SAT4Math
Proofs & Chromatic Number of the Plane

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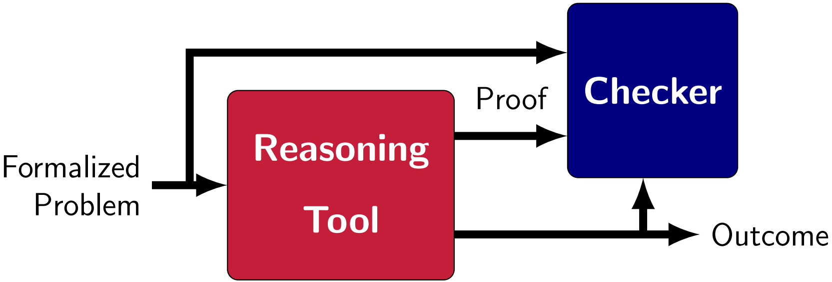
sat4math.com

Proof-Generating Automated Reasoning Programs



Proof-Generating Tools

- Only need to prove individual executions, not entire program
- Can have bugs in tool but still trust result
- Can we trust the checker?
  - Simple algorithms and implementation
  - Ideally formally verified

Clausal Proofs of Unsatisfiability

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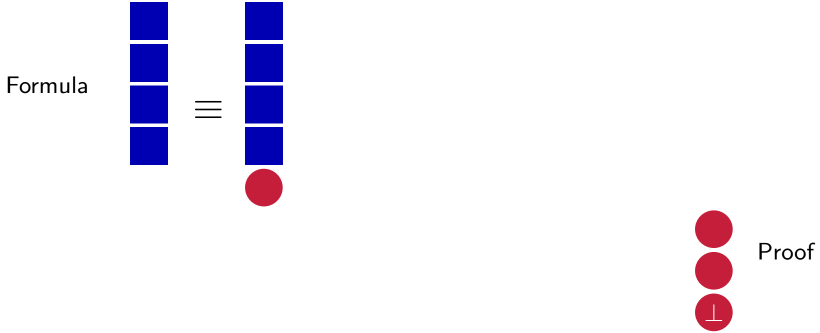

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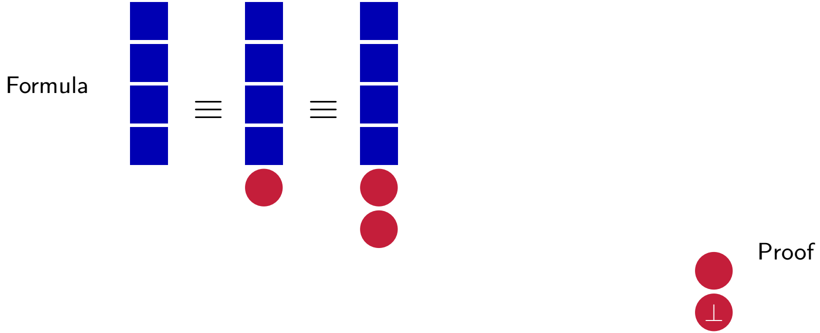

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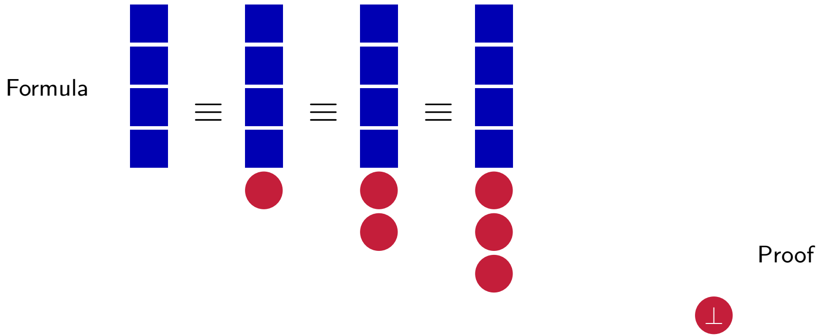

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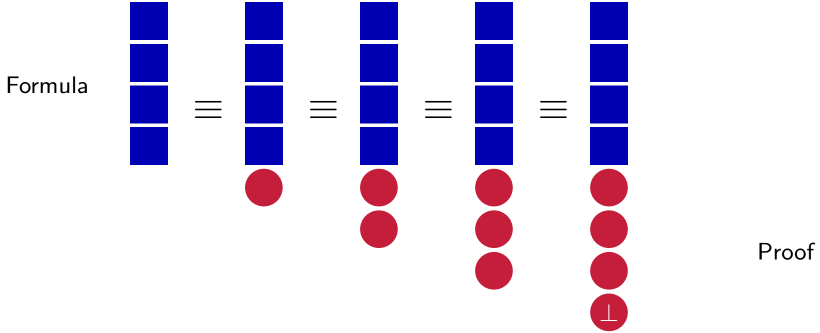

