Proof Systems and Proof Complexity

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Automated Reasoning and Satisfiability

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Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Challenges

Certifying Satisfiability and Unsatisfiability

- Certifying satisfiability of a formula is easy:

$$(x \lor y) \land (\overline{x} \lor \overline{y}) \land (\overline{y} \lor \overline{z})$$

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  - Just consider a satisfying assignment: $x\overline{y}z$

$$(x \vee y) \wedge (\overline{x} \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$$

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  - We can easily check that the assignment is satisfying: Just check for every clause if it has a satisfied literal!

- Certifying unsatisfiability is not so easy:
  - If a formula has n variables, there are 2^n possible assignments.
  - ➥ Checking whether every assignment falsifies the formula is costly.
  - More compact certificates of unsatisfiability are desirable.
    - ➥ Proofs

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- Example: Resolution proofs
  - A resolution proof is a sequence $C_1, \ldots, C_m$ of clauses.
  - Every clause is either contained in the formula or derived from two earlier clauses via the resolution rule:

$$\frac{C \vee x \qquad \overline{x} \vee D}{C \vee D}$$

  - C_m is the empty clause (containing no literals), denoted by $\perp$.
  - There exists a resolution proof for every unsatisfiable formula.

Resolution Proofs

- Example: $\Gamma = (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{z}) \wedge (x \vee \overline{y}) \wedge (\overline{u} \vee y) \wedge (u)$
- Resolution proof:
  $(\overline{x} \vee \overline{y} \vee z), (\overline{z}), (\overline{x} \vee \overline{y}), (x \vee \overline{y}), (\overline{y}), (\overline{u} \vee y), (\overline{u}), (u), \bot$

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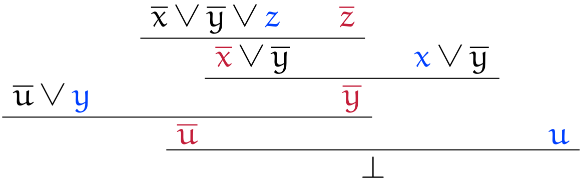

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$$\dfrac{\dfrac{\bar{u} \lor y \qquad \dfrac{\dfrac{\bar{x} \lor \bar{y} \lor z \qquad \bar{z}}{\bar{x} \lor \bar{y}} \qquad x \lor \bar{y}}{\bar{y}}}{\bar{u}} \qquad u}{\bot}$$

- Drawbacks of resolution:
  - For many seemingly simple formulas, there are only resolution proofs of exponential size.
  - State-of-the-art solving techniques are not succinctly expressible.

Clausal Proofs

Reduce the size of the proof by only storing added clauses

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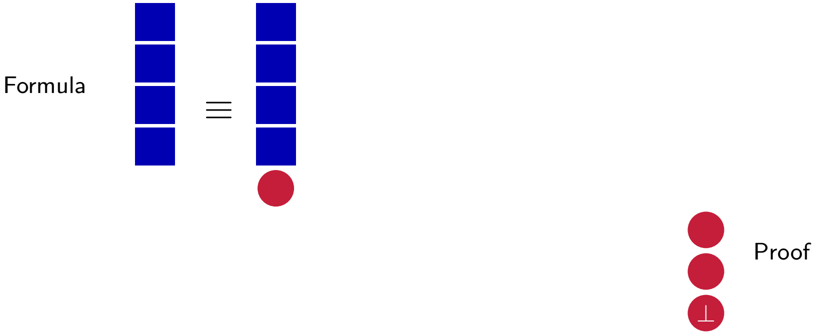

