# SAT4Math
## Beyond NP & Collatz Conjecture

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sat4math.com

# 50 Years of Successes in Computer-Aided Mathematics

1976 Four-Color Theorem

1998 Kepler Conjecture

2014 Boolean Erdős discrepancy problem (using a SAT solver)

2016 Boolean Pythagorean triples problem (using a SAT solver)

2018 Schur Number Five (using a SAT solver)

2019 Keller's Conjecture (using a SAT solver)

2021 Kaplansky's Unit Conjecture (using a SAT solver)

2022 Packing Number of Square Grid (using a SAT solver)

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# A Counterexample to the Unit Conjecture

**Theorem ([Gardam 2021])**
*Let $P = \langle a, b \mid b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2} \rangle$ be a torsion-free group and set $x = a^2$, $y = b^2$, $z = (ab)^2$. Set*

$$\begin{aligned} p &= (1+x)(1+y)(1+z^{-1}) \\ q &= x^{-1}y^{-1} + x + y^{-1}z + z \\ r &= 1 + x + y^{-1}z + xyz \\ s &= 1 + (x + x^{-1} + y + y^{-1})z^{-1}. \end{aligned}$$

*Then $p + qa + rb + sab$ is a non-trivial unit in the ring $F_2[P]$.*

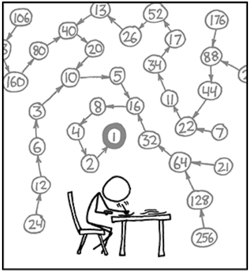
- Giles Gardam guessed $P$
- The SAT solver found $p$, $q$, $r$, and $s$

# The Collatz Conjecture

The Collatz map is defined as follows:

$$Col(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Does `while` $(n > 1)$ $n = Col(n)$; terminate?




THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

source: xkcd.com/710

## Example

27, 41, 62, 31, 47, 71, 107, 161, 242, 121, 182, 91, 137, 206, 103, 155, 233, 350, 175, 263, 395, 593, 890, 445, 668, 334, 167, 251, 377, 566, 283, 425, 1276, 638, 319, 479, 719, 1079, 1619, 2429, 3644, 18224, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 46, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1

# How to (Dis)Prove Collatz Using SAT?

Collatz is a termination problem (thus way way beyond NP)

Three possible outcomes:

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   - existing methods showed no cycle for $n$ up to $2^{70}$
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# How to (Dis)Prove Collatz Using SAT?

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1. There exists an $n$ for which the sequence goes to infinity
   - unlikely
2. There exists a cycle
   - existing methods showed no cycle for $n$ up to $2^{70}$
   - so searching for a cycle is too hard for SAT
3. The conjecture holds
   - search for a ranking function of a certain size/shape
   - satisfiable: conjecture solved
   - unsatisfiable: try another size or shape

# A Collatz Ranking Function?

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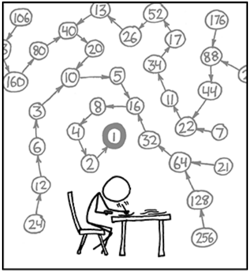
Find a non-negative function $f(n)$ s.t.

$$\forall n > 1 : f(n) > f(Col(n))$$

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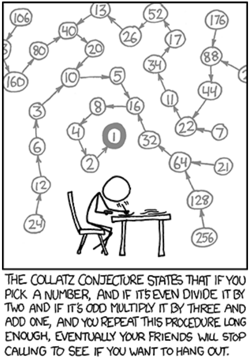


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How can we construct such a function using SAT?

| $f(3)$ | $f(5)$ | $f(8)$ | $f(4)$ | $f(2)$ | $f(1)$ |
|---|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 | 0 |

# Motivating Example: String Rewriting

| | |
|---|---|
| $\Sigma$ | alphabet |
| $\ell \to r$ | rewrite rule |
| $R$ | rewriting system |
| $\to_R$ | rewrite relation |

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Example

- $R = \{ab \to ba\}$

$$\underline{ab}bab \to_R b\underline{ab}ab \to_R bba\underline{ab} \to_R bb\underline{ab}a \to_R bbbaa$$

# Motivating Example: Termination Examples

$R$ is terminating if there is no infinite chain with $X_i \in \Sigma^*$:

$$X_0 \quad \to_R \quad X_1 \quad \to_R \quad X_2 \quad \to_R \quad \cdots$$

Example

- $R = \{ab \to ba\}$ terminating
- $S = \{a \to aa\}$ nonterminating
- $T = \{aa \to a\}$ terminating
- $P = \left\{\begin{matrix} a \to b \\ b \to a \end{matrix}\right\}$ nonterminating
- $Z = \left\{\begin{matrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{matrix}\right\}$ ?