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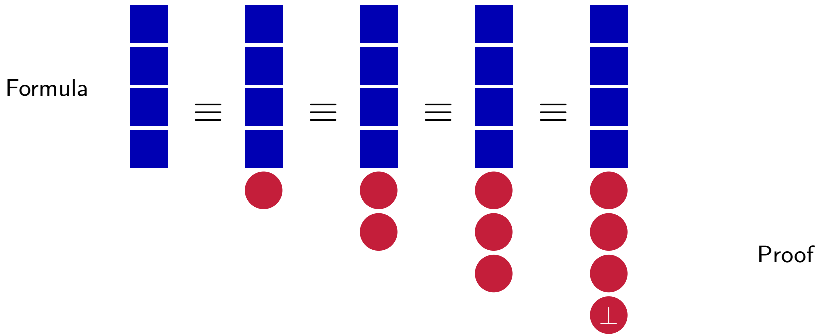


- Checking the redundancy of a clause in polynomial time
- Clausal proofs are easy to emit from modern SAT solvers
- A clausal proof usually covers many resolution proofs

Proof Checking Techniques Advances

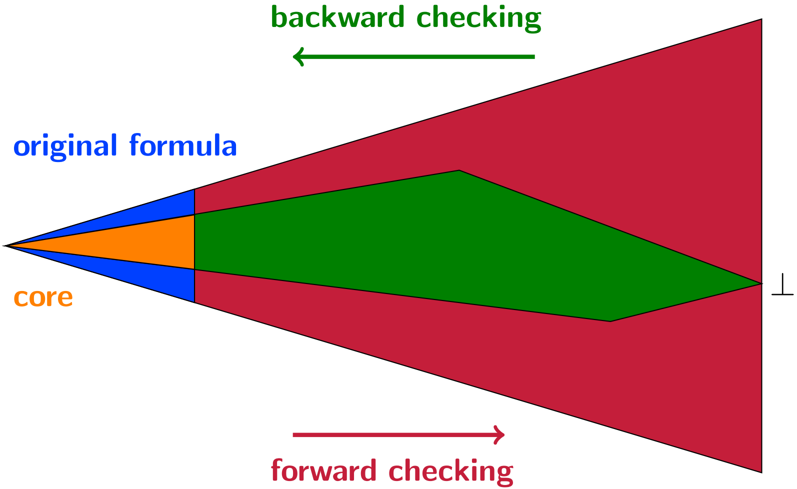
Proof checking techniques have improved significantly in recent years.

Clausal proofs of petabytes is size can now be validated.

Long-standing open math problems —including the Erdős discrepancy problem, the Boolean Pythagorean triples problem, and Schur number five— have solved with SAT and their proofs have been constructed and validated.

Efficient validation can even be achieved with a formally verified checker.

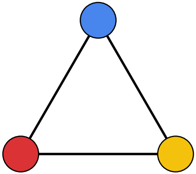
Backward Proof Checking: Remove Redundancy



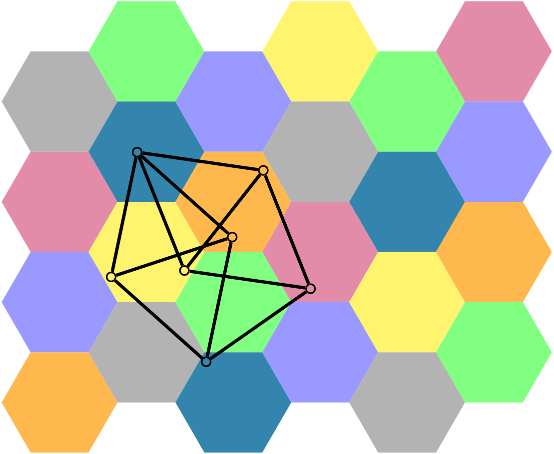
Chromatic Number of the Plane

The Hadwiger-Nelson problem:
How many colors are required to color the plane such that each pair of points that are exactly 1 apart are colored differently?

The answer must be three or more because three points can be mutually 1 apart—and thus must be colored differently.