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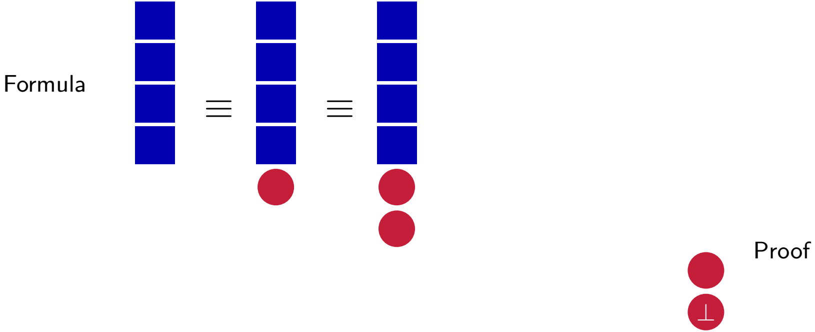

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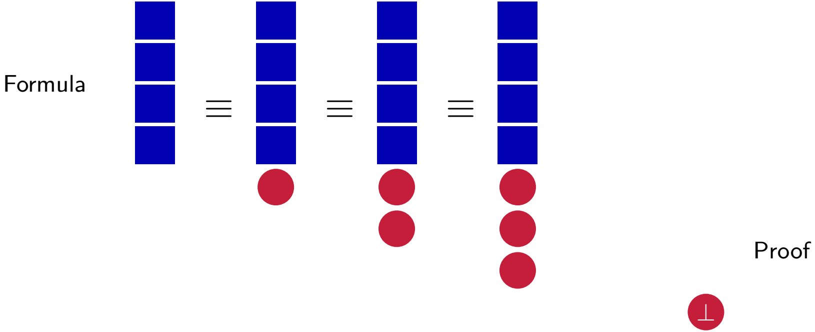

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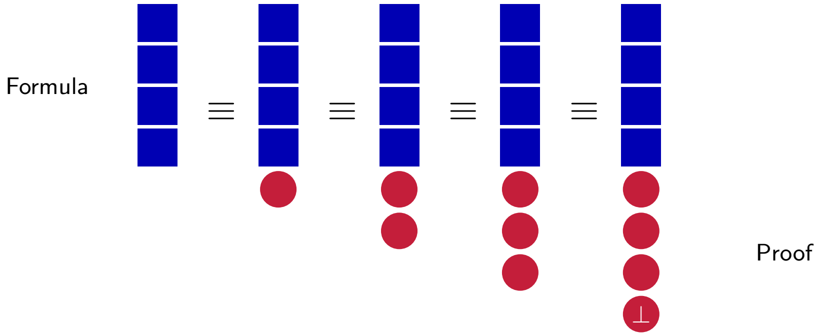

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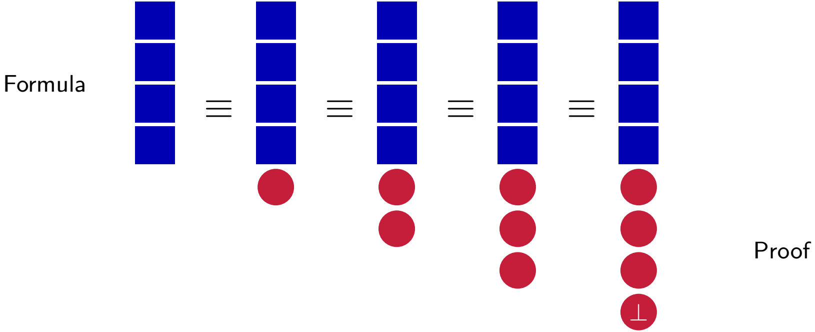

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- Checking redundancy should be efficient.

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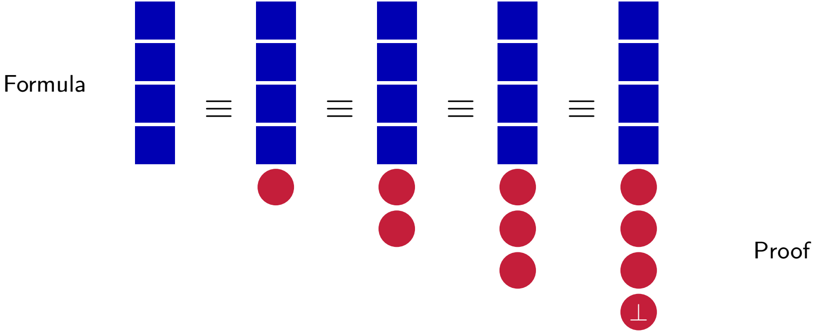

- Clauses whose addition preserves satisfiability are redundant.
- Checking redundancy should be efficient.