# Proving termination [Baader and Nipkow, 1998]

$R$ is terminating if there exists a well-founded order $\succ_{\Sigma^*}$ such that for all $X, Y \in \Sigma^*$

$$X \to_R Y \quad \Longrightarrow \quad X \succ_{\Sigma^*} Y.$$

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Translate $s \in \Sigma$ into an element $[s]$ of another domain $(A, \succ_A)$ and show that $[\ell] \succ_A [r]$ for all $\ell \to r \in R$.

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- $T = \{aa \to a\}$. Let $[a] = 1$, extend to strings additively.

$$[aa] = 2 > 1 = [a]$$

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Example

- $T = \{aa \to a\}$. Let $[a] = 1$, extend to strings additively.

$$[aa] = 2 > 1 = [a]$$

- $R = \{ab \to ba\}$. Let $[a](x) = x^2$ and $[b](x) = x + 1$, extend to strings compositionally.

$$[ab](x) = (x+1)^2 > x^2 + 1 = [ba](x)$$

# Matrix interpretations [Hofbauer and Waldmann, 2006]

Affine functions $[s]\colon \mathbb{N}^d \to \mathbb{N}^d$
(extended to strings compositionally):

$$[s](\vec{x}) = \mathbf{M}_s\vec{x} + \mathbf{v}_s$$

Well-founded order:

$$\vec{x} \succ \vec{y} \quad \iff \quad x_1 > y_1 \ \wedge \ x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}$$

Look for interpretations satisfying

$$[\ell](\vec{x}) = \mathbf{M}_\ell\vec{x} + \mathbf{v}_\ell \succ \mathbf{M}_r\vec{x} + \mathbf{v}_r = [r](\vec{x}) \text{ for all } \vec{x} \in \mathbb{N}^d.$$

Encode all the resulting constraints as a SAT instance.

# Motivating Example: Proof by SAT Solver (I)

$$Z = \begin{Bmatrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{Bmatrix}$$

# Motivating Example: Proof by SAT Solver (I)

$$Z = \left\{ \begin{matrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{matrix} \right\}$$

$$[a](\vec{x}) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[b](\vec{x}) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$[c](\vec{x}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

# Motivating Example: Proof by SAT Solver (II)

$$\vec{x} \succ \vec{y} \quad \Longleftrightarrow \quad x_1 > y_1 \ \wedge \ x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}$$

It is decidable to check the following:

$$[aa](\vec{x}) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \succ \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = [bc](\vec{x})$$

$$[bb](\vec{x}) = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \vec{x} + \begin{bmatrix} 4 \\ 0 \\ 2 \\ 6 \end{bmatrix} \succ \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix} = [ac](\vec{x})$$

$$[cc](\vec{x}) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \succ \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} = [ab](\vec{x})$$

# Collatz as Rewrite System: Symbols and Rules

Consider the following 7 symbols to express numbers:

| | | | | | | |
|---|---|---|---|---|---|---|
| $1$ | $\times 1$ | $\times 2$ | $\times 2 + 1$ | $\times 3$ | $\times 3 + 1$ | $\times 3 + 2$ |

Numbers can be expressed in multiple ways:

- 27 (binary): $1$ | $\times 2 + 1$ | $\times 2$ | $\times 2 + 1$ | $\times 2 + 1$ | $\times 1$
- 27 (ternary): $1$ | $\times 3$ | $\times 3$ | $\times 3$ | $\times 1$

# Collatz as Rewrite System: Symbols and Rules

Consider the following 7 symbols to express numbers:

[1] [×1] [×2] [×2+1] [×3] [×3+1] [×3+2]