Bounds since the 1950s



- The Moser Spindle graph shows the lower bound of 4
- A coloring of the plane showing the upper bound of 7

First progress in decades

Enormous progress after 70 years:

- Lower bound of 5 [DeGrey '18] based on a 1581-vertex graph
- This breakthrough started a polymath project
- Improved bounds of the fractional chromatic number of the plane

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We found smaller graphs with SAT:

- 874 vertices on April 14, 2018
- 803 vertices on April 30, 2018
- 610 vertices on May 14, 2018

Validation

Check 1: Are two given points exactly 1 apart? For example:

- $\left(\frac{19+3\sqrt{5}}{16}, \frac{5\sqrt{15}-7\sqrt{3}}{16}\right)$
- $\left(\frac{135+21\sqrt{5}-7\sqrt{33}+3\sqrt{165}}{96}, \frac{33\sqrt{15}-49\sqrt{3}-21\sqrt{11}-3\sqrt{55}}{96}\right)$

Our method: An approach based on Groebner basis theory developed by Armin Biere, Manuel Kauers, Daniela Kaufmann

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Check 2: Given a graph G, has it chromatic number k?

Our method: Construct two Boolean formulas: one asks whether G can be colored with $k-1$ colors (must be UNSAT) and one asks whether G can be colored with k colors (SAT).

Graph Operations

Two operations are use to construct bigger and bigger graph:

- Minkowski sum of A and B ($A \oplus B$): $\{a + b \mid a \in A, b \in B\}$
- Two rotated copies of a graph with a common point

Example

Let $A = \{(0,0),(1,0)\}$ and $B = \{(0,0),(1/2,\sqrt{3}/2)\}$

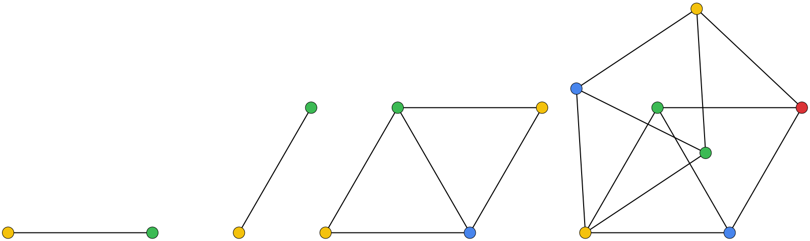
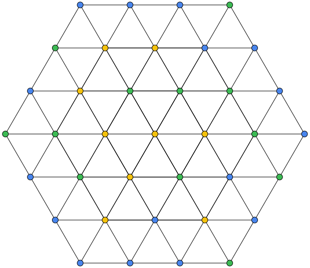


Figure: From left to right: UD-graphs A, B, $A \oplus B$, and the Moser Spindle.

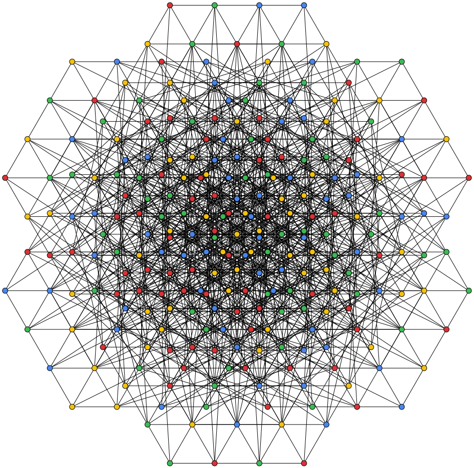
Small graphs in $\mathbb{Q}[\sqrt{3}, \sqrt{11}] \times \mathbb{Q}[\sqrt{3}, \sqrt{11}]$

Graph H_i is the 6-wheel with all edges of length i.

Graph H_i' is a copy of H_i rotated by 90 degrees.