➥ Idea: Only add clauses that fulfill an efficiently checkable redundancy criterion.

Reverse Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let Γ be a formula. A clause C is implied by Γ via UP (denoted by $\Gamma \vdash_1 C$) if UP on $\Gamma \wedge \neg C$ results in a conflict.

Example

$$\Gamma = (a \vee b \vee \overline{c}) \wedge (\overline{a} \vee \overline{b} \vee c) \wedge (b \vee c \vee \overline{d}) \wedge (\overline{b} \vee \overline{c} \vee d) \wedge (a \vee c \vee d) \wedge (\overline{a} \vee \overline{c} \vee \overline{d}) \wedge (\overline{a} \vee b \vee d) \wedge (a \vee \overline{b} \vee \overline{d})$$

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Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \lor x \qquad \overline{x} \lor D}{C \lor D}\ (\text{RES}) \qquad\qquad \frac{A \qquad A \to B}{B}\ (\text{MP})$$

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➥ Inference rules reason about the presence of facts.

- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

  ➥ This leads to interference-based proof systems.

Early work on reasoning beyond resolution

The early SAT decision procedures used the Pure Literal rule [Davis and Putnam 1960; Davis, Logemann and Loveland 1962]:

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Extended Resolution (ER) [Tseitin 1966]

- Combines resolution with the Extension rule:

$$\frac{x \notin \Gamma \qquad \overline{x} \notin \Gamma}{(x \vee \overline{a} \vee \overline{b}) \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee b)} \text{ (ER)}$$

- Equivalently, adds the definition $x := \mathrm{AND}(a, b)$
- Can be considered the first interference-based proof system
- Is very powerful: No known lower bounds

Short Proofs of Pigeon Hole Formulas [Cook 1967]

Can $n{+}1$ pigeons be in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \leq p \leq n+1} (x_{1,p} \vee \cdots \vee x_{n,p}) \wedge \bigwedge_{1 \leq h \leq n,\ 1 \leq p < q \leq n+1} (\overline{x}_{h,p} \vee \overline{x}_{h,q})$$

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However, these proofs require introducing new variables:

- Hard to find such proofs automatically
- Existing ER approaches produce exponentially large proofs
- How to get rid of this hurdle? First approach: blocked clauses...

Blocked Clauses [Kullmann 1999]

Definition (Blocked Clause)

A clause $(C \lor x)$ is a blocked on x w.r.t. a CNF formula Γ if for every clause $(D \lor \overline{x}) \in \Gamma$, resolvent $C \lor D$ is a tautology.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{c}) \wedge (\overline{a} \vee c)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{c}$.
Third clause is blocked by c

Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

Blocked Clause Addition and Blocked Clause Elimination

The Blocked Clause proof system (BC) combines the resolution rule with the addition of blocked clauses.

- BC generalizes ER [Kullmann 1999]
- Recall

$$\frac{x \notin \Gamma \qquad \overline{x} \notin \Gamma}{(x \vee \overline{a} \vee \overline{b}) \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee b)} \text{(ER)}$$

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Blocked clause elimination used in preprocessing and inprocessing

- Simulates many circuit optimization techniques
- Removes redundant Pythagorean Triples