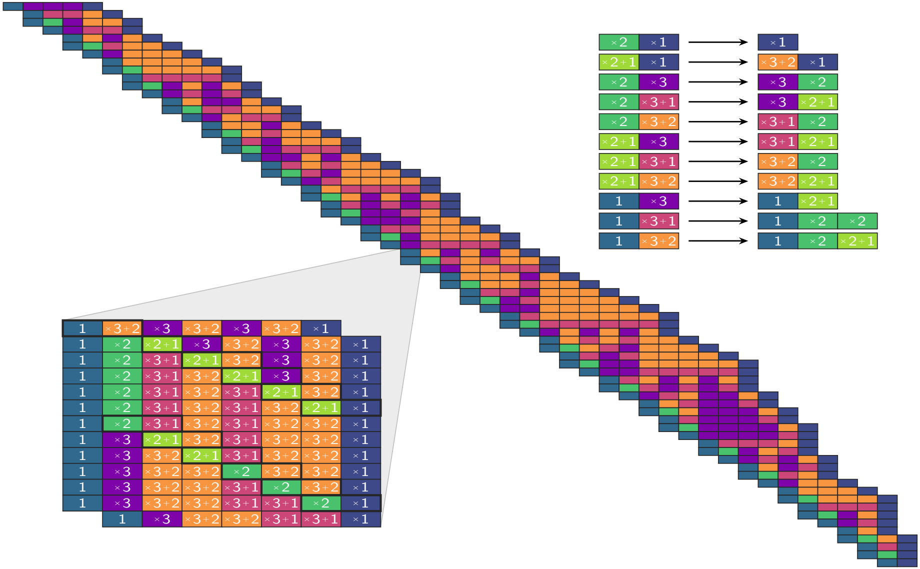
Numbers can be expressed in multiple ways:

- 27 (binary): [1] [×2+1] [×2] [×2+1] [×2+1] [×1]
- 27 (ternary): [1] [×3] [×3] [×3] [×1]

These rewrite rules express Collatz:

| | | |
|---|---|---|
| [×2] [×1] | → [×1] | $2n \rightarrow n$ |
| [×2+1] [×1] | → [×3+2] [×1] | $2n + 1 \rightarrow 3n + 2$ |
| [×2] [×3] | → [×3] [×2] | $\times 6 + 0 \rightarrow \times 6 + 0$ |
| [×2] [×3+1] | → [×3] [×2+1] | $\times 6 + 1 \rightarrow \times 6 + 1$ |
| [×2] [×3+2] | → [×3+1] [×2] | $\times 6 + 2 \rightarrow \times 6 + 2$ |
| [×2+1] [×3] | → [×3+1] [×2+1] | $\times 6 + 3 \rightarrow \times 6 + 3$ |
| [×2+1] [×3+1] | → [×3+2] [×2] | $\times 6 + 4 \rightarrow \times 6 + 4$ |
| [×2+1] [×3+2] | → [×3+2] [×2+1] | $\times 6 + 5 \rightarrow \times 6 + 5$ |
| [1] [×3] | → [1] [×2+1] | $3 \rightarrow 3$ |
| [1] [×3+1] | → [1] [×2] [×2] | $4 \rightarrow 4$ |
| [1] [×3+2] | → [1] [×2] [×2+1] | $5 \rightarrow 5$ |

# Collatz as Rewriting System: Starting at 27 Down to 1



# Collatz as Rewrite System: Theorem

$$\begin{aligned} \mathtt{f}(x) &= 2x \\ \mathtt{t}(x) &= 2x+1 \end{aligned} \qquad \begin{aligned} \mathtt{0}(x) &= 3x \\ \mathtt{1}(x) &= 3x+1 \\ \mathtt{2}(x) &= 3x+2 \end{aligned} \qquad \begin{aligned} \lhd(x) &= 1 \\ \rhd(x) &= x \end{aligned}$$

$$\begin{aligned} \mathtt{f}\rhd &\to \rhd \\ \mathtt{t}\rhd &\to \mathtt{2}\rhd \end{aligned} \qquad \begin{aligned} \mathtt{f0} &\to \mathtt{0f} \\ \mathtt{f1} &\to \mathtt{0t} \\ \mathtt{f2} &\to \mathtt{1f} \end{aligned} \qquad \begin{aligned} \mathtt{t0} &\to \mathtt{1t} \\ \mathtt{t1} &\to \mathtt{2f} \\ \mathtt{t2} &\to \mathtt{2t} \end{aligned} \qquad \begin{aligned} \lhd\mathtt{0} &\to \lhd\mathtt{t} \\ \lhd\mathtt{1} &\to \lhd\mathtt{ff} \\ \lhd\mathtt{2} &\to \lhd\mathtt{ft} \end{aligned}$$

