone. $\square$

Lemma 3. *Given an isotone, invertible mapping $f : D \to D'$, $g = f^{-1} : D' \to D$, if f is continuous, then g is continuous.*

Proof Select a point $p \in D'$. The condition we need to satisfy to establish the continuity of g is the following: For every $\epsilon > 0$ there exists a δ s.t

$$|g(p) - g(x')| < \epsilon \tag{7}$$

For all $x' \in D'$ s.t. $|p - x'| < \delta$. Let $x_p = g(p)$ s.t $f(x_p) = p$ and let $\epsilon > 0$ be given. Then set $\delta = \min\{f(x_p) - f(x_p - \epsilon), f(x_p + \epsilon) - f(x_p)\}$. We know that δ will be a positive value since f is isotone. To see that this δ satisfies (7) we make the following observations. Let $Gi = [p-\delta, p+\delta]$, $Gf = [g(p)-\epsilon, g(p)+\epsilon]$, $Fi = [x_p - \epsilon, x_p + \epsilon]$, $Ff = [f(x_p - \epsilon), f(x_p + \epsilon)]$. It should be noted that these intervals are well-defined since by Lemma 2 g is isotone so both Gi and Gf and non-empty. The same argument applies to Fi and Ff. Since f is continuous and by our selection of δ, we know that $Gi \subset Ff$. No value can be in Gi that is not in Ff since f is continuous. Applying g to Ff yields Fi, therefore applying g to Gi yields a subset of Fi. Furthermore, by the definition of f and g, $Gf = Fi$. This implies that $g(Gi) \subset Gf$ which shows that (7) is satisfied given our chosen δ.

To show continuity at the endpoints of $D' = [c, d]$ we just need to modify the above proof slightly. To show g is continuous at c, we just to need to modify the definition of Gi, Gf, Fi and Ff to exclude elements that would not be in D'. Specifically, those elements would be $p - \delta$ in Gi. Losing these "points" does not affect the rest of the argument. The same rationale applies at d as well. Therefore g is continuous over all of D'. $\square$

By Lemma 1, we know that f is either isotone or antitone. If f is isotone we apply Lemma 3 and we are done. If f is antitone, then we apply Lemma 3 to $-f$ (which is isotone and continuous by Rudin Theorem 4.9) and we are done. This is seen to be true since the inverse of $-f$ is the same as for f except for the sign switch. We know from Rudin 4.9 that sign-switching does not affect the continuity. Therefore the theorem holds for all bijections f. $\square$

3.1 Theorem. *The mapping $q : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $q(t) := 1/t$ for all $t \in \mathbb{R} \setminus \{0\}$ is continuous*

Proof We start out by writing q in terms of two new functions f and g defined as such, $f(x) = 1$, $g(t) = t$. Clearly $f(x)$ is continuous, and $g(x)$ is continuous by Theorem 4.6 from Rudin since $\lim_{x \to p} x = p$. Now by Theorem 4.9 in Rudin we know that $q(t) := f(t)/g(t)$ is continuous if f and g are continuous. Since we established the continuity of f and g above, then we have that $q(t) := 1/t$ is continuous. $\square$

3.2 Theorem. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be given, and let f and g be continuous. Let $D := \{t \in \mathbb{R} \mid g(t) \neq 0\}$ and define $h : D \to \mathbb{R}$ by $h(t) := f(t)/g(t)$ for all $t \in D$. Then h is continuous.*

Proof We consider only limit points of D since there is nothing to prove for isolated points. Since f and g are continuous, we know that $\forall\, p \in D$ $\lim_{x \to p} f(x) = f(p)$ and $\lim_{x \to p} g(x) = g(p)$ by Rudin Theorem 4.6. Now by Rudin Theorem 4.4 we know that $\lim_{x \to p} h(p) = \lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f(p)}{g(p)} = h(p)$. Applying Rudin Theorem 4.6 again, we know the function $h(x) := \frac{f(x)}{g(x)}$ is continuous because of the statement about the limit.

A few examples of uniformly continuous functions of the form $h(x) := \frac{f(x)}{g(x)}$ are:

1. $h(x) := \frac{\sin x}{x}$ - The derivative of this is bounded which means it is uniformly continuous. The limit as x gets large is 1 which further strengthens the argument. Obviously we are not considerering the case when $x = 0$.

2. $h(x) := \frac{x}{1}$ - A trivial example in case my first one is wrong. This reduces to $h(x) := x$ which is clearly uniformly continuous.

3. $h(x) := \frac{1}{1}$. - To cover my back I throw this in. $h(x) := 1$ is trivially uniform continuous and technically is usuable from the problem statement.

Some nonuniformly continuous functions:

1. $h(x) := \frac{1}{x}$. Let $\epsilon > 0$ and $\delta > 0$ be given and select a point $p \in X$ s.t. $|p| < \delta$. Now since around 0 h takes off to infinity, taking an x close to zero we can make $|h(x) - h(p)|$ larger than ϵ. We can do this however while still maintaining $|x - p| < \delta$. The key to this being true is that h is unbounded around 0, which allows me to adjust the difference between $h(x)$ and $h(p)$ without adjusting the distance between x and p.

2. $h(x) := \frac{x^2}{1}$. In class it was mentioned that x^2 is not uniformly continuous, so I put it in here too.

$\square$