DRAT: An Interference-Based Proof System

- DRAT is a popular interference-based proof system
- DRAT allows adding RATs (defined below) to a formula.
  - It can be efficiently checked if a clause is a RAT.
  - RATs are not necessarily implied by the formula.
  - But RATs are redundant: their addition preserves satisfiability.
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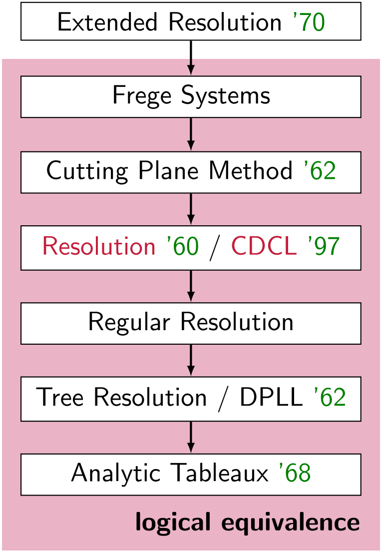
Definition (Resolution Asymmetric Tautology)

A clause $(C \lor x)$ is a resolution asymmetric tautology (RAT) on x w.r.t. a CNF formula Γ if for every clause $(D \lor \overline{x}) \in \Gamma$, $C \lor D$ is implied by Γ via unit-propagation, i.e., $\Gamma \vdash_1 C \lor D$.

Proof Search in Strong Proof Systems

Existence of Short Proofs



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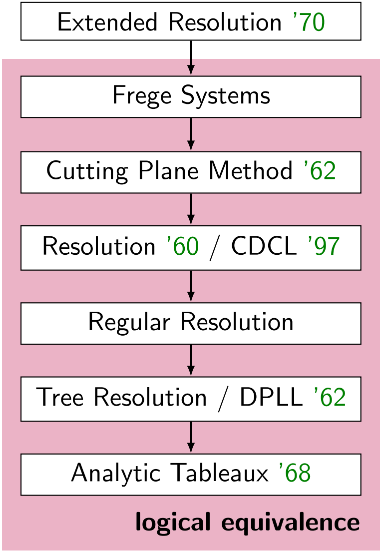
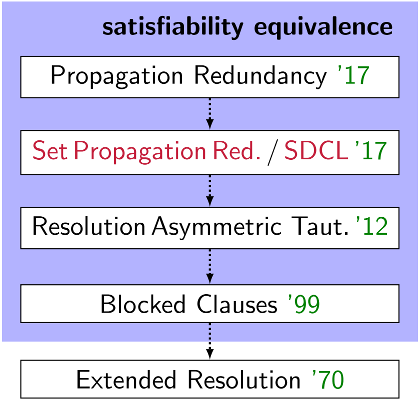


Finding Short Proofs

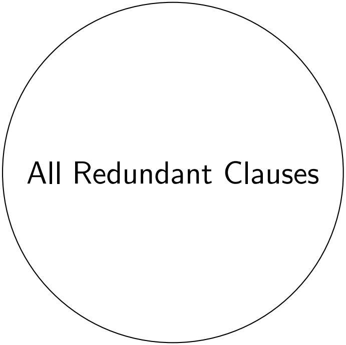


Express solving techniques compactly
[Järvisalo, Heule, and Biere '12]

Short proofs without new variables
[Heule, Kiesl, and Biere '17A]

Redundant Clauses

- Strong proof systems allow adding many redundant clauses.



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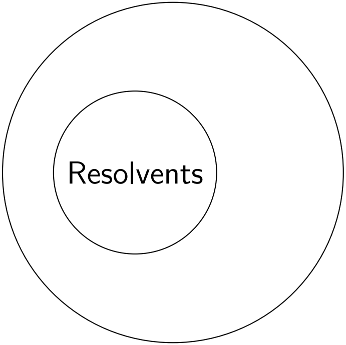

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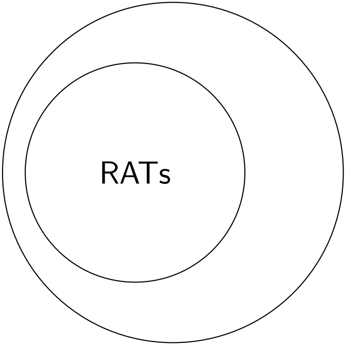