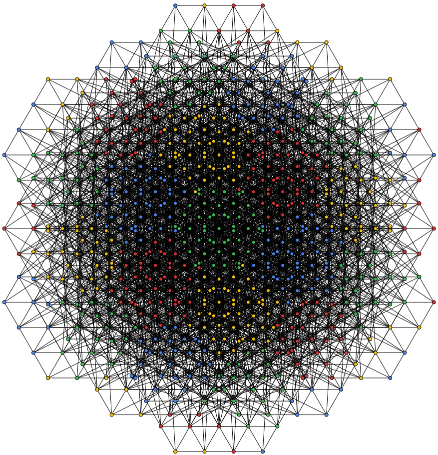


$H_{\frac{1}{3}} \oplus H_{\frac{1}{3}} \oplus H_{\frac{1}{3}}$

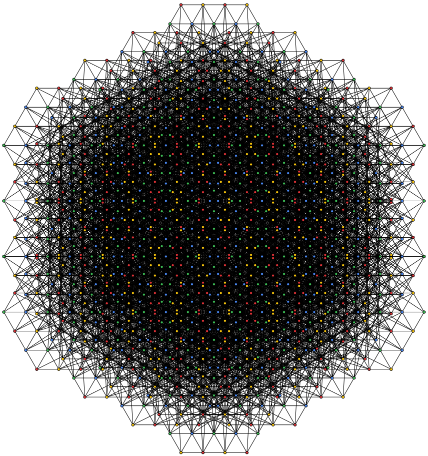


$H_{\frac{1}{3}} \oplus H_{\frac{1}{3}} \oplus H_{\frac{1}{3}} \oplus H'_{\frac{\sqrt{3}+\sqrt{11}}{6}}$

Larger graphs in $\mathbb{Q}[\sqrt{3},\sqrt{11}] \times \mathbb{Q}[\sqrt{3},\sqrt{11}]$



$$H_{\frac{1}{3}} \oplus H_{\frac{1}{3}} \oplus H_{\frac{1}{3}} \oplus H'_{\frac{\sqrt{3}+\sqrt{11}}{6}} \oplus H'_{\frac{\sqrt{3}+\sqrt{11}}{6}}$$



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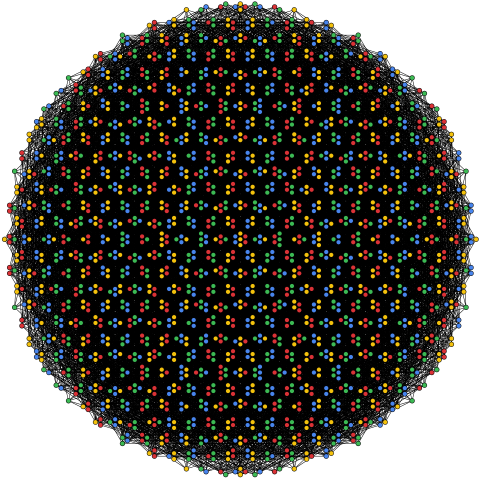
A Structured Graph on 2167 Vertices

G_{2167} is constructed by $(H_{\frac{1}{3}} \oplus H'_{\frac{\sqrt{3}+\sqrt{11}}{6}})^8$ without points > 2 from the center.

In all valid 4-colorings:

- Monochromatic lines can be observed
- Most points at distance 2 have the center color

Apply spindle to construct a 5-chromatic UD graph



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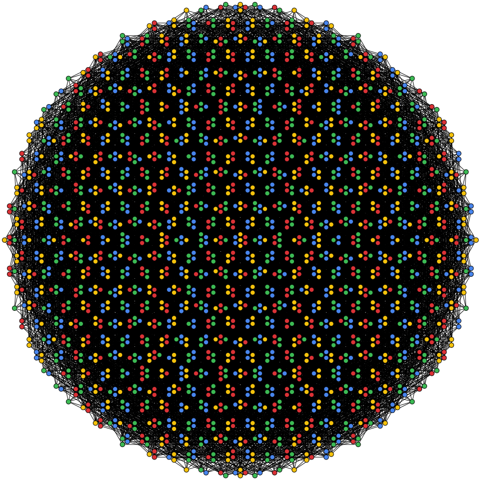

reduce each spindle part seperately

Extracting Subgraphs from a Proof of Unsatisfiability

The validation method to check whether a graph has (at least) chromatic number k construct a SAT formula asking whether the graph G can be colored with $k-1$ colors.

The resulting formula is unsatisfiable.

Most SAT solvers can emit a proof of unsatisfiability.

Proof checkers can extract an unsatisfiable core of the problem, which represents a subgraph of G.