**Theorem**

*Above system is terminating* $\iff$ *Collatz conjecture holds.*

# Collatz as Rewrite System: Theorem

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\qquad
\begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}
$$

$$
\begin{aligned} \mathtt{f}\triangleright &\to \triangleright \\ \mathtt{t}\triangleright &\to \mathtt{2}\triangleright \end{aligned}
\qquad
\begin{aligned} \mathtt{f0} &\to \mathtt{0f} \\ \mathtt{f1} &\to \mathtt{0t} \\ \mathtt{f2} &\to \mathtt{1f} \end{aligned}
\qquad
\begin{aligned} \mathtt{t0} &\to \mathtt{1t} \\ \mathtt{t1} &\to \mathtt{2f} \\ \mathtt{t2} &\to \mathtt{2t} \end{aligned}
\qquad
\begin{aligned} \triangleleft\mathtt{0} &\to \triangleleft\mathtt{t} \\ \triangleleft\mathtt{1} &\to \triangleleft\mathtt{ff} \\ \triangleleft\mathtt{2} &\to \triangleleft\mathtt{ft} \end{aligned}
$$

## Theorem

*Above system is terminating* $\iff$ *Collatz conjecture holds.*

## Example

$$
\begin{array}{ccccccccccccc}
\mathbf{12} & & \mathbf{12} & & \mathbf{6} & & \mathbf{6} & & \mathbf{3} & & \mathbf{3} & & \mathbf{5} \\
\triangleleft\mathtt{f}\underline{\mathtt{f0}}\triangleright & \to & \triangleleft\mathtt{f0}\underline{\mathtt{f}\triangleright} & \to & \triangleleft\underline{\mathtt{f0}}\triangleright & \to & \triangleleft\mathtt{0}\underline{\mathtt{f}\triangleright} & \to & \underline{\triangleleft\mathtt{0}}\triangleright & \to & \triangleleft\underline{\mathtt{t}\triangleright} & \to & \underline{\triangleleft\mathtt{2}}\triangleright
\end{array}
$$

$$
\begin{array}{cccccccccccccc}
 & \mathbf{5} & & \mathbf{8} & & \mathbf{8} & & \mathbf{8} & & \mathbf{4} & & \mathbf{2} & & \mathbf{1} \\
\to & \triangleleft\mathtt{f}\underline{\mathtt{t}\triangleright} & \to & \triangleleft\underline{\mathtt{f2}}\triangleright & \to & \underline{\triangleleft\mathtt{1}}\mathtt{f}\triangleright & \to & \triangleleft\mathtt{ff}\underline{\mathtt{f}\triangleright} & \to & \triangleleft\mathtt{f}\underline{\mathtt{f}\triangleright} & \to & \triangleleft\underline{\mathtt{f}\triangleright} & \to & \triangleleft\triangleright
\end{array}
$$

# Farkas' Variant: Definition & Rewrite System

Informal: if $n \equiv 1 \pmod 3$, then $\frac{n-1}{3}$, otherwise Collatz

$$F(n) = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{2} & \text{if } n \equiv 0 \text{ or } n \equiv 2 \pmod 6 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