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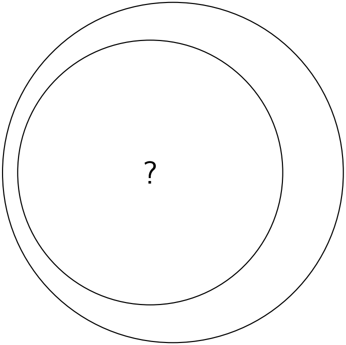

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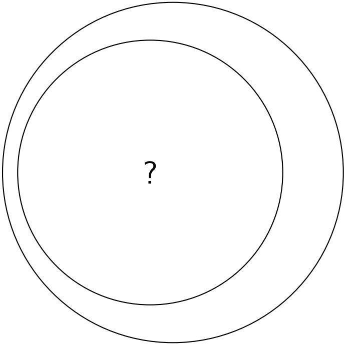
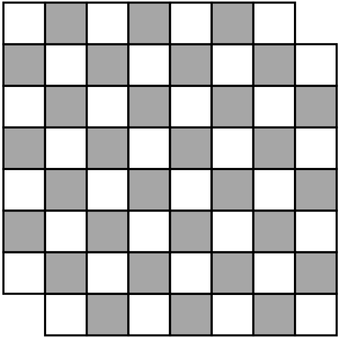
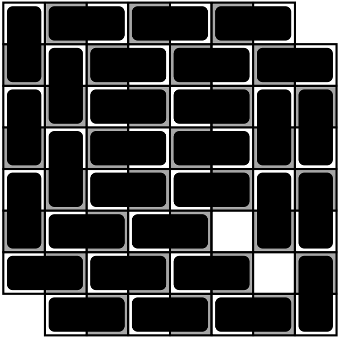


- The new proof systems can give short proofs of formulas that are considered hard.
- Are stronger redundancy notions still efficiently checkable?

Mutilated Chessboards: “A Tough Nut to Crack” [McCarthy]

Can a chessboard be fully covered with dominos after removing two diagonally opposite corner squares?





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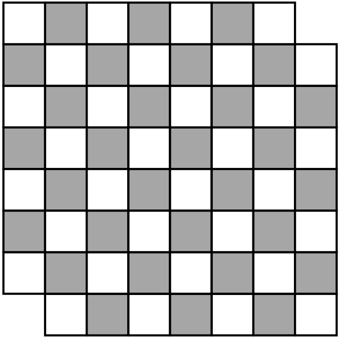
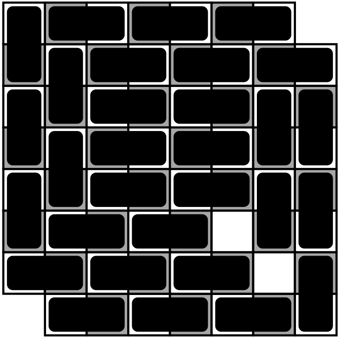


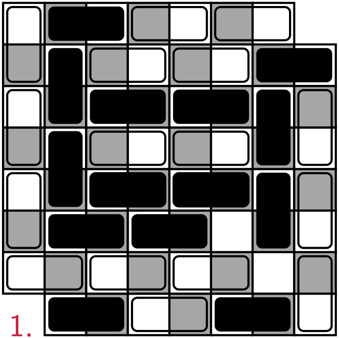


Easy to refute based on the following two observations:

- There are more white squares than black squares; and
- A domino covers exactly one white and one black square.