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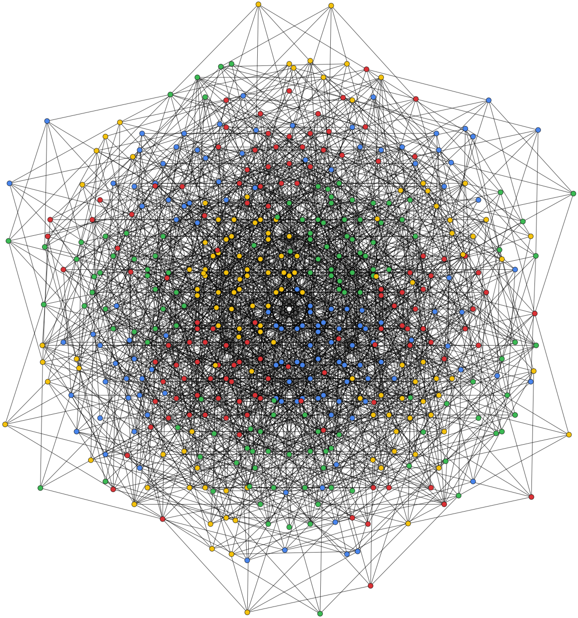
DEMO

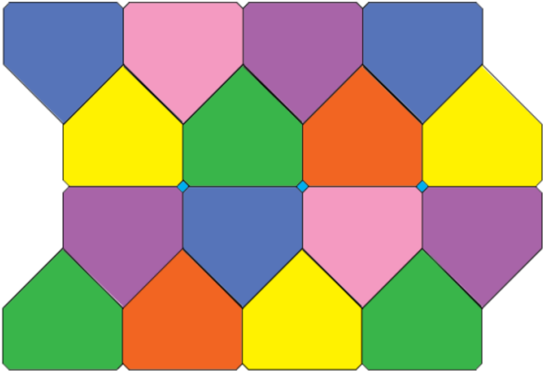
Graph G_{510}



Improve the Upper Bound?

A 7-coloring with one color covering 0.3% of the plane.

[Pritikin 1998]



Can SAT techniques be used to improve the upper bound?

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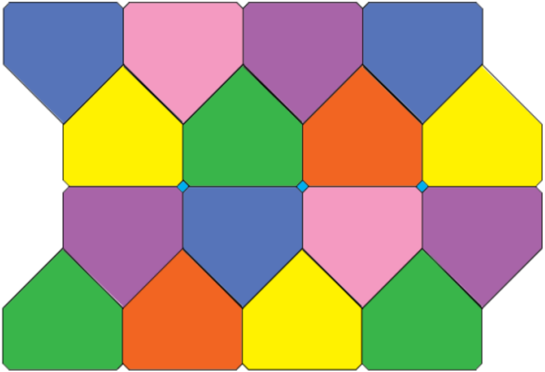


Can SAT techniques be used to improve the upper bound?

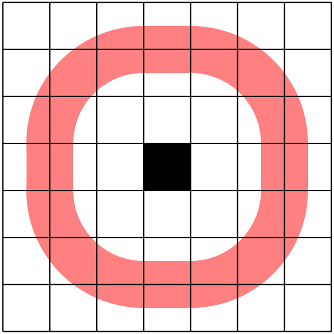
focus on height of infinite strips

Coloring Infinite Strips

What is the maximum height of an infinite strip w/o monochromic unit distance?

Strips are reduced to tilings

- square and hexagon tiles
- small tiles (0.01 UD)
- tiles at UD, different color
- observe patterns

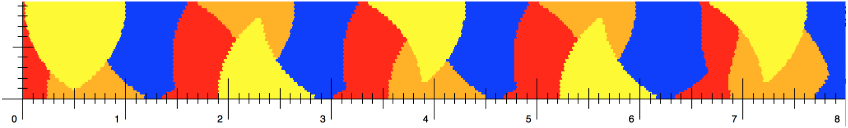


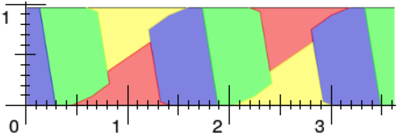
Best known bounds:

- 3 colors: $\frac{\sqrt{3}}{2} \simeq 0.866$
- 4 colors: 0.959
- 5 colors: $\frac{9}{2\sqrt{7}} \simeq 1.70084$
- 6 colors: $\frac{\sqrt{15}}{2} + \sqrt{3} \simeq 3.668$

Upper Bound Results

Pattern on 4 colors with bound 0.94:





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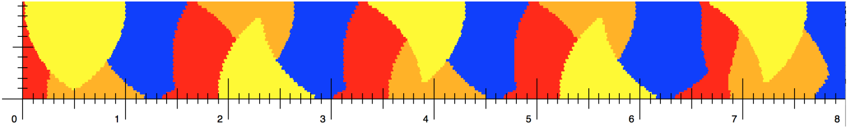
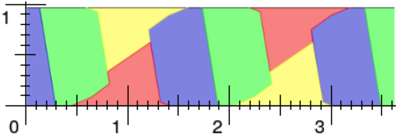
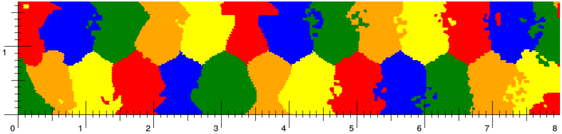
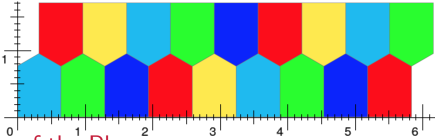




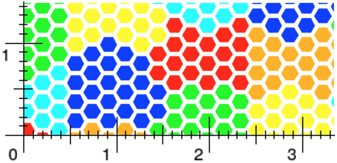
Pattern on 5 colors with improved bound 1.70:





Scaling Factor

Scaling: only add constraints of downscaled tile, so some parts of the strip are ignored



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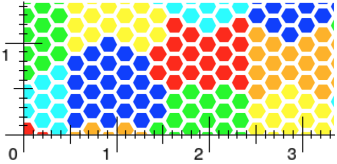
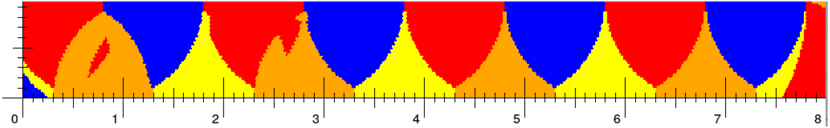


Unsuccessful on 4 colors (reaches invalid bound of 1):



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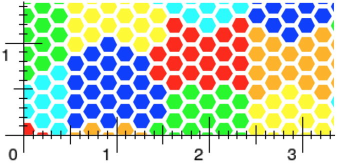

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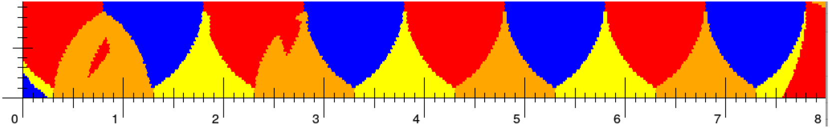
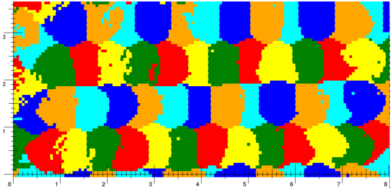


Successful on 6 colors (reaches existing bound of 3.66):



Conclusions

Aubrey de Grey showed that the chromatic number of the plane is at least 5 using a 1581-vertex unit-distance graph.

SAT technology can not only validate the result, but also reduce the size of the graph.

Our proof minimization techniques were able to construct a 510-vertex unit-distance graph with chromatic number 5.

Open questions regarding unit-distance graphs:

- What it is the smallest graph with chromatic number 5?
- Can we compute a graph that is human-understandable?
- Is there such a graph with chromatic number 6 (or even 7)?

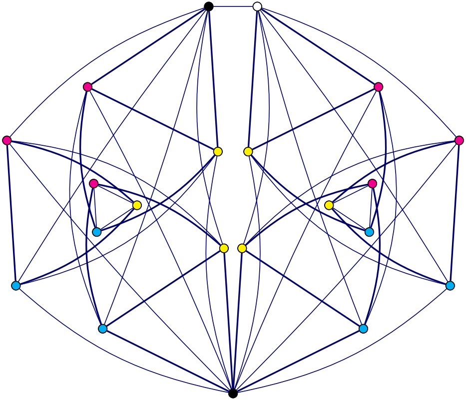
Future Work: Getting to Chromatic Number 6

Challenges:

- Need 1M+ vertices?
- Compute edges efficiently
- Which fields?

Study odd-distance graphs:

- Similar patterns observed
- Smallest-known OD 6-chromatic graph has 234 vertices [Parts]
- $\mathbb{Q}[\sqrt{3}, \sqrt{5}]^2$ is sufficient



A Page of God's Book on Theorems

"For many years now I am convinced that the chromatic number will be 7 or 6. One day, Paul Erdős said that God has an endless book that contains all the theorems and best of their evidence, and to some He shows it for a moment. If I had been awarded such an honor and I would have had a choice, I would have asked to look at the page with the problem of the chromatic number of the plane. And you?"

Alexander Soifer