# Farkas' Variant: Termination Proof (Arctic: $-$ means $-\infty$)

$$[\mathtt{f}](\vec{x}) = \begin{bmatrix} - & - & - & 2 & - \\ - & 2 & 0 & - & - \\ 2 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \vec{x} \oplus \begin{bmatrix} 0 \\ - \\ - \\ - \\ - \end{bmatrix} \qquad [\mathtt{t}](\vec{x}) = \begin{bmatrix} - & - & - & - & 2 \\ 0 & 2 & 0 & - & 0 \\ 2 & - & 2 & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \vec{x} \oplus \begin{bmatrix} 0 \\ - \\ - \\ - \\ - \end{bmatrix}$$

$$[\triangleleft](\vec{x}) = \begin{bmatrix} 0 \\ 2 \\ - \\ - \\ 4 \end{bmatrix} \quad [\triangleright](\vec{x}) = \begin{bmatrix} 0 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \vec{x} \quad [\mathtt{0}](\vec{x}) = \begin{bmatrix} 0 & 4 & 0 & - & - \\ - & 4 & - & - & - \\ - & 4 & 0 & - & - \\ 0 & 3 & 0 & - & - \\ - & - & - & - & - \end{bmatrix} \vec{x}$$

$$[\mathtt{1}](\vec{x}) = \begin{bmatrix} 1 & - & - & - & - \\ - & 4 & 0 & - & - \\ - & 4 & 0 & - & - \\ 0 & - & - & - & - \\ 0 & 3 & 0 & - & - \end{bmatrix} \vec{x} \qquad [\mathtt{2}](\vec{x}) = \begin{bmatrix} 0 & - & 0 & - & - \\ - & 4 & - & - & - \\ 0 & - & 1 & - & 0 \\ - & - & - & - & - \\ 0 & - & 0 & - & 0 \end{bmatrix} \vec{x}$$

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$$[\triangleleft](\vec{x}) = \begin{bmatrix} 0 \\ 2 \\ - \\ - \\ 4 \end{bmatrix} \quad [\triangleright](\vec{x}) = \begin{bmatrix} 0 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \vec{x} \quad [\mathtt{0}](\vec{x}) = \begin{bmatrix} 0 & 4 & 0 & - & - \\ - & 4 & - & - & - \\ - & 4 & 0 & - & - \\ 0 & 3 & 0 & - & - \\ - & - & - & - & - \end{bmatrix} \vec{x}$$

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**DEMO**

# Subsystems: Introduction

The full system is too hard.

We don't need to show that all rules are decreasing

- Only one rule needs to decrease, while no other increases
- Remove that rule and continue with the remaining ones

The first step is too hard, but all remaining steps are doable

- Consider any subset of 10 out of the 11 rules
- We can prove termination of each subset

# Subsystems: Results

| | Matrix | | | Arctic | | |
|---|---|---|---|---|---|---|
| Rule removed | $D$ | $V$ | Time | $D$ | $V$ | Time |
| `f`$\triangleright \to \triangleright$ | 3 | 4 | 1.42s | 3 | 5 | 15.95s |
| `t`$\triangleright \to 2\triangleright$ | 1 | 2 | 0.27s | 1 | 3 | 0.28s |
| `f0` $\to$ `0f` | 4 | 2 | 0.92s | 3 | 4 | 2.46s |
| `f1` $\to$ `0t` | 1 | 3 | 0.50s | 1 | 4 | 0.51s |
| `f2` $\to$ `1f` | 1 | 2 | 0.38s | 1 | 3 | 0.39s |
| `t0` $\to$ `1t` | 4 | 3 | 1.20s | 3 | 4 | 0.87s |
| `t1` $\to$ `2f` | 5 | 2 | 0.89s | 4 | 3 | 0.84s |
| `t2` $\to$ `2t` | 4 | 4 | 10.00s | 2 | 5 | 0.62s |
| $\triangleleft 0 \to \triangleleft$`t` | 2 | 2 | 0.40s | 2 | 3 | 0.42s |
| $\triangleleft 1 \to \triangleleft$`ff` | 3 | 3 | 0.53s | 3 | 4 | 0.57s |
| $\triangleleft 2 \to \triangleleft$`ft` | 4 | 4 | 7.51s | 4 | 3 | 4.04s |

# Subsystems: Solver Considerations

**Phase-saving heuristic:**

- Phase-saving uses cached assignments to variables when choosing a branch to explore.
- For small values, order encoding assigns a large fraction of the variables to false.
- Disable phase-saving and use negative branching: always explore the “false” branch first.

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- Disable phase-saving and use negative branching: always explore the “false” branch first.