Without Loss of Satisfaction

One of the crucial techniques in SAT solvers is to generalize a conflicting state and use it to constrain the problem.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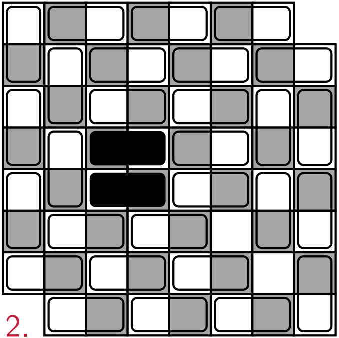
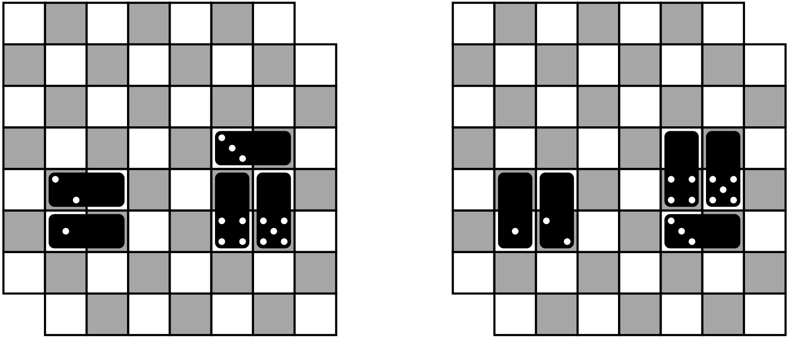

2.

The used proof system can have a big impact on the size:

1. Resolution can only reduce the 30 dominos to 14 (left); and
2. "Without loss of satisfaction" can reduce them to 2 (right).

Mutilated Chessboards: An alternative proof

Satisfaction-Driven Clause Learning (SDCL) is a new solving paradigm that finds proofs in the PR proof system [HKB '17]



SDCL can detect that the above two patterns can be blocked

- This reduces the number of explored states exponentially
- We produced SPR proofs that are linear in the formula size

Redundancy as an Implication

A formula Δ is at least as satisfiable as a formula Γ if $\Gamma \models \Delta$.

Given a formula Γ and assignment α, we denote with $\Gamma|_\alpha$ the reduced formula after removing from Γ all clauses satisfied by α and all literals falsified by α.

Theorem

A clause C is redundant w.r.t. a formula Γ iff there exists an assignment ω such that

$$\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\omega$$

This is the strongest notion of redundancy. However, entailment ($\models$) cannot be checked in polynomial time (assuming $P \neq NP$), unless bounded.

Checking Redundancy Using Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let Γ be a formula, C a clause, and α the smallest assignment that falsifies C. C is implied by Γ via UP (denoted by $\Gamma \vdash_1 C$) if UP on $\Gamma|_\alpha$ results in a conflict.
- Implied by UP is used in SAT solvers to determine redundancy of learned clauses and therefore $\vdash_1$ is a natural restriction of $\models$.
- We bound $\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\omega$ by $\Gamma \wedge \neg C \vdash_1 (\Gamma \wedge C)|_\omega$
- Example:
  $\Gamma = (x \vee y \vee z) \wedge (\overline{x} \vee y \vee z) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{y} \vee z)$
  and $\Delta = (z)$. Observe that $\Gamma \models \Delta$, but that $\Gamma \nvdash_1 \Delta$.

Hand-crafted PR Proofs of Pigeon Hole Formulas

We manually constructed PR proofs of the famous pigeon hole formulas and the two-pigeons-per-hole family.

- The proofs consist only of binary and unit clauses.
- Only original variables appear in the proof.
- All proofs are linear in the size of the formula.

➥ The PR proofs are smaller than Cook's ER proofs.

- All resolution proofs of these formulas are exponential in size.

Autarkies

A non-empty assignment α is an autarky for formula Γ if every clause $C \in \Gamma$ that is touched by α is also satisfied by α.

A pure literal and a satisfying assignment are autarkies.

Example

Consider the formula $\Gamma := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
Assignment $\alpha_1 = \overline{z}$ is an autarky:
$(x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$. Assignment $\alpha_2 = x\,\overline{y}\,z$ is an autarky: $(x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

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Given an assignment α, $\Gamma|_\alpha$ denotes a formula Γ without the clauses satisfied by α and without the literals falsified by α.

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula Γ.
Then, Γ and $\Gamma|_\alpha$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a conditional autarky for formula Γ if α_{aut} is an autarky for $\Gamma|\alpha_{\text{con}}$.

Example

Consider the formula $\Gamma := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \overline{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\overline{y}$ is a conditional autarky for Γ:

$$\alpha_{\text{aut}} = \overline{y} \text{ is an autarky for } \Gamma|\alpha_{\text{con}} = (\overline{y} \vee \overline{z}).$$

Conditional Autarkies

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Example

Consider the formula $\Gamma := (x \lor y) \land (x \lor \overline{y}) \land (\overline{y} \lor \overline{z})$.
Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \overline{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\overline{y}$ is a conditional autarky for Γ:

$$\alpha_{\text{aut}} = \overline{y} \text{ is an autarky for } \Gamma|_{\alpha_{\text{con}}} = (\overline{y} \lor \overline{z}).$$

Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a conditional autarky for formula Γ.
Then Γ and $\Gamma \land (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\overline{x} \lor \overline{y})$.

Finding Redundant Clauses: The Positive Reduct

Determining whether a clause C is PR w.r.t. a formula Γ is an NP-complete problem.

How to find PR clauses? Encode it in SAT!

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Key Idea: While solving a formula Γ, check whether the positive reduct of Γ and the current assignment α is satisfiable. In that case, prune the branch α.

The Positive Reduct: An Example

Given a formula Γ and a clause C. Let α denote the smallest assignment that falsifies C. The positive reduct of Γ and α, denoted by $p(\Gamma, \alpha)$, is the formula that contains C and all $\mathit{assigned}(D, \alpha)$ with $D \in \Gamma$ and D is satisfied by α.

Example

Consider the formula $\Gamma := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

Let $C_1 = (\overline{x})$, so $\alpha_1 = x$.
The positive reduct $p(\Gamma, \alpha_1) = (\overline{x}) \wedge (x) \wedge (x)$ is unsatisfiable.

Let $C_2 = (\overline{x} \vee \overline{y})$, so $\alpha_2 = x\,y$.
The positive reduct $p(\Gamma, \alpha_2) = (\overline{x} \vee \overline{y}) \wedge (x \vee y) \wedge (x \vee \overline{y})$ is satisfiable.

Pseudo-Code of CDCL (formula Γ)

$\alpha := \emptyset$
forever do
  $\alpha :=$ Simplify (Γ, α)
  if $\Gamma|_\alpha$ contains a falsified clause **then**
    $C :=$ AnalyzeConflict $()$
    if C is the empty clause **then return** unsatisfiable
    $\Gamma := F \cup \{C\}$
    $\alpha :=$ BackJump (C, α)
  else
    $l :=$ Decide $()$
    if l is undefined **then return** satisfiable
    $\alpha := \alpha \cup \{l\}$

Pseudo-Code of SDCL (formula Γ)

```
α := ∅
forever do
   α := Simplify (Γ, α)
   if Γ|α contains a falsified clause then
      C := AnalyzeConflict ()
      if C is the empty clause then return unsatisfiable
      Γ := F ∪ {C}
      α := BackJump (C, α)
   else if p(Γ, α) is satisfiable then
      C := AnalyzeWitness ()
      Γ := F ∪ {C}
      α := BackJump (C, α)
   else
      l := Decide ()
      if l is undefined then return satisfiable
      α := α ∪ {l}
```

Proofs of Unsatisfiability

Beyond Resolution

Propagation Redundancy

Satisfaction-Driven Clause Learning

Challenges

Theoretical Challenges

Lower bounds for interference-based proof systems with new variables will be hard, but what about without new variables?

- Lower bound for BC w/o new variables? Pigeon-hole formulas?
- Lower bound for SET w/o new variables? Tseitin formulas?
- Lower bound for PR w/o new variables?!

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Can we design stronger proof systems that make it even easier to compute short proofs?

Practical Challenges

The current version of SDCL is just the beginning:

- Which heuristics allow learning short PR clauses?
- How to construct an AnalyzeWitness procedure?
- Can the positive reduct be improved?

Can local search be used to find short proofs of unsatisfiability?

Constructing positive reducts (or similar formulas) efficiently:

- Generating a positive reduct is more costly than solving them
- Can we design data-structures to cheaply compute them?