**Heavy-tailed behavior:**

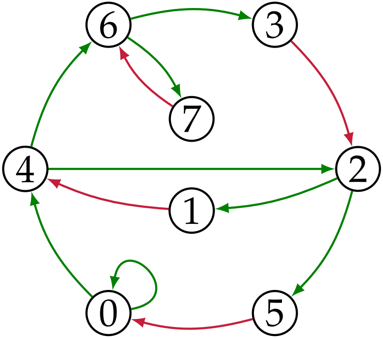
- There is a large variance in runtime across different initial conditions of the search. Some runs have a nonnegligible chance of finishing early.
- Run multiple instances of the SAT solver with varying seeds and different shufflings of the formula.

# Subsystems: Solver Configurations

| | | | Phase-saving | | Negative branching | |
|---|---|---|---|---|---|---|
| Interpretation | $D$ | $V$ | Single | Parallel | Single | Parallel |
| Arctic | 5 | 8 | 240.00s | 240.00s | 44.45s | **9.22s** |
| Arctic | 3 | 5 | 1.52s | **0.13s** | 29.95s | 13.12s |
| Arctic | 3 | 4 | 3.75s | **0.83s** | 3.27s | 1.71s |
| Natural | 4 | 4 | 75.78s | 19.12s | 29.62s | **8.13s** |
| Natural | 4 | 4 | 75.05s | **5.22s** | 24.31s | 6.43s |
| Arctic | 4 | 3 | 3.33s | **0.52s** | 11.55s | 3.84s |
| Natural | 3 | 11 | 240.00s | 240.00s | 240.00s | **79.05s** |
| Arctic | 3 | 12 | 1.94s | **0.33s** | 3.28s | 0.38s |

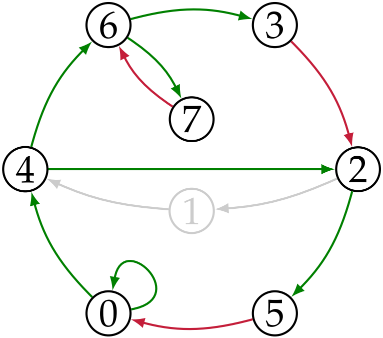
# Collatz Modulo 8: Cycles

The full system is too hard. What is we drop some states modulo 8?




| cycle | effect |
|---|---|
| $(0,0)$ | $n/2$ |
| $(0,4,2,5,0)$ | $(3n+8)/8$ |
| $(1,4,2,1)$ | $(3n+1)/4$ |
| $(2,5,0,4,2)$ | $(3n+2)/8$ |
| $(2,1,4,2)$ | $(3n+2)/4$ |
| $(4,2,5,0,4)$ | $(3n+4)/8$ |
| $(4,2,1,4)$ | $(3n+4)/4$ |
| $(5,0,4,2,5)$ | $(3n+1)/8$ |
| $(5,0,4,6,3,2,5)$ | $(9n+11)/16$ |
| $(7,6,7)$ | $(3n+1)/2$ |
| $(1,4,6,3,2,1)$ | $(9n+7)/8$ |

# Collatz Modulo 8: Terminate at 1 Mod 8



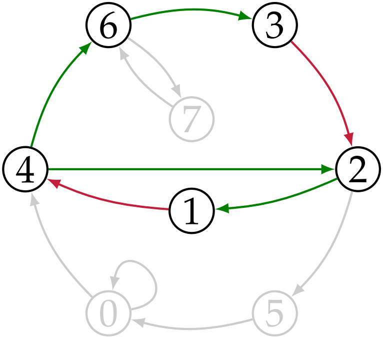

Does

$$\texttt{while}(n \not\equiv 1 \pmod 8)$$
$$n = Col(n)$$

terminate?

$500 prize

# Collatz Modulo 8: Terminate at 5 or 7 Mod 8




Even the case without 5 or 7 mod 8 is open

It is equivalent to

$$H(n) = \begin{cases} \frac{3n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{9n+1}{8} & \text{if } n \equiv 7 \pmod 8 \\ \perp & \text{otherwise} \end{cases}$$

$500 prize

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How to encode the SAT formula?

- The order encoding for multiplication is very effective
- Reduce the size of the encoding by reusing calculations

Which SAT solving techniques are effective?

- Some old SAT techniques work better than new ones
- Can local search be effective (we only look for solutions)?

# Conclusions: Starting at 27 Down to 1